Operator Theory
Advances and Applications
216

Michael Ruzhansky
Jens Wirth
Editors

Modern Aspects of the Theory of Partial Differential Equations

Birkhäuser

Birkhäuser

Operator Theory: Advances and Applications

Volume 216

Founded in 1979 by Israel Gohberg

Modern Aspects of the Theory of Partial Differential Equations

Michael Ruzhansky
Jens Wirth
Editors

A | Advances in
P | Partial
D | Differential
E | Equations

Birkhäuser

Editors
Michael Ruzhansky
Department of Mathematics
Imperial College London
180 Queen's Gate
London SW7 2AZ
UK
m.ruzhansky@imperial.ac.uk

Jens Wirth
Department of Mathematics
University of Stuttgart
Pfaffenwaldring 57
70569 Stuttgart
Germany
Jens.Wirth@mathematik.uni-stuttgart.de

2010 Mathematics Subject Classification: 35-06, 35Jxx, 35Kxx, 35Lxx, 35Pxx, 35Sxx, 31A25, 45K05, 45Q05, 47B35, 65E05, 65Nxx

ISBN 978-3-0348-0068-6 e-ISBN 978-3-0348-0069-3
DOI 10.1007/978-3-0348-0069-3

Library of Congress Control Number: 2011928227

Cover design: deblik, Berlin

Printed on acid-free paper

Springer Basel AG is part of Springer Science+Business Media

www.birkhauser-science.com

Contents

Preface

This volume contains a collection of papers devoted to modern aspects in the study of partial differential equations. A number of them stem from plenary and invited session lectures presented at the 7th International ISAAC Congress which was held at Imperial College London in the period 13–18 July 2009. ISAAC is the International Society for Analysis, its Applications, and Computation, and it is already a tradition that its biannual congresses include a wide selection of sessions devoted to the analysis of partial differential equations. Motivated by this tradition and as an instrument to further strengthen the PDE community within ISAAC an interest group *'Partial Differential Equations'* (IGPDE) was founded during the congress. The editors of this volume took this as one of the incentives to publish this collection of papers. It is aimed at a broad audience, beginners as well as specialists, and intended as a presentation of a wide range of topics addressed in contemporary research in the field.

Papers associated to the plenary lectures given by L. Boutet de Monvel, V. Kokilashvili and B.-W. Schulze appear in this volume. In addition we collected selected papers from PDE-related sessions and further contributions on closely related topics. Altogether, this volume touches upon several aspects of ordinary and partial differential equations, such as boundary value problems, maximum and extremum principles, wave, Schrödinger and parabolic equations, applications to elasticity and thermoelasticity, and further numerical aspects.

A second special collection of papers presented at the congress and devoted to the analysis of evolutionary partial differential equations has appeared as a special volume of Rendiconti dell'Istituto di Matematica dell'Università di Trieste, edited by D. Del Santo, F. Hirosawa and M. Reissig.

Last, but not least, it is our pleasure to thank F. Bucci, I. Lasiecka, V. Smyshlyaev and Y. Kurylev for all the editorial work they undertook for papers arising from their sessions. They are mentioned as communicators in this volume.

Michael Ruzhansky and Jens Wirth
London, August 2010

Operator Theory:
Advances and Applications, Vol. 216, 1–16

Toeplitz Operators and Asymptotic Equivariant Index

L. Boutet de Monvel

Abstract. This is an account of a lecture given at the 7th ISAAC Congress, where I described a joint work with E. Leichtnam, X. Tang and A. Weinstein giving a proof of the Atiyah-Weinstein index formula. This concerns the index of an operator closely related to Toeplitz operators, for which analogues of the Atiyah-Singer index formula do not make sense. Instead we used an equivariant asymptotic index formula, which does; it is an outgrowth of Atiyah and Singers theory of equivariant index for transversally elliptic pseudodifferential operators.

Mathematics Subject Classification (2000). 19L47; 32A25; 53D10; 58J40.

1. Szegö projectors, Toeplitz operators

We first describe generalized Szegö projectors and Toeplitz operators, which generalize pseudo-differential operators on arbitrary contact manifolds. An important case arises from complex (CR) analysis.

Let M be a compact manifold, and $\Sigma \subset T^\bullet M$ a symplectic subcone[1].

Definition 1. A generalized Szegö projector associated to Σ (or Σ-Szegö projector) is a self adjoint elliptic Fourier integral projector S of degree 0 ($S = S^* = S^2$), whose complex canonical relation $\mathbb{C}$ is $\gg 0$, with real part the diagonal $\operatorname{diag}\Sigma$ (elliptic means that the principal symbol of S does not vanish on Σ).

Specially useful examples are

1) Σ is the full cotangent bundle $T^\bullet M$, S is the identity operator.
2) M is the boundary of a strictly pseudoconvex bounded complex domain, S is the Szegö projector (see below). More generally, M is a compact oriented contact manifold, $\Sigma \subset T^\bullet M$ is the set of positive multiples of the contact form (a generalized Szegö projector always exists, see below).

[1] $T^\bullet$ denotes the cotangent bundle deprived of its zero section.

1.1. Example 1: Microlocal model

The following example was described in [6]. It is universal in the sense that any generalized Szegö projector is microlocally isomorphic to it, via some elliptic Fourier integral transformation (with $\dim \Sigma = 2p, \dim M = p+q$).

Let $(x, y) = (x_1, \ldots, x_p, y_1, \ldots, y_q)$ denote the variable in $\mathbb{R}^{p+q}$. Set $D = (D_j)$, with

$$D_j = \partial_{y_j} + |D_x| y_j \qquad (j = 1, \ldots, q).$$

The D_j commute; the complex involutive variety $\operatorname{char} D$ is defined by the complex equations $\eta_j - i|\xi| y_j = 0$; it is $\gg 0$, in the sense of [20, 21]. Its real part is the symplectic manifold $\Sigma : \{\eta_j = y_j = 0\}$.

The kernel of D in L^2 is the range of the Hermite operator H (in the sense of [6]) defined by its partial Fourier transform:

$$f \in L^2(\mathbb{R}^p) \mapsto Hf \quad \text{with} \quad \mathcal{F}_x Hf(\xi, y) = \left(\frac{|\xi|}{\pi}\right)^{\frac{q}{4}} e^{-\frac{1}{2}|\xi| y^2} \hat{f}(\xi).$$

The orthogonal projector on $\ker D$ is $S = HH^*$:

$$f \mapsto (2\pi)^{-p} \int_{\mathbb{R}^{2p+q}} e^{i(\langle x - x', \xi\rangle + i\frac{|\xi|}{2}(y^2 + y'^2))} \left(\frac{|\xi|}{\pi}\right)^{\frac{q}{2}} f(x', y') dx' dy' d\xi.$$

As H, it is a Fourier integral operator, whose complex canonical relation is $\gg 0$, with real part the graph of Id_Σ.[2].

1.2. Example 2: Holomorphic model

Let X be the boundary of a strictly pseudoconvex Stein complex manifold (with smooth boundary); the contact form of X is the form induced by $\operatorname{Im} \partial\phi$ where ϕ is any defining function ($\phi = 0, d\phi \neq 0$ on X, $\phi < 0$ inside).

E.g., if X is the unit sphere bounding the unit ball of $\mathbb{C}^n$, with defining function $\bar{z} \cdot z - 1$, the contact form is $Im\, \bar{z} \cdot dz_{|X}$.

The **Szegö projector** S is the orthogonal projector on the holomorphic subspace $\mathbb{H} = \ker \bar{\partial}_b$ of boundary values of holomorphic functions (the fact that S is Fourier integral operator as above was proved in [14]). The system of (pseudo) differential operators playing the role of D is the tangential Cauchy–Riemann system $\bar{\partial}_b$.[3]

Remark. A basic example of Toeplitz structure is $\Sigma = T^\bullet M$ (M a compact manifold), $S = \operatorname{Id}$: the Toeplitz algebra is the algebra of pseudodifferential operators acting on the sheaf of microfunctions on M. This is in fact a special case of the holomorphic case – Example 2.

[2] Fourier integral operators are described in [19]. Fourier integral operators with complex canonical relation are described in [20, 21]

[3] at least if the dimension n is > 1 – if $n = 1$, S is the Hilbert projector $\sum_{-\infty}^{\infty} f_k z^k \mapsto \sum_0^\infty f_k z^k$, it is a pseudodifferential projector.

1.3. Main properties

Cf. [11, 9, 10]

1) A Σ-Szegö projector S always exists. All such projectors have a unique microlocal model (via some elliptic FIO transformation) depending only on $\dim\Sigma$, $\dim M$.
2) Toeplitz operators defined by S are the operators on $\mathbb{H}$ of the form $u \in \mathbb{H} \mapsto T_P(u) = SPS(u)$ with P a pseudodifferential operator on M. They form an algebra $\mathcal{E}_X$ (or $\mathcal{E}_\Sigma$ or $\mathcal{E}$[4]. Modulo smoothing operators, they form a sheaf acting on $\mu\mathbb{H}$, locally isomorphic to the sheaf of pseudodifferential operators acting on the sheaf of microfunctions (in p variables if $\dim\Sigma = 2p$).
3) If S, S' are two Σ-Szegö projectors with range $\mathbb{H}, \mathbb{H}'$, S' induces a quasi isomorphism $\mathbb{H} \to \mathbb{H}'$ (the restriction of SS' to $\mathbb{H}$ is a positive (≥ 0) elliptic Toeplitz operator).

   More generally, if $\Sigma \subset T^\bullet M, \Sigma' \subset T^\bullet M'$ are two symplectic cones and $f : \Sigma \to \Sigma'$ a homogeneous symplectic isomorphism, there always exists a Fourier integral operator F from M to M', inducing an "elliptic" Fredholm map $\mathbb{H} \to \mathbb{H}'$ (such elliptic FIO exist, they were called "adapted" in [11, 9]).

The pair $(\mathcal{E}_\Sigma, \mu\mathbb{H})$ consisting of the sheaf of micro-Toeplitz operators (i.e., smoothing operators), acting on $\mu\mathbb{H}$ is well defined, up to (non unique) isomorphism: it only depends on the symplectic cone Σ, not on the embedding.

4) $\mathbb{H}$ is the set of solutions of a system (an ideal) of pseudo-differential equations described by a pseudo-differential complex D_Σ mimicking the $\bar\partial_b$ in the holomorphic case (see below).

The K-theoretic element $[D_\Sigma] \in K_X(S^*M)$ it defines is precisely the Bott element, defining the Bott periodicity isomorphism $K(X) \to K_X(S^*M)$.

5) All these constructions allow a compact group action.

We also use a vector bundle extension: an equivariant G-bundle is an invariant direct factor E of a trivial G vector-bundle $X \times V$, defined by an invariant projector p (V a finite representation of G). The corresponding Toeplitz space (or Toeplitz bundle) $\mathbb{H}_E$, with symbol E, is the range of an equivariant Toeplitz projector P of degree 0 in $\mathbb{H} \otimes V$, with symbol p. Here again $\mathbb{H}_E$ is only defined up to a Fredholm map. Equivalently, $\mathbb{H}$ is defined by a 'good' projective $\mathcal{E}$ module $\mathcal{M}$, i.e., the range of Toeplitz projector P' of degree 0 in some free left-module $\mathcal{E}^N$: $\mathbb{E} = \mathrm{Hom}_{\mathcal{E}}(\mathcal{M}, \mathbb{H})$.

If $\mathbb{E}, \mathbb{F}$ are two equivariant Toeplitz bundles, there is an obvious notion of Toeplitz operator $P : \mathbb{E} \to \mathbb{F}$, and of its principal symbol $\sigma_d(P)$ if it is of degree d, which is a homogeneous vector-bundle homomorphism $E \to F$ on Σ.

P is elliptic of degree d if its symbol is invertible; then it is a Fredholm operator $\mathbb{E}^{(s)} \to \mathbb{F}^{(s-d)}$ and has an index (which does not depend on s)[5].

---

[4] if M is a manifold one writes $\mathcal{E}_M$ for $\mathcal{E}_{S^*M}$.
[5] $\mathbb{E}^{(s)}$ its space of Sobolev H^s sections of $\mathbb{E}$.

1.4. Miscellaneous

Toeplitz-Fourier integral operators. The analogue of Fourier integral transformations is the following: let X, X' be two contact manifolds, S, S' generalized Szegö projectors, and $f : X \to X'$ a contact isomorphism. The pushforward map $u \mapsto u \circ f^{-1}$ does not send $\mathbb{H}$ to $\mathbb{H}'$: we correct it as for Toeplitz operators $T_f(u) = S'(u \circ f^{-1})$; this behaves as an elliptic Fourier operator attached to the contact map f. Other analogues of F.I.O attached to f are of the form $u \mapsto A'T_f u$, A' a Toeplitz operator on X'.

Atiyah-Weinstein problem. The Atiyah-Weinstein problem can be described as follows: If X is a compact contact manifold, and S, S' two Szegö projectors defined by two embeddable CR structures giving the same contact structure, then the restriction of S' to $\mathbb{H}$ is a Fredholm operator $\mathbb{H} \to \mathbb{H}'$ (SS' induces an elliptic Toeplitz operator on $\mathbb{H}$). In this case the spaces $\mathbb{H}, \mathbb{H}'$ and the index are well defined. The Atiyah-Weinstein conjecture computes the index in terms of topological data of the situation (topology of the holomorphic fillings of which X is the boundary).

2. Equivariant Toeplitz algebra

In the sequel we use the following notations:

- G a compact Lie group, with Haar measure dg ($\int dg = 1$), Lie algebra $\mathfrak{g}$.
- Σ a G-symplectic cone, basis X (a compact oriented contact G-manifold).
- ω its symplectic form,
- λ the Liouville form ($\omega = d\lambda$) (G-invariant).
- Σ is canonically identified with the set of positive multiples of λ_X in T^*X.
- S a G-invariant generalized Szegö projector, with range $\mathbb{H} = \hat{\bigoplus}\mathbb{H}_\alpha$ (where α runs over the set of irreducible representations, and $\mathbb{H}_\alpha$ is the corresponding isotypic component of $\mathbb{H}$).

2.1. Equivariant trace

The G-trace and G-index were introduced by M.F. Atiyah in [4] for equivariant pseudo-differential operators on a G-manifold. The G-trace of P is a distribution on G, describing $\operatorname{tr}(g \circ P)$. We adapt this to Toeplitz operators.

Any $v \in \mathfrak{g}$ defines a vector field L_v on X and a Toeplitz operator T_v on $\mathbb{H}$ (or any Toeplitz bundle $\mathbb{E}$).

Definition 2. $\operatorname{char}\mathfrak{g}$ (characteristic set of $\mathfrak{g}$) denotes the closed subcone of Σ where all symbols of infinitesimal operators $T_v, \xi \in \mathfrak{g}$ vanish.

The base Z of $\operatorname{char}\mathfrak{g}$ is the set of points of X where all Lie generators $L_v, v \in \mathfrak{g}$ are orthogonal to the Liouville form λ_X. $\operatorname{char}\mathfrak{g}$ contains the fixed point set Σ^G, whose basis is the fixed point set X^G because G is compact. Note that Σ^G is always a smooth symplectic cone and its base X^G a smooth contact manifold; $\operatorname{char}\mathfrak{g}$ and Z may be singular.

Let $\mathbb{E}$ be an equivariant Toeplitz bundle as above, $\mathbb{E} = \bigoplus \mathbb{E}_\alpha$ its the decomposition in isotypic components. If $P : \mathbb{E} \to \mathbb{E}$ is a Toeplitz operator of trace class ($\deg P < -n$), the trace function $\mathrm{Tr}_P^G(g) = \mathrm{tr}\,(g \circ P)$ is a continuous function on G (it is smooth if P is of degree $-\infty$), and we have

$$\mathrm{Tr}_P^G(g) = \sum_\alpha \frac{1}{d_\alpha}\ \mathrm{tr}\, P|_{\mathbb{H}_\alpha}\ \chi_\alpha\,; \tag{1}$$

χ_α is the character of α, d_α the dimension (the Fourier coefficient is $\frac{1}{d_\alpha}\mathrm{tr}\, P|_{\mathbb{E}_\alpha}$).

The following result is an immediate adaptation of the similar result of [4] for pseudo-differential operators.

Theorem 3. *Let $P : \mathbb{E} \to \mathbb{E}$ be a Toeplitz operator, with $P \sim 0$ near $\mathrm{char}\,\mathfrak{g}$. Then $Tr_P^G(g) = tr\, g \circ P$ is defined as a distribution on G; $P|_{\mathbb{E}_\alpha}$ is of trace class for each α and formula* (1) *holds.*

We have $\mathrm{Tr}_{PQ}^G(g) = \mathrm{Tr}_{QP}^G(g)$ if one of the two operators is equivariant and one ~ 0 near $\mathrm{char}\,\mathfrak{g}$; so Tr^G defines a trace map on the algebra of equivariant Toeplitz operators.

Proof. This is true if P is of trace class. For the general case, we choose a bi-invariant elliptic operator D of order $m > 0$ on G, e.g., the Casimir of a faithful representation, with $m = 2$; it defines an invariant Toeplitz operator $D_X : \mathbb{E} \to \mathbb{E}$, elliptic outside of $\mathrm{char}\,\mathfrak{g}$. If $P \sim 0$ near Σ, we can divide it repeatedly by D_X (mod. smoothing operators) and get for any N:

$$P = D_X^N Q + R \quad (\text{with R} \sim 0)$$

Then $\mathrm{Tr}_P^G = D^N \mathrm{Tr}_Q^G + \mathrm{Tr}_R^G$: this is well defined as a distribution since Q is of trace class if N is large, and it does not depend on the choice of D, N, Q, R.

The series is convergent in distribution sense, i.e., the coefficients have at most polynomial growth with respect to the eigenvalues of D.

More generally if we have an equivariant Toeplitz complex of finite length:

$$(\mathbb{E}, d): \quad \cdots \to \mathbb{E}_j \xrightarrow{d} \mathbb{E}_{i+1} \to \cdots,$$

i.e., $\mathbb{E}$ is a finite sequence $\mathbb{E}_k$ of equivariant Toeplitz bundles, $d = (d_k : \mathbb{E}_k \to \mathbb{E}_{k+1})$ a sequence of Toeplitz operators such that $d^2 = 0$. Then for a Toeplitz operator $P : \mathbb{E} \to \mathbb{E}, P \sim 0$ near $\mathrm{char}\,\mathfrak{g}$, its equivariant supertrace $\mathrm{Tr}_P^G = \sum(-1)^k \mathrm{Tr}_{P_k}^G$ is well defined; it vanishes if P is a supercommutator $[A, B]$ where A, B are equivariant, and one of them vanishes near $\mathrm{char}\,\mathfrak{g}$.

2.2. Equivariant index

Let $\mathbb{E}_0, \mathbb{E}_1$ be two equivariant Toeplitz bundles.

Definition 4. We will say that an equivariant Toeplitz operator $P : \mathbb{E}_0 \to \mathbb{E}_1$ is G-elliptic (transversally elliptic in [4]) if it is elliptic on $\mathrm{char}\,\mathfrak{g}$, i.e., the principal symbol $\sigma(P)$, which is a homogeneous equivariant vector bundle homomorphism $E_0 \to E_1$, is invertible on $\mathrm{char}\,\mathfrak{g}$.

If P is G-elliptic it has a G-parametrix Q, i.e., $Q : \mathbb{F} \to \mathbb{E}$ is equivariant, and $QP \sim 1_{\mathbb{E}}, PQ \sim 1_{\mathbb{F}}$ near $\operatorname{char} \mathfrak{g}$.

The G-index Ind_P^G is then defined as the distribution

$$\operatorname{Ind}_P^G = \operatorname{Tr}^G_{1-QP} - \operatorname{Tr}^G_{1-PQ}. \tag{2}$$

More generally, an equivariant complex $(\mathbb{E}, d)$ as above is G-elliptic if the principal symbol $\sigma(d)$ is exact on $\operatorname{char} \mathfrak{g}$. Then there exists an equivariant Toeplitz operator $s = (s_k : \mathbb{E}_k \to \mathbb{E}_{k-1})$ such that $1-[d,s] \sim 0$ near $\operatorname{char} \mathfrak{g}$ ($[d,s] = ds+sd$). The index (Euler characteristic) is the super trace

$$I^G_{(\mathbb{E},d)} = \operatorname{supertr}(1-[d,s]) = \sum (-1)^j \operatorname{Tr}^G_{(1-[d,s])_j}.$$

If P is G-elliptic, the restriction $P_\alpha : \mathbb{E}_{0,\alpha} \to \mathbb{E}_{1,\alpha}$ is a Fredholm operator for any irreducible representation α. Its index I_α is finite (resp. more generally the cohomology H^*_α of $d|_{\mathbb{E}_\alpha}$ is finite dimensional), and we have

$$\operatorname{Ind}_P^G = \sum \frac{1}{d_\alpha} I_\alpha \, \chi_\alpha \quad \left(\text{or } \operatorname{Ind}^G_{(\mathbb{E},d)} = \sum \frac{(-1)^j}{d_\alpha} \dim H^j_\alpha \, \chi_\alpha \right). \tag{3}$$

2.3. Asymptotic index

The G-index Ind_P^G is obviously invariant under compact perturbation and deformation, so for fixed $\mathbb{E}_j$ it only depends on the homotopy class of the symbol $\sigma(P)$. But it does depend on the choice of Szegö projectors: the Toeplitz bundles $\mathbb{E}_j$ are known in practice only through their symbols E_j, and are only determined up to a space of finite dimension, just as the Toeplitz spaces $\mathbb{H}$.

However if $\mathbb{E}, \mathbb{E}'$ are two equivariant Toeplitz bundles with the same symbol, there exists an equivariant elliptic Toeplitz operator $U : \mathbb{E} \to \mathbb{E}'$ with quasi-inverse V (i.e., $VU \sim 1_{\mathbb{E}}, UV \sim 1'_{\mathbb{E}}$). This may be used to transport equivariant Toeplitz operators from $\mathbb{E}$ to $\mathbb{E}'$: $P \mapsto Q = UPV$. Then if $P \sim 0$ on X_0, $Q = UPV$ and VUP have the same G-trace, and since $P \sim VUP$, we have $T_P - T_Q \in C^\infty(G)$. Thus the equivariant G-trace or index are ultimately well defined up to a smooth function on G.

Definition 5. We define the asymptotic G-trace Tras_P^G as the singularity of the distribution Tr_P^G (i.e., Tr_P^G mod. $C^\infty(G)$).

If $P \sim 0$, we have $\operatorname{Tr}_P^G \sim 0$, i.e., the sequence of Fourier coefficients is of rapid decrease, $O(c_\alpha)^{-m}$ for all m, where c_α is the eigenvalue of D_G in the representation α.

Definition 6. If P is elliptic on $\operatorname{char} \mathfrak{g}$, the asymptotic G-index Indas_P^G is defined as the singularity of Ind_P^G.

It can also be viewed as a virtual trace-class representation or character $\sum n_\alpha \chi_\alpha$ of G, mod finite representations.

It only depends on the homotopy class of the principal symbol $\sigma(A)$, and since it is obviously additive we get:

Theorem 7 (Main theorem).

1) *The asymptotic index defines an additive map from* $K^G(X-Z)$ *to* $\mathrm{Sing}\,(G) = C^{-\infty}/C^{\infty}(G)$ $(Z \subset X$ *denotes the basis of* $\mathrm{char}\,\mathfrak{g})$.
2) *If* $u : X \to X'$ *is a contact map, then the asymptotic index map* Indas *commutes with the Bott periodicity map* $K^G(X-Z) \to K^G(X' - u(Z))$.

The Bott periodicity map is described below.

$K^G(X-Z)$ denotes the equivariant K-theory of X with compact support in $X-Z$, i.e., the group of stable classes of triples (E, F, u) where E, F are equivariant G-bundles on X, u an equivariant isomorphism $E \to F$ defined near Z, with the usual equivalence relations ($(E, F, a) \sim 0$ if a is stably homotopic near Z to an isomorphism on the whole of X).

The asymptotic index is as well defined for equivariant Toeplitz complexes, exact on Z.

Example. Let Σ be a symplectic cone, with free positive elliptic action of $U(1)$, i.e., the Toeplitz generator $A = \frac{1}{i}\partial_\theta$ is elliptic with positive symbol (this is the situation studied in [11]). Then the algebra of invariant Toeplitz operators (mod. C^∞) is a deformation star algebra, setting as "deformation parameter" $\hbar = A^{-1}$. $\mathrm{char}\,\mathfrak{g}$ is empty and the asymptotic trace or index is always defined. The asymptotic trace of any element A is the series $\sum_{-\infty}^{\infty} a_k e^{ki\theta}$, $a_k = \mathrm{tr}\, A|_{\mathbb{H}_k}$, mod smooth functions of θ, i.e., the sequence (a_k) is known mod rapidly decreasing sequences. It is standard knowledge that the sequence (a_k) has an asymptotic expansion in (negative) powers of k:

$$a_k \sim \sum_{j \le j_0} \alpha_j k^j. \tag{4}$$

In this case the asymptotic trace is as well defined by this asymptotic expansion; it encodes the same thing as the residual trace, viewed as a power series of $\hbar = k^{-1}$.

Remark. For a general circle group action, with generator $A = e^{i\theta}$, all simple representations are powers of the identity representation, denoted T, and all representations occurring as indices can be written as formal power series with integral coefficients:

$$\sum_{k \in \mathbb{Z}} n_k T^k \quad \text{(mod. finite sums)}.$$

In fact, using the sphere embedding below, it can be seen that the positive and negative parts of the series are "weakly periodic", of the form

$$\frac{P_\pm(T, T^{-1})}{(1 - T^{\pm k})^k}$$

for suitable polynomials $P_\pm$ and some integer k, i.e., both the positive and negative parts are the Taylor series of rational functions whose poles are roots of 1; the asymptotic index corresponds to the polar parts.

2.4. *K*-theory and embedding

It is convenient (even though not technically indispensable), in particular to follow the index in an embedding (Lemma 10), to reformulate some constructions above in terms of sheaves of Toeplitz algebras and modules. In the C^∞ category $\mathcal{E}$ is not coherent and general $\mathcal{E}$-module theory is not practical. We will just stick to two useful examples.[6]

As above we use the following notation: for distributions, $f \sim g$ means that $f - g$ is C^∞; for operators, $A \sim B$ (or $A = B$ mod. C^∞) means that $A - B$ is of degree $-\infty$, i.e., has a smooth Schwartz kernel; if M is a manifold, $T^\bullet M$ denotes the cotangent bundle deprived of its zero section; it is a symplectic cone with base the cotangent sphere $S^*M = T^\bullet M/\mathbb{R}_+$. As mentioned earlier, if Σ is a G-symplectic cone, the sheaf $\mathcal{E}_\Sigma$ of Toeplitz operators (mod C^∞) acting on $\mu\mathbb{H}$ is well defined, with the action of G, up to isomorphism, independently of any embedding $\Sigma \to T^\bullet M$. The asymptotic trace Tras_P^G resp. index Indas_P^G are well defined for a section P of $\mathcal{E}_\Sigma$ vanishing (resp. invertible) near $\mathrm{char}\,\mathfrak{g}$. (If M is a G-manifold and $X = S^*M$ ($\Sigma = T^\bullet M$), $\mathcal{E}_\Sigma$ identifies with the sheaf of pseudodifferential operators acting on the sheaf $\mu\mathbb{H}$ of microfunctions on X; even in that case the exact index problem does not make sense: a Toeplitz bundle $\mathbb{E}$ corresponds to a vector bundle E on the cotangent sphere $X = S^*M$, not necessarily the pullback of a vector bundle on M, and $\mathbb{E}$ is in general at best defined up to a space of finite dimension.)

An $\mathcal{E}$-module $\mathcal{M}$, corresponds to a system of Toeplitz operators, whose sheaf of micro-solutions is $\mathrm{Hom}_{\mathcal{E}}(\mathcal{M}, \mu\mathbb{H})$; likewise a locally free complex (L, d) of $\mathcal{E}$-modules defines a Toeplitz complex $(\mathbb{E}, D) = \mathrm{Hom}\,(L, \mathbb{H})$.

We will say that the $\mathcal{E}$-module $\mathcal{M}$ is "good" if it is finitely generated, equipped with a filtration $\mathcal{M} = \bigcup \mathcal{M}_k$ (i.e., $\mathcal{E}_p\mathcal{M}_q = \mathcal{M}_{p+q}$, $\bigcap \mathcal{M}_k = 0$) such that the symbol $\sigma(M) = \mathcal{M}_0/\mathcal{M}_{-1}$ has a finite locally free resolution (as a $C^\infty(X)$-modul[7]). A locally free resolution of $\sigma(\mathcal{M})$ lifts to a "good resolution" of $\mathcal{M}$ (i.e., locally free and whose symbol is a resolution of $\sigma(\mathcal{M})$).[8] Two resolutions of $\sigma(\mathcal{M})$ are homotopic, and if $\sigma(\mathcal{M})$ has locally finite locally free resolutions it also has a global one (because on compact X (or on the cone Σ with compact basis) we dispose of smooth (homogeneous) partitions of unity); this lifts to a global good resolution of $\mathcal{M}$.

Similarly we will say that a G-elliptic complex $(\mathbb{E}, d)$ is "good" if its symbol is exact on $\mathrm{char}\,\mathfrak{g}$. Note that "good" is not indispensable to define the asymptotic index, but it is to define the K-theoretic element $[(\mathbb{E}, d)] \in K^G(X - Z)$.

[6] In the proof of the Atiyah-Weinstein conjecture we need to patch together two smooth embedded manifolds near their boundaries: this cannot be done in the real analytic category, even if things work slightly better there.

[7] The symbol map identifies $\mathcal{E}_0/\mathcal{E}_{-1}$ with $C^\infty(X)$; since there exist global elliptic sections of $\mathcal{E}$, $\mathrm{gr}\,\mathcal{M}$ is completely determined by the symbol, same for the resolution.

[8] The converse is not true: if d is a locally free resolution of $\mathcal{M}$ its symbol is not necessarily a resolution of the symbol of $\mathcal{M}$ – if only because filtrations must be defined to define the symbol and can be modified rather arbitrarily.

All this works just as well in presence of a G-action (one must choose invariant filtrations etc.).

The asymptotic trace and index extend in an obvious manner to endomorphisms of good complexes or modules:

- if $\mathcal{M} = \mathcal{E}^N$ is free, $\mathrm{End}_{\mathcal{E}}(\mathcal{M})$ identifies with the ring of $N \times N$ matrices with coefficients in the opposite ring $\mathcal{E}^{\mathrm{op}}$, and if $A = (A_{ij})$ vanishes near $\mathrm{char}\,\mathfrak{g}$ we set $\mathrm{Tras}^G(A) = \sum \mathrm{Tras}^G(A_{jj})$.
- If $\mathcal{M}$ is isomorphic to the range $P\mathcal{N}$ of a projector P in a free module $\mathcal{N}$ (this does not depend on the choice of $\mathcal{N}$) and if $A \in \mathrm{End}_{\mathcal{E}}(M)$ we set $\mathrm{Tras}^G(A) = \mathrm{Tras}^G(PA)$.
- If (L, d) is a locally free complex and $A = (A_k)$ is a endomorphism, vanishing near $\mathrm{char}\,\mathfrak{g}$, we set $\mathrm{Tras}^G(A) = \sum(-1)^k \mathrm{Tras}^G(A_k)$ (the Euler characteristic or super trace; if A, B are endomorphisms of opposite degrees $m, -m$, we have $\mathrm{Tras}^G[A, B] = 0$, where $[A, B] = AB - (-1)^{m^2} BA$ is the superbracket).
- If $\mathcal{M}$ is a good $\mathcal{E}$-module, (L, d) a good locally free resolution of $\mathcal{M}$, $A \in \mathrm{End}_{\mathcal{E}}(\mathcal{M})$, we set $\mathrm{Tras}^G(A) = \mathrm{Tras}^G(\widetilde{A})$, where $\widetilde{A}$ is any extension of A to (L, d) (such an extension exists, and is unique up to homotopy, i.e., up to a supercommutator).
- Finally if $\mathcal{M}$ is a locally free complex with symbol exact on $\mathrm{char}\,\mathfrak{g}$, or a good $\mathcal{E}$-module with support outside of $\mathrm{char}\,\mathfrak{g}$, it defines a K-theoretical element $[\mathcal{M}] \in K_Z^G(X)$, and its asymptotic index (the supertrace of the identity), is the image by the index map of Theorem 7 of $[\mathcal{M}]$.

Remark. The equivariant trace or index are defined just as well for modules admitting a projective resolution (projective meaning direct summand of some $\mathcal{E}^N$, with a projector not necessarily of degree 0). What does not work for these more general objects is the relation to topological K-theory.

2.5. Embedding and transfer

Let Σ be a G-symplectic cone, embedded equivariantly in $T^\bullet M$ with M a compact G-manifold, and S an equivariant Szegö projector. As recalled in §1, the range $\mu\mathbb{H}$ of S is the sheaf of solutions of an ideal $I \subset \mathcal{E}_M$. The corresponding $\mathcal{E}_M$-module is $\mathcal{M} = \mathcal{E}_M/I$; it is "good", as is obvious on the microlocal model or the holomorphic model (for which a good resolution near Σ is $\bar{\partial}_b$).

Endomorphisms of $\mathcal{M}$ are induced by right multiplications $m \mapsto ma$ where $aI \subset I$ ($a \in [I : I]$, so $\mathcal{E}' = \mathrm{End}\,\mathcal{M}^{\mathrm{op}} \simeq [I : I]/I$. The map which to $a \in [I : I]$ associates the Toeplitz operator T_a gives an isomorphism from $\mathrm{End}_{\mathcal{E}}(\mathcal{M})^{\mathrm{op}}$ to the Toeplitz algebra (mod C^∞). (This is easily seen by successive approximations since the symbol of T_a is $\sigma(a)|_\Sigma$, or because, as indicated in [11], any Toeplitz operator is also of the form T_P where P commutes with the Szegö projector.)

If $\mathcal{P}$ is a Toeplitz module, i.e., a left $\mathcal{E}'$-module supported by Σ, the transferred module is $\mathcal{M} \otimes_{\mathcal{E}'} \mathcal{P}$ (also supported by Σ); it has the same solution sheaf as $\mathcal{P}$, since we have $\mathrm{Hom}\,(\mathcal{M} \otimes \mathcal{P}, \mathbb{H}) = \mathrm{Hom}\,(\mathcal{P}, \mathrm{Hom}\,(\mathcal{M}, \mathbb{H}))$ and $\mathrm{Hom}\,(\mathcal{M}, \mathbb{H}) =$

$\mathbb{H}'$. In this equality we can replace $\mathcal{P}$ by its global good resolution (i.e., replace Hom by Rhom^0), because this resolution is locally isomorphic to $\bar\partial_b$ which has no cohomology mod C^∞ near Σ in degree > 0. Thus **the transfer preserves asymptotic traces and indices.**

This extends obviously to the case where Σ is embedded equivariantly in another symplectic cone $\Sigma \subset \Sigma'$: the Toeplitz sheaf $\mu\mathbb{H}$ is $\mathrm{Hom}_{\mathcal{E}_\Sigma}(\mathcal{M}, \mu\mathbb{H}')$, with $\mathcal{M} = \mathcal{E}/I$ and $I \subset \mathcal{E}$ is the annihilator of the Szegö projector S of Σ.

Theorem 8. *Let X', X be two compact contact G-manifolds and $f : X \to X'$ be an equivariant embedding. Then the K-theoretical push-forward (Bott homomorphism) $K^G(X - Z) \to K^G(X' - Z')$ commutes with the asymptotic G index of G-elliptic equivariant Toeplitz operators.*

Let $F : \mathcal{E}_\Sigma \to \mathcal{E}_{\Sigma'}$ be an equivariant embedding of the corresponding Toeplitz algebras (over f), and let $\mathcal{M}$ be the $\mathcal{E}'_\Sigma$-module associated with the Szegö projector S_Σ (transfer module). We have seen that transfer $\mathcal{P} \mapsto \mathcal{M} \otimes \mathcal{P}$ preserves the asymptotic index.

Lemma 9. *Notations being as above, the K-theoretical element (with support in Σ) $[\mathcal{M}] \in K^G_\Sigma(T^\bullet M)$ is precisely the Bott element used to define the Bott isomorphism $K^G(X) \to K^G_X(X')$; $[\mathcal{M} \otimes \mathcal{P}]$ is the Bott image of $[\mathcal{P}]$.*[9]

Proof. The transfer module $\mathcal{M}$ is good: it has, locally (and globally), a good resolution. Its symbol is a locally free resolution of $\sigma(\mathcal{M}) = C^\infty(X)/\sigma(I)$. We may identify a small equivariant tubular neighborhood of Σ with the normal tangent bundle N of Σ in Σ'; N is a symplectic bundle; the ideal I endows it with a compatible positive complex structure N^c (for which the first-order jet of elements of $\sigma(I)$ are holomorphic in the fibers of N^c). In such a neighborhood a good symbol resolution is homotopic to the Koszul complex of N (or the symbol of $\bar\partial_b$ in the holomorphic case): the K-theoretical element it defines is precisely the Bott element.

Example. Let $X = S^{2N-1}$ be the unit sphere of $\mathbb{C}^N$, $\mathbb{H}$ the space of holomorphic functions (the symplectic cone Σ can be identified with $\mathbb{C}^N$). Similarly $X' = \mathbb{C}^{2k-1}$ and $\mathbb{H}'$. We can embed X' as a subsphere of X (equivariantly if we are given suitable unitary group actions).

We can identify $\mathbb{H}'$ with the subset of functions independent of $z_{k+1}, \dots, z_N$. The corresponding operators are the $\partial_{z_j}, k < j \le N$ and the corresponding complex of Toeplitz operators is the partial De Rham complex.

[9] If $f : X \to Y$ is a map between manifolds (or suitable spaces), the K-theoretical push-forward is the topological translation of the Grothendieck direct image in K-theory (for algebraic or holomorphic coherent modules). Its definition requires a spin^c structure on the virtual normal bundle of f (cf [11], §1.3) and this always exists canonically if X, Y are almost symplectic or almost complex, or as here if f is an immersion whose normal tangent bundle is equipped with a symplectic or complex structure.

Another way of relating the two is to identify $\mathbb{H}'$ to $\mathbb{H}/\sum_{k+1}^N z_j\mathbb{H}$, identifying $\mathbb{H}'$ with the cohomology of the Koszul complex.

Note that we have $\partial_{z_m} = (N + \sum_1^N z_j\partial_j)T_{\bar{z}_m}$ so up to a positive factor, the De Rham complex is the adjoint of the Koszul complex, and both define the same K-theoretical (equivariant) element.

Remark. It is always possible to embed equivariantly a compact contact manifold in a canonical contact sphere with linear G-action (this reduces the problem of computing asymptotic indices to the case where the base space is a sphere – but if $G \neq 1$ this is still complicated):

Lemma 10. *Let Σ be a G cone (with compact base), λ a horizontal 1-form, homogeneous of degree 1 ($L_\rho\lambda = \lambda, \rho\lrcorner\lambda = 0$, where ρ is the radial vector field, generating homotheties). Then there exists a homogeneous embedding $x \mapsto Z(x)$ of Σ in a complex representation V^c of G such that $\lambda = \operatorname{Im}\bar{Z}.dZ$*

In this construction, Z is homogeneous of degree $\frac{1}{2}$ as above. This applies of course if Σ is a symplectic cone, λ its Liouville form (the symplectic form is $\omega = d\lambda$ and $\lambda = \rho\lrcorner\omega$. We first choose a smooth equivariant function $Y = (Y_j)$, homogeneous of degree $\frac{1}{2}$, realizing an equivariant embedding of Σ in $V - \{0\}$, where V is a real unitary G-vector space (this always exists if the basis is compact).

Then there exists a smooth function $X = (X_j)$ homogeneous of degree $\frac{1}{2}$ such that $\lambda = 2X.dY$. We can suppose X equivariant, replacing it by its mean $\int g.X(g^{-1}x)\,dg$ if need be. We have $2\rho\lrcorner dY = Y$ (Y is of degree $\frac{1}{2}$) so $X.Y = \rho\lrcorner X.dY = 0$. Finally we get $\lambda = \operatorname{Im}\bar{Z}.dZ$ with Z = X + iY (the coordinates z_j on V are homogeneous of degree $\frac{1}{2}$ so that the canonical form $\operatorname{Im}\bar{Z}.dZ$ is of degree 1).

3. Relative index

Let Ω, Ω' be two strictly pseudo convex Stein domains with smooth boundaries X, X'. Let f be a smooth contact isomorphism $X \to X'$. Then the holomorphic push-forward

$$W: \quad u \in \mathbb{H} \mapsto S'(u \circ f^{-1}) \in \mathbb{H}' \tag{5}$$

is well defined, and is a (Toeplitz FIO) Fredholm map. The Atiyah-Weinstein formula computes its index in terms of the geometrical data.

The original Atiyah question was: if M, M' are two smooth manifolds, $f : S^*M \to S^*M'$ a contact isomorphism, F an elliptic FIO associated to f, then F has an index, which should be given by a similar formula.

This reduces to the former problem since ΨDO on M are the same thing as Toeplitz operators on the boundary of a small tubular neighborhood of M in a complexification M^c (cf. [7]).[10]

[10] Except one should also take into account the homotopy class ("winding number") of the principal symbol.

The main difficulty in this problem is that, with a fixed contact structure, we are changing the CR structure, hence the Szegö projectors, and there is no formula, using only the contact boundary data, telling how the index behaves.

To overcome this, we enlarge the spaces of holomorphic boundary values in such a manner that the index is repeated infinitely many times and can be interpreted as an asymptotic index, which can be handled geometrically.

3.1. Enlargement

Let Ω be as above, with defining function $-\phi$ ($\phi > 0$, *I have changed the sign*). We denote the boundary by X_0 *rather than* X.

$\widetilde{\Omega} \subset \mathbb{C} \times \bar{\Omega}$ denotes the ball $|t|^2 < \phi$. Its boundary X is strictly pseudoconvex, provided that $\operatorname{Log} \frac{1}{\phi}$ is strictly psh. (e.g., $-\phi$ strictly psh. on $\bar{\Omega}$[11]). We still denote by $\Sigma \supset \Sigma_0$ the symplectic cones.

The circle group $U(1)$ acts on $\widetilde{X}$: $(t,x) \mapsto (e^{i\theta}t, x)$.

The volume element on $\widetilde{X}$ is $d\theta\, dv$ (smooth, positive, invariant) with dv a smooth positive density on $\bar{\Omega}$; S denotes the Szegö projector, $\mathbb{H}$ its range (space of boundary values of holomorphic functions of moderate growth near X).

D denotes the Toeplitz operator defined by $\frac{1}{i}\partial_\theta$ on $\mathbb{H}$. It is self-adjoint, ≥ 0, equal to $T_t T_{\partial_t}$.

The expansion of a function in the Fourier decomposition

$$\mathbb{H} = \sum_{k\geq 0} \mathbb{H}_k \quad (\mathrm{\mathbb{H}_k = ker\,(D-k)}\ )$$

is equivalent to its Taylor expansion:

$$f = \sum f_k(x) t^k.$$

$\mathbb{H}_0$ identifies with the set of holomorphic distributions on X_0 (set of boundary values of holomorphic functions on Ω with moderate growth at $\partial\Omega$).

Note that the L^2 norm of a holomorphic function $t^k f(x)$ on X is

$$\int_X |t^k f|^2 = 2\pi \int_\Omega \phi^k |f^2| dv$$

(because $|t|^2 = \phi$ on X and the measure on X is $d\theta\, dv$)

If we decompose S in its equivariant components $S = \sum S_k$, we get a sequence closely related to that of Berezin (see [5, 16]).

It will be convenient to replace the Toeplitz FIO operator W by a unitary multiple

$$E_0 = (WW^*)^{-\frac{1}{2}} W : \mathbb{H}_0 \to \mathbb{H}_0' \tag{6}$$

with the convention that $(WW^*)^{-\frac{1}{2}}$ vanishes on the kernel of W^*; E_0 is in any case unitary mod smoothing operators and has obviously the same index as W. We are using the norm of $\mathbb{H}_0$, i.e., the L^2 norm of X (or of $\bar{\Omega}$), which is not the

[11] ϕ can always be chosen so.

L^2 norm of X_0 (it is rather related to the Sobolev $H^{-\frac{1}{2}}$ norm) – but for the index this makes no difference).

As mentioned above the Toeplitz operator corresponding to rotations is

$$D = t\partial_t \quad \left(= \frac{1}{i}\partial_\theta\right).$$

we have $D = D^* = T^*_{\partial_t} T^*_t$; it follows that

$$\partial_t^* = tC, (t^* = C^{-1}\partial_t).$$

for an invariant Toeplitz operator $C > 0$ (unique)[12]

$$\text{We set} \quad \tau = tC^{\frac{1}{2}}. \tag{7}$$

This is a Toeplitz operator of degree $\frac{1}{2}$, not an integer, but for the commutation constructions below this does not matter

$$D = \tau\tau^*, [D,\tau] = \tau, [\tau^*,\tau] = 1 \tag{8}$$

$$\tau \text{ is globally defined, a positive multiple of } t \ , \ \tau\mathbb{H}_k = \mathbb{H}_{k+1}$$

τ is uniquely defined by by these conditions.[13]

There is a similar construction for Ω'.

Theorem 11 (embedding). *There exists an equivariant Toeplitz FIO:* $E : X \to X'$ *(with microsupport close to* X_0*) such that* $(\text{mod}\, C^\infty)$*.*

1) *E is unitary elliptic* $(\text{mod}\, C^\infty)$ *near X_0.*
2) *E induces E_0 on $\mathbb{H}_0$* (mod *smoothing operators*)*.*
3) *$E\tau = \tau' E$.*

Then the $E_k = \mathbb{H}_k \to \mathbb{H}'_k$ all have the same index $\text{Index}\, E_0$*.*

If 2) holds, E is elliptic on X_0 hence G-elliptic (because here $G = U(1)$ acts freely, with a positive action, on the "interior" $X - X_0$). The last assertion follows: we have $E - \tau' E_k = E_{k+1}\tau$ and since τ is a bijection $E_k \to E_{k+1}$ (same for τ'), E_k, E_{k+1} have same index.

The theorem replaces the relative index $\text{Index}\,(E_0)$ by the G-asymptotic index $\text{Indas}\,(E)$.

[12] If we have a factorization $D = PQ$ with $[D,P] = P$, there exists a (unique) invariant invertible Toeplitz operator U such that $P = tU, Q = U^{-1}\partial_t$. Here we have $D = tCt^*$, so $C = C^* > 0$ since D is ≥ 0 and T_t injective.

[13] In fact we need a little less than that: τ should be globally defined over Ω, and $\tau_k : \mathbb{H}_k \to \mathbb{H}_{k+1}$ should have index zero; the Hamiltonians of the real and imaginary parts of τ should commute.

3.2. Collar isomorphism

The geometric counterpart is: there is a (unique) equivariant homogeneous symplectic isomorphism f of some equivariant neighborhood of Σ_0 in Σ to (same for Σ') such that $f|_{\Sigma_0} = \mathrm{Id}$, and $\sigma(\tau) \circ f = \sigma(\tau')$, i.e., f commutes with the Hamiltonians of the real and imaginary parts of τ, τ'. This works because the Hamiltonians of $\mathrm{Re}\,\tau, \mathrm{Im}\,\tau$ commute.[14]

The operator statement follows from the geometric one in the usual manner. Notice that E is at first only defined mod smoothing operators near X_0. We extend it globally using any Toeplitz cut-off.

3.3. Embedding

We have mentioned that any G-contact manifold (compact) can be embedded in a standard contact sphere with linear unitary action of G. Here we choose embeddings more precisely.

Let $\widetilde{X} = S^{2N+1} \subset \mathbb{C}^{N+1}$ be a large sphere, with variables (T, Z).

The circle group $G = U(1)$ acts by

$$(T, Z) \mapsto (e^{i\theta}T, Z).$$

The base of char $\mathfrak{g}$ is the diameter $Z(T = 0)$; it is equal to the fixed point set.

Theorem 12. *There exist equivariant contact embeddings F, F' of X, X' in the sphere S^{2N+1} (with $U(1)$-action as above) such that $F = F' \circ f$ near the boundary X_0.*

We are now reduced to the case where X, X' sit in a large sphere S and coincide near the fixed points. The trivial bundle of X defines, via the transfer homomorphism, a complex A of Toeplitz operators on the large sphere $\widetilde{X}$, whose K-theoretical element in $K_X^G(S)$ is the equivariant Bott image. Same for X'.

The Toeplitz FIO E of Theorem 11 provides a Toeplitz isomorphism $A \to A'$ near the boundary X_0, thus defining a G-elliptic complex on $\widetilde{X}$, whose asymptotic G-index is precisely what we want to compute.

3.4. Index

Now $U(1)$ acts freely on $S - S^0$ and $U(1)\backslash(S - S^0)$ is the open unit ball $B \subset \mathbb{C}^N$, so the pull back is an isomorphism $K_0(\mathbb{C}^N) = \mathbb{Z} \to K^G_{S-S_0}(S)$ (the generator is the symbol of the partial De Rham complex ∂_X, or of the Koszul complex).

We may now go back to the original situation: Ω and Ω' are complex manifolds, glued together by the symplectic map f_0; the result Y is not a manifold, but the K-theoretical index is well defined: $\chi : K_{\mathrm{comp}}(Y) \to \mathbb{Z}$:

Theorem 13. *The relative index is $\chi(1_\Omega - 1_{\Omega'})$; χ is the K-theoretical character defined by the Bott periodicity theorem; the two trivial bundles $1_\Omega - 1_{\Omega'}$ are glued together along the boundary to give an element of compact support.*

[14] This would not work if we replaced τ by t because the Hamiltonians of $\mathrm{Re}\,t, \mathrm{Im}\,t$ do not commute in general.

The K-theoretical element defined by the complex above is the difference element between the K-theoretical (spin^c) images of Ω and Ω' defined by F_0 on the boundary (or its extension near the boundary defined by F; any symplectic diffeomorphism near the boundary would do as well since these are all isotopic.

This can be readily translated in terms of cohomology, using the Chern characters and Todd class, as done in [12]; the Todd class appears when comparing the Chern class of the Bott element with the Euler class used for integration along fibers.

References

[1] Atiyah, M.F. *K-theory.* W.A. Benjamin, Inc., New York-Amsterdam. 1967.

[2] Atiyah, M.F.; Singer; I.M. *The index of elliptic operators.* Ann. of Math. (2) 87 1968 484–530, 531–545, 546–604.

[3] Atiyah, M.F. *Bott periodicity and the index of elliptic operators.* Quart. J. Math. Oxford Ser. (2), 19 (1968), 113–140.

[4] Atiyah, M.F. *Elliptic operators and compact groups.* LN in Math., 401. Springer-Verlag, 1974.

[5] Berezin, F.A. *General concept of quantization.* Comm. Math. Phys. 40 (1975), 153–174.

[6] Boutet de Monvel, L. *Hypoelliptic operators with double characteristics and related pseudodifferential operators.* Comm. Pure Appl. Math. 27 (1974), 585–639.

[7] Boutet de Monvel, L. *Convergence dans le domaine complexe des séries de fonctions propres.* C.R.A.S. 287 (1978), 855–856.

[8] Boutet de Monvel, L. *On the index of Toeplitz operators of several complex variables.* Inventiones Math. 50 (1979) 249–272.

[9] Boutet de Monvel, L. *Symplectic cones and Toeplitz operators* (congrès en l'honneur de Trèves, Sao Carlos), Contemporary Math., vol. 205 (1997) 15–24.

[10] Boutet de Monvel, L. *Vanishing of the logarithmic trace of generalized Szegö projectors.* Proceedings of the Conference "Algebraic Analysis of Differential Equations" in honor of Prof. T. Kawai, Springer-Verlag, 2007. arXiv:math.AP/0604166 v1.

[11] Boutet de Monvel, L.; Guillemin, V. *The Spectral Theory of Toeplitz Operators.* Ann. of Math Studies no. 99, Princeton University Press, 1981.

[12] Boutet de Monvel, L.; Leichtnam E.; Tang, X.; Weinstein A. *Asymptotic equivariant index of Toeplitz operators and relative index of CR structures* arXiv:0808.1365v1; to appear in 'Geometric Aspects of Analysis and Mechanics, a conference in honor of Hans Duistermaat', Progress in Math., Birkhäuser (2010?)

[13] Boutet de Monvel, L.; Malgrange, B. *Le théorème de l'indice relatif.* Ann. Sci. ENS., 23 (1990), 151–192.

[14] Boutet de Monvel, L.; Sjöstrand, J. *Sur la singularité des noyaux de Bergman et de Szegö.* Astérisque 34-35 (1976), 123–164.

[15] Duistermaat, J.J.; Hörmander, L. *Fourier integral operators II.* Acta Math. 128 (1972), 183–269.

[16] Englis, M. *Berezin quantization and reproducing kernels on complex domains.* Trans. Amer. Math. Soc. 349, pages 411–479, 1996.

[17] Epstein, C. *Subelliptic Spin$_c$ Dirac operators, III, the Atiyah-Weinstein conjecture.* Ann. of Math., 168 (2008), 299–365.

[18] Epstein, C.; Melrose, R. *Contact degree and the index of Fourier integral operators.* Math. Res. Lett. 5 (1998), no. 3, 363–381.

[19] Hörmander, L. *Fourier integral operators* Acta Math. 127 (1971), 79–183.

[20] Melin, A.; Sjöstrand, J. *Fourier Integral operators with complex-valued phase functions* Lecture Notes 459 (1975) 120–223.

[21] Melin, A.; Sjöstrand, J. *Fourier integral operators with complex phase functions and parametrix for an interior boundary value problem* Comm. P.D.E. 1:4 (1976) 313–400.

[22] Weinstein, A. *Some questions about the index of quantized contact transformations* RIMS Kokyuroku No. 1014, pages 1–14, 1997.

L. Boutet de Monvel
UPMC Univ Paris 6
F-75005, Paris, France

Operator Theory:
Advances and Applications, Vol. 216, 17–39

Boundary Value Problems of Analytic and Harmonic Functions in a Domain with Piecewise Smooth Boundary in the Frame of Variable Exponent Lebesgue Spaces

Vakhtang Kokilashvili

Abstract. We present a survey paper on boundary value problems for analytic and harmonic functions in weighted classes of Cauchy type integrals in a simply connected domain not containing $z = \infty$ and having a density from variable exponent Lebesgue spaces. It is assumed that the domain boundary is a piecewise smooth curve. The solvability conditions are established and solutions are constructed. The solution is found to essentially depend on the coefficients from the boundary condition, the weight, space exponent values at the angular points of the boundary curve and also on the angle values. The non-Fredholmian case is investigated. An application of the obtained results to the Neumann problem is given.

This survey is based on the joint research with V. Paatashvili.

Mathematics Subject Classification (2000). 35Q15, 30E25, 31A25, 31B20.

Keywords. Cauchy type integrals, Riemann–Hilbert problem, weighted Lebesgue space with variable exponent, Log-Hölder condition, piecewise smooth boundary, non Fredholmian case, Riemann–Hilbert–Poincaré problem, Vekua's representation of holomorphic functions.

1. Introduction

The study of boundary value problems and related boundary integral equations in domains with intricate geometrical structure of the boundary is one of challenging problem nowadays. The first part of the present survey deals with the Riemann–Hilbert problem

$$\operatorname{Re}\left[(a(t) + ib(t))\,\Phi^{+}(t)\right] = c(t), \quad t \in \Gamma, \tag{1}$$

in a domain with nonsmooth boundary and in the frame of Banach function spaces with nonstandard growth condition. Several authors studied this problem in domains with sufficiently smooth boundaries under various assumptions with regard to the coefficients and unknown functions (see, e.g., [18], [5] and references therein).

In this section the Riemann–Hilbert problem is considered in the following setting: find a function $\Phi \in K^{p(\cdot)}(D;\omega)$ whose boundary values satisfy (1) a.e. on Γ. Here D is a simply connected domain not containing $z=\infty$ and bounded by a simple piecewise smooth closed curve Γ, and $K^{p(\cdot)}(D;\omega)$ is the set of functions

$$\Phi(z)=\frac{1}{\omega(z)}\int_\Gamma \frac{\varphi(t)\,dt}{t-z}\,,\quad z\in D,\quad \varphi\in L^{p(\cdot)}(\Gamma),$$

when $\omega(z)$ is an arbitrary function of the form

$$\omega(z)=\prod_{k=1}^{\nu}(z-t_k)^{\alpha_k},\quad t_k\in\Gamma,\quad \alpha_k\in\mathbb{R}.$$

Special properties of Banach function spaces with nonstandard growth were singled out from the Banach space theory in the 30s of the last century. The initial works related to this topic belong to W. Orlicz and J. Musielak. At that time these works were of purely theoretical value, but nowadays there has arisen a necessity to investigate these spaces as they play an essential role in mathematical models of nonlinear elasticity and mechanics of incompressible fluids. They are also important for the investigation of various physical phenomena via variational models (e.g., V. Zhikov's study of Lavrentiev's phenomena), the construction of models of the mechanics of incompressible fluids (M. Ružička), also for the study of the related integral operators and Sobolev spaces with variable exponent (H. Hudzik, O. Kováčik and J. Rakosnik, S. Samko, L. Diening, X. Fan and D. Zhao). Research of $p(x)$-Laplacian nonlinear differential equations and function spaces associated with them that unable to describe physical events by "point variable" characteristics, for example, in the elasticity theory of nonhomogeneous media (E. Acerbi and G. Mingione, P. Marcelini, X. Fan, H. Zhang and others).

Let $\Gamma=\{t\in\mathbb{C}:\ t=z(s),\ 0\le s\le l<\infty\}$ be a simple closed rectifiable curve with arc-length measure $\nu(t)=s$. Let $C^1_D(A_1,\dots,A_i;\nu_1,\dots,\nu_i)$ be the set of simple piecewise smooth curves Γ having angular points $A_1,\dots,A_i$ whose angle values with respect to the domain D with boundary Γ are equal to $\pi\nu_k$, $k=\overline{1,i}$, $0\le\nu_k\le 2$. The set of piecewise-Lyapunov curves contained in this class is denoted by $C^{1,L}_D(A_1,\dots,A_i;\nu_1,\dots,\nu_i)$.

Let p be a measurable function on Γ such that $p:\Gamma\to(1,\infty)$. By $\mathcal{P}(\Gamma)$ we denote the set of functions $p(t)$ satisfying the conditions:

$$1<p_-:=\operatorname*{ess\,inf}_{t\in\Gamma}p(t)\le\operatorname*{ess\,sup}_{t\in\Gamma}p(t)=:p_+<\infty \tag{2}$$

and there exists a constant A such that

$$|p(t)-p(\tau)|\le\frac{A}{-\ln|t-\tau|}\,,\quad t\in\Gamma,\ \ \tau\in\Gamma. \tag{3}$$

The set of functions p satisfying the conditions (2) and (3) we denote by $\mathcal{P}(\Gamma)$.

A generalized Lebesgue space with variable exponent is defined via the modular

$$I_\Gamma^p(f) := \int_\Gamma |f(t)|^{p(t)}\, d\nu(t)$$

by the norm

$$\|f\|_{L^{p(\cdot)}(\Gamma)} = \inf\left\{\lambda > 0:\ I_\Gamma^p\left(\frac{f}{\lambda}\right) \le 1\right\}. \tag{4}$$

For a given weight function ω we denote by $L^{p(\cdot)}(\Gamma,\omega)$ a weighted Banach function space of all measurable functions $f:\Gamma\to\mathbb{C}$ such that

$$\|f\|_{L^{p(\cdot)}(\Gamma;\omega)} = \|f\omega\|_{L^{p(\cdot)}(\Gamma)} < \infty.$$

For such spaces see, e.g., [1], [2], [9], [11]–[14], etc.

There naturally arises the question of studying boundary value problems of the function theory, including problem (1), too, in the classes of holomorphic functions representable by a Cauchy type integral with a density from $L^{p(\cdot)}(\Gamma,\omega)$. The investigation of problems in this setting not only generalizes the previously considered cases, but is more naturally, so far as also makes it possible to take into consideration the integral behavior of the solution not only on the boundary as a whole, but also locally, near any point of the boundary.

In [12], [7], [8], [10], [3] and other works, the boundary value problems are studied under the assumption that the boundary values of a sought solutions belongs to a variable exponent Lebesgue space.

In our paper [8], problem (1) was investigated in a simply connected domain D not containing $z=\infty$ and bounded by a simple piecewise-Lyapunov curve with nonzero angles in the class $K^{p(\cdot)}(D;\omega)$, i.e., in the class of analytic functions Φ representable in the form

$$\Phi(z) = \frac{1}{\omega(z)}\,\frac{1}{2\pi i}\int_\Gamma \frac{\varphi(t)\,dt}{t-z}, \quad z\in D, \quad \varphi\in L^{p(\cdot)}(\Gamma), \tag{5}$$

where $\omega(z)$ is an arbitrary power function. With regard to the coefficients $a(t)$, $b(t)$ it was assumed that they are piecewise-Hölder and $\inf\limits_{t\in\Gamma}(a^2(t)+b^2(t))>0$, and $c\in L^{p(\cdot)}(\Gamma,\omega)$. An analogous problem was investigated in [17] for $p(t)=\text{const}$.

In the present paper, the investigation of problem (1) is carried out under the following assumptions:

i) Γ is a closed piecewise-smooth curve from the class $C^1(A_1,\dots,A_i;\nu_1,\dots,\nu_i)$, where $A_1,\dots,A_i$ are all angular points of the curve Γ, while the values of angles, which are internal with respect to a finite domain bounded by Γ, are equal to $\pi\nu_k$, $0<\nu_k\le 2$;

ii) the coefficients $a(t)$, $b(t)$ are piecewise-continuous;

iii) a function $p(t)$ belongs to the class

$$\widetilde{\mathcal{P}}(\Gamma) = \bigcup_{\varepsilon>0} \mathcal{P}_{1+\varepsilon}(\Gamma),$$

where $\mathcal{P}_{1+\varepsilon}(\Gamma)$ denotes the set of those real functions p, for which condition (2) is fulfilled and there exists positive numbers A and ε such that

$$|p(t_1) - p(t_2)| < \frac{A}{|\ln|t_1 - t_2||^{1+\varepsilon}} \tag{6}$$

for arbitrary points t_1 and t_2 on Γ.

Let $z = z(w)$ be the conformal mapping of the circle $U = \{w : \ |w| < 1\}$ into D and let $w = w(z)$ be the inverse function.

Assume that $l(\tau) = p(z(\tau))$, $\tau_k = w(t_k)$, $a_k = w(A_k)$.

Let

$$D(t,r) = \Gamma \cap B(t,r), \quad t \in \Gamma, \quad r > 0$$

where $B(t,r) = \{z \in \mathbb{C} : \ |z - t| < r\}$.

We remind that a curve is called the Carleson curve (regular curve), if there exists a constant $c_0 > 0$ nondepending on t and r, such that

$$\nu D(t,r) \le c_0 r.$$

In the sequel we consider the power weights of the form

$$w(t) = \prod_{k=1}^{n} |t - t_k|^{\beta}, \quad t_k \in \Gamma, \quad t_i \ne t_j \ \text{ when } \ i \ne j.$$

It is well known that to solve boundary value problems for analytic and harmonic functions boundedness in weighted spaces of the Cauchy singular integral

$$S_\Gamma f(t) = \frac{1}{\pi i} \int_\Gamma \frac{f(\tau)}{\tau - t} d\tau$$

is crucial.

One basic result of our investigation is the following

Theorem A [14],[9]**.** *Let $p \in \mathcal{P}(\Gamma)$. The Cauchy singular operator S_Γ is bounded in $L^{p(\cdot)}_w(\Gamma)$, if and only if Γ is a Carleson curve and*

$$-\frac{1}{p(t_k)} < \beta_k < \frac{1}{p'(t_k)} \quad k = 1, 2, \dots, n.$$

When investigating problem (1) in the case of a piecewise-smooth boundary, we have to deal with the following problems:

1) Denote by $W^{p(\cdot)}(\gamma)$ the set of weight functions ρ for which the operator

$$T : f \to Tf, \quad (Tf)(\tau) = \frac{\rho(\tau)}{\pi i} \int_\gamma \frac{f(\zeta)\, d\zeta}{(\zeta - \tau)\rho(\zeta)}, \quad \tau \in \gamma, \ \ f \in L^{p(\cdot)}(\gamma), \tag{7}$$

is continuous in $L^{p(\cdot)}(\gamma)$ (γ is a unit circle).

As is well known, if $-[\ell(\tau_k)]^{-1} < \alpha_k < [\ell'(\tau_k)]^{-1}$, $\ell \in \mathcal{P}(\gamma)$, $\ell'(\tau) = \frac{\ell(\tau)}{\ell(\tau)-1}$, and φ is a continuous real function on γ, then

$$\rho(\tau) = \prod_{k=1}^{\nu} (\tau - \tau_k)^{\alpha_k} \exp\left(\frac{1}{\pi}\int_\gamma \frac{\varphi(\zeta)\,d\zeta}{\zeta - \tau}\right) \in W^{\ell(\cdot)}(\gamma) \tag{8}$$

(see [10, Corollary 6.2]).

This result was used in investigating problem (1) when Γ is a piecewise-Lyapunov curve [8]. If however Γ is a piecewise-smooth curve, then for the investigation we need to know whether the function

$$\rho(\tau) = \prod_{k=1}^{\nu} (\tau - \tau_k)^{\alpha_k} \exp\left(\frac{1}{\ell(\tau)}\frac{1}{\pi}\int_\gamma \frac{\varphi(\zeta)\,d\zeta}{\zeta - \tau}\right), \quad -\frac{1}{\ell(\tau_k)} < \alpha_k < \frac{1}{\ell'(\tau_k)} \tag{9}$$

possesses the same property.

2) If $X(w)$ is a canonical function for a piecewise-continuous function, then we do not know whether the functions $X^+(\tau)$ and $\rho(\tau)X^+(\tau)$, where $\rho(\tau)$ is given by equality (9), belong to the class $W^{\ell(\cdot)}(\gamma)$.

3) When Γ is a piecewise-smooth curve from $C^1(A_1,\dots,A_i;\nu_1,\dots,\nu_i)$, and $z(a_k) = A_k$, where $z = z(w)$ is the conformal mapping of the unit circle $U = \{w : |w| < 1\}$ on D, then we must know the weight properties of the function $z(\tau) - z(a_k)$.

4) We must know whether the Log-Hölder condition holds for the function $\ell(\tau) = p(z(\tau))$ on γ when $p(t)$ satisfies the Log-Hölder condition on Γ.

The reasoning in [8] clearly implies that positive answers to the above four questions make if possible to establish the validity of the results obtained there also for the case where Γ is piecewise-smooth, while $a(t)$, $b(t)$ are piecewise-continuous.

As to questions 1) and 2), we succeeded in showing that they both have a positive answer if $p \in \widetilde{\mathcal{P}}(\Gamma)$.

As to question 3), the following was clarified: if we follow the method of investigation of the functions z' and z, which is used in [4] (Ch. III) and apply inclusion (8) given above, then it can be shown that

$$\prod_{k=1}^{i} (z(\tau) - z(a_k)) \in W^{\ell(\cdot)}(\gamma)$$

for $0 < \nu_k < \frac{1}{\ell'(a_k)}$.

As to question 4), we can state that the function $\ell(\tau) = p(z(\tau))$ belongs to the class $\widetilde{\mathcal{P}}(\gamma)$ if so does the function $p(t)$ on $\Gamma \in C^1(A_1,\dots,A_i;\nu_1,\dots,\nu_i)$. The proof is obtained by a slight modification of the proof of Lemma 1 from [7] if instead of Warshavski's result on the derivative of the conformal mapping of a circle onto a domain with piecewise-Lyapunov boundary we use the result from [4] on the behavior of z' in the case of piecewise-smooth curves.

2. The Riemann–Hilbert problem in the class $K^{p(\cdot)}(D;\omega)$ in the case of piecewise-smooth boundaries and piecewise-continuous coefficients

2.1. Reducing of problem (1) to a linear conjugation problem with an additional condition

We will use the well-known method of N. Muskhelishvili by which the Riemann–Hilbert problem is reduced to the Riemann problem (see [18, §§ 36–43]).

Assume $\Gamma \in C^1(A_1,\dots,A_i;\nu_1,\dots,\nu_i)$, $0<\nu_i\le 2$, $k=\overline{1,i}$, bounds the domain D not containing $z=\infty$. Let further $t_k\in\Gamma$, $k=\overline{1,\nu}$, $\alpha_k\in\mathbb{R}$ and

$$\omega(z)=\prod_{k=1}^{\nu}(z-t_k)^{\alpha_k} \tag{10}$$

be an arbitrary fixed branch of analytic function in D. Denote by $K^{p(\cdot)}(D;\omega)$ the set of analytic functions in D, that are representable by equality (5). If $\omega\in W^{p(\cdot)}(\Gamma)$, then the class $K^{p(\cdot)}(D;\omega)$ coincides with the class $K^{p(\cdot)}(\Gamma;\omega)$ (see [8, Theorem 1]), where

$$K^{p(\cdot)}(\Gamma;\omega)=\bigg\{\Phi:\ \Phi(z)=\frac{1}{2\pi i}\int_\Gamma\frac{\varphi(\tau)\,d\tau}{t-z},\ \ \varphi\in L^{p(\cdot)}(\Gamma,\omega),\ \ z\in D\bigg\}. \tag{11}$$

Let, further, $c\in L^{p(\cdot)}(\Gamma,\omega)$, $a(t)$ and $b(t)$ be piecewise-continuous functions on Γ, with the condition $\inf\limits_{t\in\Gamma}(a^2(t+b^2(t))>0$, and B_k, $k=\overline{1,\lambda}$, be the discontinuity points of the functions $\widetilde{G}(t)=-[a(t)-ib(t)]\,[a(t)+ib(t)]^{-1}$.

Let us consider the Riemann–Hilbert problem: find a function $\Phi\in K^{p(\cdot)}(D;\omega)$ satisfying the boundary condition (1). Let $\Psi(w)=\Phi(z(w))$. Then

$$\begin{gathered}\Psi^+(\tau)=-[A(\tau)-iB(\tau)]\,[A(\tau)+iB(\tau)]^{-1}\overline{\Psi^+(\tau)}\\ +2C(\tau)[A(\tau)+iB(\tau)]^{-1},\\ A(\tau)=a(z(\tau)),\quad B(\tau)=b(z(\tau)),\quad C(\tau)=c(z(\tau)).\end{gathered} \tag{12}$$

For the function $F(w)$, analytic outside γ, we assume

$$F_*(w)=\begin{cases}F(w), & |w|<1,\\ \overline{F\big(\frac{1}{\overline{w}}\big)}, & |w|>1.\end{cases}$$

Let us introduce the new unknown function

$$\Omega(w)=\begin{cases}\Psi(w), & |w|<1,\\ \overline{\Psi\big(\frac{1}{\overline{w}}\big)}, & |w|>1.\end{cases} \tag{13}$$

Then (12) takes the form

$$\begin{gathered}\Omega^+(\tau)=G(\tau)\Omega^-(\tau)+C_1(\tau),\\ G(\tau)=-[A(\tau)-iB(\tau)]\,[A(\tau)+iB(\tau)]^{-1},\\ C_1(\tau)=C(\tau)[A(\tau)+iB(\tau)]^{-1},\end{gathered} \tag{14}$$

and thereby we have to find a solution of the problem

$$\begin{cases} \Omega^+(\tau) = G(\tau)\Omega^-(\tau) + C_1(\tau), \quad \tau \in \gamma, \quad C_1 \in L^{p(\cdot)}(\gamma), \\ \Omega_*(w) = \Omega(w), \quad |w| \neq 1, \quad \Omega(\infty) = \text{const}. \end{cases} \tag{15}$$

2.2. Solution of problem (15)

In Problem (15), for the time being the question in what class the function Ω should be sought for remains open.

Let $G(B_k^-)/G(B_k^+) = \exp 2\pi i u_k$, $k = \overline{1,\lambda}$ ($u_k \in \mathbb{R}$ since $|G| = 1$). Let $w = w(z)$ be the inverse function to $z = z(w)$ and $b_k = w(B_k)$. The scheme of our investigation will be as follows.

Step I. Write G in the form $G(\tau) = X^+(\tau)[X^-(\tau)]^{-1}$, where

$$\begin{aligned} &X(w) = r(w)X_1(w), \quad r(w) = \prod_{k=1}^{\lambda} r_k(w), \\ &r_k(w) = \begin{cases} (w - b_k)^{u_k}, & |w| < 1, \\ \left(\frac{1}{\overline{w}} - b_k\right)^{u_k}, & |w| > 1, \end{cases} \end{aligned} \tag{16}$$

$$X_1(w) = \begin{cases} C \exp\left\{\dfrac{1}{2\pi i}\displaystyle\int_\gamma \dfrac{\ln G_1(\tau)\tau^{-\varkappa_1}\, d\tau}{\tau - w}\right\}, & C = \text{const}, \\ Cw^{-\varkappa_1} \exp\left\{\dfrac{1}{2\pi i}\displaystyle\int_\gamma \dfrac{\ln G_1(\tau)\tau^{-\varkappa_1}\, d\tau}{\tau - w}\right\}. \end{cases} \tag{17}$$

Here $G_1(\tau) = \prod_{k=1}^{\lambda} r_k^+(\tau)[r_k(\tau)]^{-1}G(\tau)$ is a continuous function and $\varkappa_1 = \operatorname{ind} G_1(\tau)$.

Step II. Rewrite the boundary condition from (15) in the form

$$\frac{\Omega^+(\tau)}{X^+(\tau)} = \frac{\Omega^-(\tau)}{X^-(\tau)} + \frac{C_1(\tau)}{X^+(\tau)}.$$

Step III. Construct a rational function $Q(w)$ with zeros and poles on γ, such that the function $F(w) = Q(w)\Omega(w)[X(w)]^{-1}$ belongs to $K^{\ell(\cdot)}(\gamma;\rho)$, where $\ell(\tau) = p(z(\tau))$ and $\rho(\tau)$ is a power function from $W^{\ell(\cdot)}(\gamma)$. For this construction, the numbers α_k, ν_k, u_k together must satisfy a certain condition (see condition (21) below).

To construct $Q(w)$, we proceed as follows.

Let

$$\begin{gathered} T = \{\tau_k : \ \tau_k = w(t_k)\}, \\ A = \{a_k : \ a_k = w(A_k)\}, \quad B = \{b_k : \ b_k = w(B_k)\}. \end{gathered} \tag{18}$$

These sets may have common sets. Let us enumerate the points of the set $T\cup A\cup B$ as follows:

$$\begin{aligned} & w_1=\tau_1=a_1=b_1,\dots,w_\mu=\tau_\mu=a_\mu=b_\mu,\\ & w_{\mu+1}=\tau_{\mu+1}=a_{\mu+1},\dots,w_{\mu+r}=\tau_{\mu+r}=a_{\mu+r},\\ & w_{\mu+r+1}=\tau_{\mu+r+1}=b_{\mu+1},\dots,w_{\mu+r+q}=\tau_{\mu+r+q}=b_{\mu+q},\\ & w_{\mu+r+q+1}=a_{\mu+r+1}=b_{\mu+q+1},\dots,w_{\mu+r+q+p}=a_{\mu+r+p}=b_{\mu+q+p},\\ & w_{\mu+r+q+p+1}=\tau_{\mu+r+q+1},\dots,w_{\mu+r+q+p+m}=\tau_{\mu+r+q+m},\\ & w_{\mu+r+q+p+m+1}=a_{\mu+r+p+1},\dots,w_{\mu+r+q+p+m+n}=a_{\mu+r+p+n},\\ & w_{\mu+r+q+p+m+n+1}=b_{\mu+q+p+1},\dots,w_{\mu+r+q+p+m+n+s}=b_{\mu+p+q+s}. \end{aligned} \tag{19}$$

Thus we have $j=\mu+r+q+p+m+n+s$ points w_k.

Let

$$\delta_k=\begin{cases} \alpha_k\nu_k+\frac{\nu_k}{\ell(a_k)}+u_k, & k=\overline{1,\mu},\\ \alpha_k\nu_k+\frac{\nu_k-1}{\ell(a_k)}\,, & k=\overline{\mu+1,\mu+r},\\ \alpha_k+u_{k-r}, & k=\overline{\mu+r+1,\mu+r+q},\\ \frac{\nu_{k-q}-1}{\ell(a_{k-q})}+u_{k-r}, & k=\overline{\mu+r+q+1,\mu+r+q+p},\\ \alpha_{k-p}, & k=\overline{\mu+r+q+p+1,\mu+r+q+p+m},\\ \frac{\nu_{k-q-m}-1}{\ell(w_{k-q-M})}\,, & k=\overline{\mu+r+q+p+m+1,\mu+r+q+p+m+n},\\ u_{k-r-m-n}, & k=\overline{\mu+r+q+p+m+n+1,\mu+r+q+p+m+n+s}. \end{cases} \tag{20}$$

For the real number x we write $x=[x]+\{x\}$, where $0\le\{x\}<1$.

Assume

$$\{\delta_k\}\neq\frac{1}{\ell'(w_k)}\,. \tag{21}$$

Assume

$$\left.\begin{aligned} \gamma_k&=[\delta_k], && \text{if } \{\delta_k\}<\frac{1}{\ell'(w_k)}\,,\\ \gamma_k&=[\delta_k]+1, && \text{if } \{\delta_k\}>\frac{1}{\ell'(w_k)}\,, \end{aligned}\right\}\quad k=\overline{1,j}. \tag{22}$$

Then

$$-\frac{1}{\ell(w_k)}<\delta_k-\gamma_k<\frac{1}{\ell'(w_k)}\,,\quad k=\overline{1,j}. \tag{23}$$

Let

$$Q(w)=\prod_{k=1}^{j}(w-w_k)^{\gamma_k}. \tag{24}$$

2.3. Main result

Let:

1) D be an internal domain bounded by a curve $\Gamma \in C^1(A_1, \dots, A_i; \nu_1, \dots, \nu_i)$, $0 < \nu_k \le 2$;
2) $p \in \widetilde{\mathcal{P}}(\Gamma)$;
3) $\omega(z) = \prod\limits_{k=1}^{\nu} (z - t_k)^{\alpha_k}$, $-\frac{1}{p(t_k)} < \alpha_k < \frac{1}{p'(t_k)}$;
4) $a(t)$, $b(t)$ be piecewise-continuous real functions with the condition $\inf(a^2(t) + b^2(t)) > 0$ such that the function $G(t) = -[a(t) - ib(t)][a(t) + ib(t)]^{-1}$ has discontinuity points B_k, $G(B_k^-)/G(B_k^+) = \exp 2\pi i u_k$, $u_k \in \mathbb{R}$;
5) $c(t)\omega(t) \in L^{p(\cdot)}(\Gamma)$.

Theorem 1. *Let*

i) *the points* τ_k, a_k, b_k *be enumerated according to* (19)*; the numbers* δ_k *be defined by equalities* (20) *and* $\{\delta_k\} \neq \frac{1}{\ell'(w_k)}$, *and the integer numbers* γ_k *be chosen according to* (22)*;*

ii) $Q(w)$ *be the rational function defined by equality* (24), *and* $\varkappa_0$ *be its order at the point* $w = \infty$*;*

iii) *the functions* $r(w)$ *and* $X_1(w)$ *be defined by equalities* (16), (17) *and therefore* X_1 *have order* $(-\varkappa_1)$ *at infinity.*

Assume that $\varkappa = \varkappa_0 + \varkappa_1$. *Then:*

a) *if* $\varkappa < 0$, *then for problem* (1) *to be solvable in the class* $K^{p(\cdot)}(D;\omega)$ *it is necessary and sufficient that the conditions*

$$\int_\gamma \frac{c(z(\tau))Q(\tau)}{X^+(\tau)[a(z(\tau)) + ib(z(\tau))]}\, \tau^k d\tau = 0, \quad k = \overline{0, \varkappa}, \tag{25}$$

would be fulfilled. Thus there exists a unique solution

$$\Phi_0(z) = \Omega(w(z)) = \widetilde{\Omega}_c(w(z)) = \frac{1}{2}\left(\Omega_c(w(z)) + (\Omega_c)_*(w(z))\right), \tag{26}$$

where

$$\Omega_c(w) = \frac{X(w)}{Q(w)}\, \frac{1}{2\pi i} \int_\gamma \frac{c(z(\tau))Q(\tau)}{X^+(\tau)[a(z(\tau)) + ib(z(\tau))]}\, \frac{d\tau}{\tau - w}\,; \tag{27}$$

b) *if* $\varkappa \ge 0$, *then problem* (1) *is solvable unconditionally and all its solutions are given by the equality*

$$\Phi(z) = \Phi_0(z) + X(w(z))Q^{-1}(w(z))P_\varkappa(w(z)), \tag{28}$$

where $P_\varkappa(w)$ *is an arbitrary polynomial* $P_\varkappa(w) = \sum\limits_{k=0}^{\varkappa} h_k w^k$ *whose coefficients satisfy the condition*

$$\overline{h}_k = A h_{\varkappa - k}\,, \quad k = \overline{0, \varkappa}, \quad A = (-1)^{\varkappa_0} \prod_{k=1}^{\varkappa_j} w_k^{-\gamma_k}.$$

3. Some particular cases

3.1. The Riemann–Hilbert problem with Hölder coefficients $a(t)$, $b(t)$ in the class $K^{p(\cdot)}(\Gamma;\omega)$ for $\omega\in W^{p(\cdot)}(\Gamma)$

In this case there are no points b_k and $-\frac{1}{p(t_k)}<\alpha_k<\frac{1}{p'(t_k)}$. Only the points τ_k and a_k may coincide; to simplify numeration (19) it is assumed that $w_1=\tau_1,\dots,w_m=\tau_m$, $w_{m+1}=a_1,\dots,w_{m+n}=a_n$, $w_{m+n+1}=\tau_{m+1}=a_{n+1},\dots,w_{m+n+r}=\tau_{m+r}=a_{n+r}$. Accordingly, $\delta_1=\dots=\delta_m=0$, $\delta_{m+k}=\frac{\nu_k-1}{p(A_k)}$, $k=\overline{1,n}$, $\delta_{m+n+k}=\alpha_{n+k}\nu_{n+k}+\frac{\nu_{n+k}-1}{p(A_{n+k})}$, $k=\overline{1,r}$. Therefore $\gamma_1=\gamma_2=\dots=\gamma_m=0$,

$$\gamma_{m+k}=\begin{cases}0 & \text{if } \nu_k<\frac{1}{p'(A_k)}\,,\\ 1 & \text{if } \nu_k>\frac{1}{p'(A_k)}\,,\end{cases}\qquad k=\overline{1,n},$$

$$\gamma_{m+n+k}= \tag{29}$$

$$\begin{cases}\Big[\alpha_{n+k}\nu_{n+k}+\frac{\nu_{n+k}-1}{\ell'(a_{n+k})}\Big] & \text{if } \Big\{\alpha_{n+k}\nu_{n+k}+\frac{\nu_{n+k}-1}{\ell'(a_{n+k})}\Big\}<\frac{1}{p'(A_{n+k})}\,,\\ \Big[\alpha_{n+k}\nu_{n+k}+\frac{\nu_{n+k}-1}{\ell'(a_{n+k})}\Big]+1 & \text{if } \Big\{\alpha_{n+k}\nu_{n+k}+\frac{\nu_{n+k}-1}{\ell'(a_{n+k})}\Big\}>\frac{1}{p'(A_{n+k})}\,,\end{cases}$$

$k=\overline{1,r}$.

The latter numbers can be considered in more detail: let $\beta=\gamma_{m+n+k}$, $\alpha\in\{\alpha_{n+1},\dots,\alpha_{n+r}\}$, $\nu\in\{\nu_{n+1},\dots,\nu_{n+r}\}$, $\ell=p(A_{n+k})=\ell(a_{n+k})$, $\ell'=p'(A_{n+k})=\ell'(a_{n+k})$ and $v=\alpha\nu+\frac{\nu-1}{\ell}$. Then (29) takes the form

$$\beta=\begin{cases}[v] & \text{if } \{v\}<\frac{1}{\ell'}\,,\\ [v]+1 & \text{if } \{v\}>\frac{1}{\ell'}\,.\end{cases}$$

Since $-\frac{1}{\ell}<\alpha<\frac{1}{\ell'}$, $\frac{1-\ell-\nu}{\ell\nu}<-\frac{1}{\ell}$ and $\frac{2\ell-\nu}{\ell\nu}\ge\frac{1}{\ell'}$, it is sufficient to consider the following possible cases:

i) $\frac{1-\ell-\nu}{\ell\nu}<\alpha\le\frac{1-\nu}{\ell\nu}$; **ii)** $\frac{1-\nu}{\ell\nu}<\alpha<\frac{\ell-\nu}{\ell\nu}$;
iii) $\frac{\ell-\nu}{\ell\nu}<\alpha<\frac{1+\ell-\nu}{\ell\nu}$; **iv)** $\frac{1+\ell-\nu}{\ell\nu}\le\alpha<\frac{2\ell-\nu}{\ell\nu}$.

Case i) We have $-1<v\le 0$. If $v=0$, then $[v]=\{v\}=0$ and therefore $\beta=0$. If $-1<v<0$, then $[v]=-1$, $\{v\}=1+v=\alpha\nu+\frac{\nu-1}{\ell}+1=\alpha\nu+\frac{\nu}{\ell}+\frac{1}{\ell'}>\big(-\frac{1}{\ell}\big)\nu+\frac{\nu}{\ell}+\frac{1}{\ell'}=\frac{1}{\ell'}$ and therefore $\beta=-1+1=0$.

Case ii) We have $0<v<\frac{1}{\ell'}$. Hence $[v]=0$, $\{v\}=v=\alpha\nu+\frac{\nu-1}{\ell}<\frac{\ell-\nu}{\ell}+\frac{\nu-1}{\ell}=\frac{1}{\ell'}$; thus $\beta=0$.

Case iii) We have $\frac{1}{\ell'}<v<1$, i.e., $[v]=0$, $\{v\}=v=\alpha\nu+\frac{\nu-1}{\ell'}>\frac{1}{\ell'}$ and therefore $\beta=[v]+1=1$.

Case iv) We have $1\le v<2-\frac{1}{\ell'}$ (<2). If $v=1$, then $[v]=1$, $\{v\}=0$, and therefore $\beta=1$. If $v>1$, then $[v]=1$, $\{v\}=v-1=\alpha\nu+\frac{\nu-1}{\ell}-1<\frac{2\ell-\nu}{\ell}+\frac{\nu-1}{\ell}-1=\frac{1}{\ell'}$. Therefore $\beta=[v]=1$.

Thus

$$\begin{cases} \beta = 0 \quad \text{if} \quad \dfrac{1-\ell-\nu}{\ell\nu} < \alpha < \dfrac{\ell-\nu}{\ell\nu}\,, \\ \beta = 1 \quad \text{if} \quad \dfrac{\ell-\nu}{\ell\nu} < \alpha < \dfrac{2\ell-\nu}{\ell\nu}\,. \end{cases} \tag{30}$$

Equalities (30) can be written in the form

$$\begin{cases} \beta = 0 \quad \text{if} \quad 0 < \nu < \dfrac{\ell}{1+\alpha\ell}\,, \\ \beta = 1 \quad \text{if} \quad \dfrac{\ell}{1+\alpha\ell} < \nu < \dfrac{2\ell}{1+\alpha\ell}\,. \end{cases}$$

By virtue of (30), it follows from (24) that $Q(\tau)$ is a polynomial and its order is equal to

$$\begin{aligned} \varkappa_0 &= \mathcal{N}\{w_k : \nu_k > \ell(w_k)\} \\ &\quad + \mathcal{N}\left\{\tau_k = a_k : \frac{\ell(a_k)}{1+\alpha_k\ell(a_k)} < \nu_k < \frac{2\ell(a_k)}{1+\alpha_k\ell(a_k)}\right\} \end{aligned} \tag{31}$$

where $\mathcal{N}(E)$ denotes the number of elements of a set E.

Thus we derive the following

Proposition 2. *If problem* (1) *is considered in the class* $K^{p(\cdot)}(\Gamma;\omega)$, $\omega \in W^{p(\cdot)}(\Gamma)$ *and* $a(t)$, $b(t)$ *belong to the Hölder class, the number* $\varkappa_0$ *in Theorem* 1 *is calculated by equality* (31)*.*

3.2. The Dirichlet problem in the class $\mathrm{Re}K^{p(\cdot)}(\Gamma;\omega)$

In what follows, the set of functions $u(z) = \mathrm{Re}\phi(z)$ where $\phi \in K^{p(\cdot)}(\Gamma;\omega)$ is denoted by $\mathrm{Re}K^{p(\cdot)}(\Gamma,\omega)$. Let $a(t) = 1$, $b(t) = 0$, ω is defined by (5), $\omega \in W^{p(\cdot)}(\Gamma)$, $c \in L^{p(\cdot)}(\Gamma;\omega)$. Then by virtue of Theorem 1 we have $K^{p(\cdot)}(D;\omega) = K^{p(\cdot)}(\Gamma;\omega)$. Problem (1) is posed as follows: find a function $\phi \in K^{p(\cdot)}(\Gamma;\omega)$ that satisfies the condition $\mathrm{Re}\,\phi^+(t) = c(t)$ a.e. on Γ, i.e., we deal with the Dirichlet problem: find a function u for which

$$\begin{cases} \triangle u = 0, \quad u = \mathrm{Re}\,\phi, \quad \phi \in K^{p(\cdot)}(\Gamma;\omega), \\ u^+(t) = c(t), \quad t \in \Gamma, \quad c\omega(t) \in L^{p(\cdot)}(\Gamma). \end{cases} \tag{32}$$

Then $r(w) = 1$,

$$X_1(w) = \begin{cases} -i, & |w| < 1, \\ i, & |w| > 1 \end{cases}$$

(we need this to have $(X_1)_*(w) = X_1(w)$). Thus $\varkappa_1 = 0$, i. e. $\varkappa = \varkappa_0$, where $\varkappa_0$ is calculated by formula (31) and therefore $\varkappa \geq 0$.

3.3. The Dirichlet problem in the class $\mathrm{Re}K^{p(\cdot)}(\Gamma;\omega)$ for $w \equiv 1$

In that case, r, X_1 and $\varkappa_1$ are calculated as in the previous case. From condition (21) we obtain $\nu_k \neq p(A_k)$, $k = \overline{1,i}$. The order $\varkappa_0$ of $Q(w)$ at infinity is equal to the number of angular points for which $\nu_k > p(A_k)$.

Let us find the solution when $i = 1$, $\nu > p(A_1) = \ell(a_1)$ and $c(t) = 0$.

From (28) we have

$$\Psi(w) = i\,\frac{P_1(w)}{w - w_1} = i\,\frac{h_0 + h_1 w}{w - w_1}\,, \quad w_1 = w(A_1),$$

where the coefficient $h_1 = is_1$ with $s_1 \in \mathbb{R}$ (this fact follows from equality $\Omega(\infty) = \widetilde{\Omega}(\infty) = \frac{1}{2}(\Psi(0) + \overline{\Psi(0)})$). From the equality

$$\overline{h}_k = A h_{\varkappa - k}\,, \quad k = \overline{0,\varkappa}, \quad A = (-1)^{\varkappa_0} \prod_{k=1}^{\varkappa_j} w_k^{-\gamma_k}$$

we have $\overline{h}_0 = -Ah_1$, where $A = -\frac{1}{w_1}$ (see (23)). Hence $h_0 = \frac{is_1}{\overline{w}_1} = is_1 w_1$ and therefore

$$\phi(w) = i\,\frac{is_1 w_1 + is_1 w}{w - w_1} = -s_1\,\frac{w + w_1}{w - w_1}\,, \quad s_1 \in \mathbb{R}.$$

Thus for $\nu > p(A_1)$ the problem

$$\begin{cases} \triangle u = 0, \quad u \in \mathrm{Re}\,K^{p(\cdot)}(\Gamma), \quad p \in \mathcal{P}(\Gamma), \quad \Gamma \in C_L(A,\nu), \\ u^+(t) = 0, \quad t \in \Gamma, \end{cases}$$

has a solution

$$u(z) = s_1\ \mathrm{Re}\,\frac{w(z) + w(A_1)}{w(z) - w(A_1)}$$

depending on one real parameter.

If $\nu < p(A_1)$, then the problem has only a trivial solution.

Remark 3. In [7] the Dirichlet problem was investigated in the Smirnov class

$$e^{1,p(\cdot)}(D) = \Big\{u:\ u = \mathrm{Re}\phi,\ \ \phi \in E^{1,p(\cdot)}(D)\Big\}$$

where

$$E^{1,p(\cdot)}(D) = \Big\{\phi:\ \phi \in E^1(D),\ \ \phi^+ \in L^{p(\cdot)}(\Gamma)\Big\}$$

and assumed that Γ is an arbitrary piecewise-Lyapunov curve.

It is clear, that $e^{1,p(\cdot)}(D) = \mathrm{Re}K^{p(\cdot)}(\Gamma)$ and, therefore, the results of this subsection are contained in Theorem 3 from [7].

3.4. The Neumann problem

Define a harmonic function $u = \operatorname{Re}\Phi$, where Φ is an analytic function such that $\Phi' \in K^{p(\cdot)}(D)$, which satisfies the condition

$$\left(\frac{\partial u}{\partial n}\right)(t) = f(t), \quad f \in L^{p(\cdot)}(\Gamma),$$

a.e. on Γ.

Let

$$\widetilde{K}^{p(\cdot)}(\Gamma;\omega) = \left\{\Phi :\ \Phi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\varphi(\tau)\,d\tau}{t-z} + \text{const}\,,\ \ \varphi \in L^{p(\cdot)}(\Gamma,\omega),\ \ z \in D\right\}.$$

Following [18, pp. 243–248], we come to the problem in the class $\widetilde{K}^{p(\cdot)}(D)$

$$\operatorname{Re}\left[ie^{-2i\vartheta}\Psi^+(t)\right] = f(t), \quad \Psi = \Phi', \tag{33}$$

where ϑ is the angle formed by the tangent to Γ at the point t and the abscissa axis. Here the sets of points $\{a_k\}$ and $\{b_k\}$ coincide, while the set $\{\tau_k\}$ is empty. Furthermore, $u_k = 1-\nu_k$ and thus condition (21) takes the form $\left\{\frac{\nu_k-1}{p'(A_k)}\right\} \neq \frac{1}{p'(A_k)}$. It is not difficult to verify that this is equivalent to the condition $\nu_k \neq p'(A_k)$. For problem (33) to be solvable it is necessary that $f(t)$ would be orthogonal to solutions of the class $\widetilde{K}^{p'(\cdot)}(D)$ of the problem

$$F^+ = e^{2i\vartheta}F^-$$

(which is the conjugate problem to the Riemann problem corresponding to problem (33)). It has $\varkappa(p')+1$ solutions, where

$$\varkappa(p') = N\{A_k :\ \nu_k > p'(A_k)\}$$

(see [8, Subsection 7]).

Thus we have the following statement:

If $0 < \nu_k \leq 2$, $\nu_k \neq p'(A_k)$, then for the posed Neumann problem to be solvable it is necessary that $\varkappa(p')+1$ conditions would be fulfilled. If these conditions are fulfilled, then the problem has solutions depending on $\varkappa(p)+1 = N\{A_k : \nu_k > p(A_k)\}+1$ arbitrary constants. To find them we proceed as follows: using the formulas from Theorem 1, we find Φ', then integrate it and separate the real part.

4. Non-Fredholm case

In the assumptions we have considered above, problem (1) has turned out to be Fredholmian in the sense that the homogeneous problem has a finite number of linearly independent solutions, while the set of functions $c(t)$, for which it is solvable, is closed in $L^{p(\cdot)}(\Gamma;\omega)$.

Let us now consider the case where condition (21) is violated at individual points w_k and assume that $a(t)$, $b(t)$ are piecewise-Hölder functions.

Let

$$\{\delta_{k_i}\} = \frac{1}{\ell'(w_{k_i})}\,. \tag{34}$$

In that case, we choose integer numbers γ_{k_i} such that we have

$$\delta_{k_i} - \gamma_{k_i} = \frac{1}{\ell'(w_{k_i})}\,. \tag{35}$$

Then it is not difficult to verify that all possible solutions of the considered problem lie in the set of functions given by equality (26) (for $\varkappa < 0$) and by equality (28) (for $\varkappa \geq 0$). In order that the functions defined by these equalities would indeed be solutions, it is necessary and sufficient that their boundary functions would belong to the class $L^{p(\cdot)}(\gamma;\rho)$. This is equivalent to the requirement that the function

$$(M_c)(t) = \frac{X^+(t)}{Q(t)} \int_\gamma \frac{c(z(\tau))Q(\tau)}{X^+(\tau)(a(z(\tau)) + ib(z(\tau)))}\,\frac{d\tau}{\tau - t}\,, \quad t \in \Gamma, \tag{36}$$

would belong to the class $L^{p(\cdot)}(\gamma;\rho)$, where

$$\rho(\tau) \sim \omega(z(\tau))r(\tau)|z'(\tau)|^{\frac{1}{\ell(\tau)}}/Q(\tau) \tag{37}$$

(see [8, equality (25)]). Under our assumptions we have

$$\rho(\tau) = \prod_{k=1}^{j}(\tau - w_k)^{\beta_k}, \quad \beta_k = \delta_k - \gamma_k, \quad \beta_{k_i} = \frac{1}{\ell'(w_{k_i})}\,. \tag{38}$$

Thus it is required that the condition

$$\rho(t)(M_c)(t) \in L^{\ell(\cdot)}(\gamma) \tag{39}$$

or, which is same, the conditions

$$\frac{\omega(z(t))r(t)|z'(t)|^{\frac{1}{\ell(t)}}}{Q(t)} \int_\gamma \frac{c(z(\tau))Q(\tau)}{X^+(\tau)(a(z(\tau)) + ib(z(\tau)))}\,\frac{d\tau}{\tau - t} \in L^{\ell(\cdot)}(\gamma)$$

be fulfilled.

Since in the case of piecewise-Hölder coefficients we have $X^+(t) \sim r(t)$, the latter condition can be written in the form

$$\frac{\omega(z(t))r(t)|z'(t)|^{\frac{1}{\ell(t)}}}{Q(t)} \int_\gamma \frac{c(z(\tau))\omega(z(\tau))|z'(\tau)|^{\frac{1}{\ell(\tau)}}Q(\tau)}{\omega(z(\tau))|z'(\tau)|^{\frac{1}{\ell(\tau)}}X^+(\tau)(a(z(\tau)) + ib(z(\tau)))}\,\frac{d\tau}{\tau - t}$$
$$= \rho(t)\int_\gamma \frac{g(\tau)}{\rho(\tau)}\,\frac{d\tau}{\tau - t} \in L^{\ell(\cdot)}(\gamma), \tag{40}$$

where

$$g(\tau) = c(z(\tau))\omega(z(\tau))|z'(\tau)|^{\frac{1}{\ell(\tau)}}\,(a(z(\tau)) + ib(z(\tau)))^{-1} \in L^{\ell(\cdot)}(\gamma). \tag{41}$$

Assume that

$$(Tg)(t) = \rho(t)\int_\gamma \frac{g(\tau)}{\rho(\tau)}\,\frac{d\tau}{\tau - t}\,, \quad t \in \gamma. \tag{42}$$

Finally, we have

Lemma 4. *For problem* (1) *to be solvable it is necessary and sufficient that the function* Tg *would belong to the class* $L^{\ell(\cdot)}(\gamma)$, *where* ρ *is given by equality* (38), *and* g *by equality* (41).

Under assumptions (34)–(35) this condition is not fulfilled for all $c \in L^{p(\cdot)}(\Gamma;\omega)$; otherwise the Cauchy operator S_γ would turn out to be continuous in $L^{\ell(\cdot)}(\gamma;\rho)$, where ρ is given by equality (38), which is impossible because condition $-\frac{1}{l(a)} < \nu < \frac{1}{l'(a)}$ is violated.

We naturally pose the problem of indicating wide subclasses of a function $g \in L^{p(\cdot)}(\gamma)$, for which $Tg \in L^{\ell(\cdot)}(\gamma)$.

If $p(t) = p = \text{const} > 1$, then one of such possible classes is a family of those functions g, for which

$$g(\tau)\ln|\tau - \tau_{k_i}| \in L^{\ell(\cdot)}(\gamma)$$

(see [4, p. 163]).

If on Γ there is a point t_0, for which $\delta(w_0) = \frac{1}{\ell'(w_0)}$ ($w_0 = w(t_0)$) and also $p(t_0) = \underline{p} = \min\limits_{t\in\Gamma} p(t)$, then, using the condition

$$g(\tau)\ln|\tau - w_0| \in L^{\ell(\cdot)}(\gamma) \tag{43}$$

we have $Tg \in L^{\ell(\cdot)}(\gamma)$ (see [6]).

As a result of the above consideration we come to

Theorem 5. *If* $a(t)$, $b(t)$ *are piecewise-Hölder functions and the conditions of Theorem* 1, *except condition* (21), *are fulfilled, then for problem* (1) *to be solvable it is necessary and sufficient that, in addition to conditions* (25) (*for* $\varkappa < 0$), *the conditions*

$$(M_c) \in L^{\ell(\cdot)}(\gamma;\rho)$$

would be fulfilled, where M_c *is the function given by equality* (36), *and* ρ *by equality* (38).

If $p(t)$ *attains a minimum at the point* t_0, *at which condition* (21) *is violated* (*i.e.*, $\delta(w_0) = \frac{1}{\ell'(w_0)}$, $w_0 = w(t_0)$) *and if* $c(t)$ *is such a function that the function* g *constructed by means of it according to formula* (41) *satisfies condition* (43), *then for this function* $c(t)$, *problem* (1) *is solvable.*

Remark 6. Condition (43) is equivalent to the condition

$$c(t)\omega(t)\ln|w(t) - w(t_0)| \in L^{p(\cdot)}(\Gamma),$$

and if $\Gamma \in C^{1,L}(A_1,\ldots,A_i;\nu_1,\ldots,\nu_i)$, $0 < \nu_k < 2$, then to the condition

$$c(t)\omega(t)\ln|t - t_0| \in L^{p(\cdot)}(\Gamma).$$

5. Generalization of Vekua's integral representations of holomorphic functions

In many boundary value problems of function theory and mathematical physics the boundary conditions contain not only the sought function, but also its derivatives up to certain order. Therefore it is useful to have formulas giving an integral representation of this holomorphic function. One form of such representations, quite convenient for applications, was proposed by I. Vekua ([16], [17]). N. Muskhelishvili expounded them in his book, where they are called I. Vekua's integral representations (see [18, pp. 224–232]).

Theorem [I. Vekua]**.** *Let D^+ be a finite domain bounded by a simple closed Lyapunov curve Γ and $\Phi(z)$ be a holomorphic function in D^+, whose derivative of order m is continuous in D^+ and the boundary belong to the Hölder class H. Then, assuming that the origin is in D^+, the function Φ is representable for $m = 0$ as*

$$\Phi(z) = \int_\Gamma \frac{\varphi(t)\,dt}{1 - \frac{z}{t}} + id \tag{44}$$

and for $m \geq 1$ as

$$\Phi(z) = \int_\Gamma \varphi(t) \left(1 - \frac{z}{t}\right)^{m-1} \ln\left(1 - \frac{z}{t}\right) ds + \int_\Gamma \varphi(t)\,ds + id, \tag{45}$$

where $\varphi(t)$ is a real function from the class H, and d is a real constant; $\varphi(t)$ and d are defined uniquely with respect to $\Phi(z)$.

Subsequently, B. Khvedelidze [5] gave a generalization of this theorem to the case where a derivative of order m of the function $\Phi(z)$ is representable in D^+ by a Cauchy type integral with a density from the Lebesgue space $L^p(\Gamma;\omega)$, where $p > 1$ and

$$\omega(t) = \prod_{k=1}^{n} |t - t_k|^{\alpha_k}, \ \ t_k \in \Gamma, \ \ -\frac{1}{p} < \alpha_k < \frac{1}{p'}\,, \ \ p' = \frac{p}{p-1}\,. \tag{46}$$

In that case, φ belongs to $L^p(\Gamma;\omega)$.

I. Vekua used these representations for investigating quite a general boundary value problem, namely, the Riemann–Hilbert–Poincaré problem.

Definition 7. If $m \geq 0$ is an integer number, then we denote by $K^{p(\cdot)}_{D,m}(\Gamma;\omega)$ the set of functions Φ holomorphic in D for which $\Phi^{(m)}(z) \in K^{p(\cdot)}_D(\Gamma;\omega)$. It is assumed that $\Phi^{(0)}(z) = \Phi(z)$ and thus $K^{p(\cdot)}_{D,0}(\Gamma;\omega) = K^{p(\cdot)}_D(\Gamma;\omega)$.

Theorem 8. *Let*

i) *Γ be a curve of the class $C^1_{D^+}(A_1,\dots,A_i;\nu_1,\dots,\nu_i)$, $0 \leq \nu_k < 2$, $k = \overline{1,i}$ and $p \in \widetilde{\mathcal{P}}(\Gamma)$ or $\Gamma \in C^{1,L}_{D^+}(A_1,\dots,A_i;\nu_1,\dots,\nu_i)$, $0 \leq \nu_k < 2$, $k = \overline{1,i}$ and $p \in \mathcal{P}(\Gamma)$;*

ii) *ω be a power function of form* (46)*;*

iii) *the point $z=0$ lie in D^+; $z=z(w)$ be a conformal mapping of the circle U onto the domain $D^-=CD^+$; $z(0)=\infty$ and $w=w(z)$ be its inverse mapping. Let $a_k=w(A_k)$, $k=\overline{1,i}$, $\tau_k=w(t_k)$, $k=\overline{1,n}$, and the points of the set $\{a_1,\dots,a_i\}\cup\{\tau_1,\dots,\tau_n\}$ be numbered so that*

$$\begin{aligned}&w_1=a_1=\tau_1,\dots,w_\mu=a_\mu=\tau_\mu,\\&w_{\mu+1}=a_{\mu+1},\dots,w_{\mu+p}=a_{\mu+p},\\&w_{\mu+p+1}=\tau_{\mu+1},\dots,w_{\mu+p+M}=\tau_{\mu+M}\end{aligned}$$

and

$$\delta_k=\begin{cases}\alpha_k\lambda_k+\frac{\nu_k-1}{\ell(w_k)}+\nu_k-1, & k=\overline{1,\mu},\quad \lambda_k=2-\nu_k,\\ \frac{\nu_k-1}{\ell(w_k)}+\nu_k-1, & k=\overline{\mu+1,\mu+p},\\ \alpha_{k-p}, & k=\overline{\mu+p+1,\mu+p+M},\end{cases}$$

where $\ell(\tau)=p(z(\tau))$, $|\tau|=1$;

iv) *$\Phi\in K^{p(\cdot)}_{D^+,m}(\Gamma;\omega)$.*

If

$$\{\delta_k\}\neq\frac{1}{\ell'(w_k)}\,,\quad k=\overline{1,j},\quad j=n+i-\mu=\mu+p+M,\quad \ell'(\tau)=\frac{\ell(\tau)}{\ell(\tau)-1}\,,$$

then there exist a real function $\varphi\in L^{p(\cdot)}(\Gamma;\omega)$ and a real constant d such that the representations (1) *and* (2) *are valid.*

The function φ and the constant d are defined in a unique manner.

Representations (1) and (2) are valid in the following particular cases:

I. a) Γ is a smooth curve and $p\in\widetilde{\mathcal{P}}(\Gamma)$ or Γ is a Lyapunov curve and $p\in\mathcal{P}(\Gamma)$;

b) ω is a weight function of form (46).

II. a) Γ is a curve of the class $C^1(A_1,\dots,A_i;\nu_1,\dots,\nu_i)$ and $p\in\widetilde{\mathcal{P}}(\Gamma)$ or Γ is a curve of the class $C^{1,L}(A_1,\dots,A_i;\nu_1,\dots,\nu_i)$ and $p\in\mathcal{P}(\Gamma)$ (in both cases $0<\nu_k<2$, $k=\overline{1,i}$);

b) $\omega(t)=1$;

c) $\left\{\frac{\nu_k-1}{p(A_k)}+\nu_k-1\right\}\neq\frac{1}{p'(A_k)}$.

The problem of representation of holomorphic functions in terms of new assumptions reduces to the investigation of the Riemann–Hilbert problem in the class $K^{p(\cdot)}_{D^-}(\Gamma;\omega)$. In Section 2 this problem is solved in the class $K^{p(\cdot)}_{D}(\Gamma;\omega)$ when D is a bounded domain with the boundary Γ. The case of an unbounded domain can be easily investigated by reducing it to the considered one.

6. The Riemann–Hilbert–Poincaré problem in the class $K^{p(\cdot)}_{D^+,m}(\Gamma;\omega)$

I. Vekua applied the representations (1), (2) to the investigation of the Riemann–Hilbert–Poincaré problem

$$\operatorname{Re}\left[\sum_{k=0}^{m}\left(a_k(t)\Phi^{(k)}(t)+\int_\Gamma H_k(t,\tau)\Phi^{(k)}(\tau)\,d\tau\right)\right]=f(t), \tag{47}$$

where a_k, H_k, f are the given functions of Hölder's class, Γ is a Lyapunov curve bounding the finite domain D^+, and the sought function Φ has a continuous derivative of order m in $\overline{D^+}$ and with boundary values from H ([19], [20]). In [5] this problem is considered when $a_k(t)$ are continuous and $\Phi^{(m)}(z)$ is representable by a Cauchy type integral with a density from $L^p(\Gamma;\omega)$, where $p>1$ and ω is a power function.

Here we assume that for p, Γ and ω the conditions of Theorem 3 are fulfilled. We want to solve problem (47) in the class $K^{p(\cdot)}_{D^+.m}(\Gamma;\omega)$, therefore it is assumed that $f\in L^{p(\cdot)}(\Gamma;\omega)$. Since $\Phi^{(m)}\in K^{p(\cdot)}_{D^+}(\Gamma;\omega)\subset E^1(D^+)$, the functions $\Phi^{(0)}(z)=\Phi(z)$, $\Phi'(z),\dots,\Phi^{(m-1)}(z)$ are continuous in $\overline{D^+}$ and absolutely continuous on Γ with respect to the arc abscissa. Thus it is natural to assume that in condition (47) the coefficients $a_k(t)$, $k=\overline{0,m-1}$, belong to $L^{p(\cdot)}(\Gamma;\omega)$. As to the coefficient $a_m(t)$, we should assume that it is bounded. However this is not enough. Following [17], [18], [20], we reduce the problem to a singular integral equation in the class $L^{p(\cdot)}(\Gamma;\omega)$, which is investigated in various conditions depending on the assumptions made for p, Γ and ω ([3], [9]). It is assumed for simplicity that $a_m(t)$ is piecewise-continuous on Γ and $\inf|a_m(t)|>0$.

So, let Γ be a curve of the class $C^1_{D^+}(A_1,\dots,A_i;\nu_1,\dots,\nu_i)$, $0\le\nu_k<2$, ω be the power function (46), the coefficients $a_0,a_1,\dots,a_{m-1}$ belong to $L^{p(\cdot)}(\Gamma;\omega)$, $p\in\widetilde{\mathcal{P}}(\Gamma)$, $a_m\in C(\widetilde{B}_1,\dots,\widetilde{B}_\lambda)$ (i.e., a_m is piecewise-continuous on Γ with discontinuity points $\widetilde{B}_k$), and the operators

$$\mathcal{H}_k\varphi=\int_\Gamma H_k(t_0,t)\varphi(t)\,dt,\quad t_0\in\Gamma,$$

be compact in $L^{p(\cdot)}(\Gamma;\omega)$.

It is required to find a function $\Phi\in K^{p(\cdot)}_{D^+,m}(\Gamma;\omega)$ for which equality (47) holds a.e. on Γ. Note that the compactness of the operators $\mathcal{H}_k$ is provided, for instance, by the fulfillment of the conditions

$$|H_k(t_0,t)|<\frac{A}{[s(t_0,t)]^\lambda},\quad k=\overline{0,m},$$

where A, $\lambda\in[0,1)$, are constants and $s(t_0,t)$ is the length of the smallest of two arcs connecting the points t_0 and t on Γ (see [11]).

Following [18, p. 233] this problem will sometimes be called Problem V.

Since under the above assumptions the conditions of Theorem 3 are fulfilled, the sought solution Φ is representable by equality (1) for $m = 0$ and by equality (2) for $m \geq 1$.

Assuming first that $m \geq 1$, we calculate the derivatives of the function Φ given by equality (2) and substitute them into (47). Thus we obtain (see [18, pp. 234–235]) that the function φ satisfies the condition

$$N\varphi = A(t_0)\varphi(t_0) + \int_\Gamma N(t_0,t)\varphi(t)\,ds = f(t_0) - d\sigma(t_0), \tag{48}$$

where

$$\begin{aligned}
A(t_0) &= \operatorname{Re}\left[(-1)^m(m-1)!\pi i t_0^{1-m} t_0' a_m(t_0)\right], \qquad (49)\\
\sigma(t_0) &= \operatorname{Re}\left[ia_0(t_0) + i\int_\Gamma h_0(t_0,t)\,ds\right],\\
N(t_0,t) &= \sum_{l=0}^{m} \operatorname{Re}\left[a_l(t_0)N_l(t_0,t_1)N_l(t_1,t)\,ds_1\right]\\
&\quad + \operatorname{Re}\left[(-1)^m(m-1)!\pi i h_m(t_0,t)t^{1-m}\overline{t'}\right],\\
N_0(t_0,t) &= \left(1-\frac{t_0}{t}\right)^{m-1}\ln\left(1-\frac{t_0}{t}\right)+1, \quad N_m(t_0,t) = \frac{(-1)^m(m-1)!}{t^{m-1}(t-t_0)},\\
N_l(t_0,t) &= (-1)^l\,\frac{(m-1)\cdots(m-l)}{t^l}\left(1-\frac{t_0}{t}\right)^{m-l-1}\\
&\quad \times\left(\ln\left(1-\frac{t_0}{t}\right)+\frac{1}{m-1}+\cdots+\frac{1}{m-l}\right), \quad l=\overline{1,m-1}.
\end{aligned}$$

It is evident that

$$N\varphi = N^0\varphi + T\varphi,$$

where

$$N^0\varphi = A(t_0)\varphi(t_0) + \frac{B(t_0)}{\pi i}\int_\Gamma \frac{\varphi(t)\,dt}{t-t_0}, \tag{50}$$

$$B(t_0) = \frac{1}{2}(-1)^m(m-1)!\pi i\left[t_0^{1-m}\overline{t_0'}a_m(t_0) + \overline{t_0^{1-m}}t_0'\overline{a_m(t_0)}\right], \tag{51}$$

$$\begin{aligned}
T\varphi &= \int_\Gamma \sum_{l=0}^{m-1}\operatorname{Re}\left[a_l(t_0)N_l(t_0,t) + \int_\Gamma H_l(t_0,t_1)N_l(t_1,t)\,ds_1\right]\varphi(t)\,ds\\
&\quad + \int_\Gamma \operatorname{Re}\left[(-1)^m(m-1)!\pi i H_m(t_0,t)t^{1-m}\overline{t'}\right]\varphi(t)\,ds.
\end{aligned}$$

Moreover, (49) implies that the function $A(t_0)$ can be written in the form

$$A(t_0) = \frac{1}{2}(-1)^m(m-1)!\pi i\left[t_0^{1-m}\overline{t_0'}a_m(t_0) - \overline{t_0^{1-m}}t_0'\overline{a_m(t_0)}\right]. \tag{52}$$

By virtue of our assumptions about Γ, the coefficients $a_0(t_0),\dots,a_{m-1}(t_0)$ and operators $\mathcal{H}_k$ and using the above-mentioned result from [11] it is not difficult to establish that the operator T is compact in $L^{p(\cdot)}(\Gamma;\omega)$.

The operator

$$(N'g)(t) = A(t)g(t) + \int_\Gamma N(t_0,t)g(s_0)\,ds_0$$

considered in the space $L^{p(\cdot)}(\Gamma;\omega^{-1})$ is the conjugate operator to N. We prove that the operator N is Noetherian in $L^{p'(\cdot)}(\Gamma;\omega)$ and we calculate the index ind$N = \varkappa$.

Theorem 9. *Let the conditions of Theorem* 3 *be fulfilled. Then for Problem V to be solvable in the class* $K^{p(\cdot)}_{D^+,m}(\Gamma;\omega)$ *it is necessary and sufficient that for some real* d *the function* $\widetilde{f}(t_0) = f(t_0) - d\sigma(t_0)$ *should satisfy the conditions*

$$\int_\Gamma \widetilde{f}(t_0)g_k(s_0)\,ds_0, \quad k = \overline{1,n'},$$

where $g_1,\dots,g_{n'}$ *are linearly independent solutions from the class* $L^{p'(\cdot)}(\Gamma;\omega^{-1})$ *of the equation* $N'g = 0$*, where* N' *is the adjoint operator to the operator* N*.*

In order that Problem V has a solution, for any right-hand part f *it is necessary and sufficient that* $n' = 0$ *or* $n' = 1$ *and in the latter case the solution* g *of the equation* $N'g = 0$ *must satisfy the condition*

$$(g,\sigma) = \int_\Gamma g(t_0)\sigma(t_0)\,ds_0 \neq 0.$$

In both cases the homogeneous problem has $\varkappa + 1$ *linearly independent solutions (where* $\varkappa \geq -1$*).*

If these conditions are misobserved, then: if $(g_k,\sigma) = 0$ *for any* $k = \overline{1,n'}$*, then the homogeneous problem has* $\varkappa + n'$ *linearly independent solutions, and if among the numbers* (g_k,σ) *there is at least one nonzero number, then it has* $\varkappa + n' + 1$ *solutions.*

If $\sigma(t) = 0$*, then problem* (47) *is solvable for any right-hand part* $f(t_0)$ *if and only if* $n' = 0$*; in that case the homogeneous problem has* $\varkappa+1$ *linearly independent solutions.*

7. The Poincaré problem

We will consider this problem formulated as follows:

Find, in the domain D^+, a harmonic function u from the set

$$e^{p(\cdot)}_{D^+,1}(\Gamma;\omega) = \Big\{u:\ u = \operatorname{Re}\Phi,\ \ \Phi \in K^{p(\cdot)}_{D^+,1}(\Gamma;\omega)\Big\},$$

for which a.e. on Γ we have

$$\alpha(s)\,\frac{\partial u}{\partial n} + \beta(s)\,\frac{\partial u}{\partial s} + \gamma(s)u = f(s). \tag{53}$$

Here $\alpha(s)$, $\beta(s)$, $\gamma(s)$, $f(s)$ are the real functions given on Γ, s is an arc abscissa, $\frac{\partial u}{\partial n}$ is normal derivative. It is assumed that $p \in \widetilde{\mathcal{P}}(\Gamma)$, $\Gamma \in C^1(A_1, \dots, A_i; \nu_1, \dots, \nu_i)$, $0 < \nu_k < 2$, $k = \overline{1,i}$; α and β belong to Hölder's class, while γ and f belong to $L^{p(\cdot)}(\Gamma; \omega)$.

Let

$$a(t) = -\alpha(s) \sin \vartheta(s) + \beta(s) \cos \vartheta(s), \quad b(t) = \alpha(s) \cos \vartheta(s) + \beta(s) \sin \vartheta(s),$$

where $\vartheta(s)$ is the angle formed between the tangent to Γ at the point $t(s)$ and the axis of abscissa. Then condition (30) takes the form

$$a(t) \frac{\partial u}{\partial x} + b(t) \frac{\partial u}{\partial y} + \gamma(t) u = f(t), \quad t \in \Gamma,$$

which can be rewritten as

$$\operatorname{Re} \left[(a(t_0) + ib(t_0)) \Phi'(t_0) + \gamma(t_0) \Phi(t_0) \right] = f(t_0), \quad t_0 \in \Gamma.$$

Using representation (2), for $m = 1$ we obtain the equality

$$\begin{aligned} N\varphi = & \operatorname{Re} \left[-\pi i \overline{t_0'} (a(t_0) + ib(t_0)) \right] \varphi(t_0) \\ & + \int_\Gamma \operatorname{Re} \left[\gamma(t_0) \ln e \left(1 - \frac{t_0}{t} \right) - \frac{a(t_0) + ib(t_0)}{t - t_0} \right] \varphi(t)\, ds = f(t_0). \end{aligned}$$

Let us assume that $(a^2 + b^2) > 0$ (or, which is the same, $\alpha^2 + \beta^2 > 0$).

Assume that

$$n = \frac{1}{2\pi} \left[\arg(\alpha(t) + i\beta(t)) \right]_\Gamma , \tag{54}$$

where $[f]_\Gamma$ denotes an increment of the function $f(t)$ when the point t performs one-time movement along the curve Γ. In that case the index of the operator N is calculated by the equality $\varkappa = \varkappa_0 + \varkappa_1$, where $\varkappa_1 = 2n$ and

$$\begin{aligned} \varkappa_0 = & \mathcal{N} \left\{ A_k : A_k \notin \bigcup \{t_j\}, \ \nu_k > p(A_k) \right\} \\ & + \mathcal{N} \left\{ t_k = A_k : \frac{p(A_k)}{1 + \alpha_k p(A_k)} < \nu_k < \frac{2p(A_k)}{1 + \alpha_k p(A_k)} \right\} \end{aligned} \tag{55}$$

(recall that A_k are the angular points of Γ and α_k are power exponents from weight (46)).

Theorem 10. *For the Poincaré problem to have a solution in the class* $e^{p(\cdot)}_{D^+,1}(\Gamma; \omega)$ *for any right-hand part* $f(t)$ *it is necessary and sufficient that the equation*

$$\begin{aligned} N' g = & \operatorname{Re} \left\{ -\pi i \overline{t_0'} \left[a(t_0) + ib(t_0) \right] \right\} g(t_0) \\ & + \int_\Gamma \operatorname{Re} \left\{ \gamma(t_0) \ln e \left(1 - \frac{t_0}{t} \right) + \frac{a(t) + ib(t)}{t - t_0} \right\} g(t)\, ds = 0 \end{aligned}$$

would not have nonzero solutions in the class $L^{p'(\cdot)}(\Gamma; \omega^{-1})$.

When this condition is fulfilled, the problem has $\varkappa + 1$ *linearly independent solutions, where* $\varkappa = 2n + \varkappa_0$.

References

[1] L. Diening, Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$. *Math. Inequal. Appl.* **7**(2004), No. 2, 245–253.

[2] L. Diening and M. Růžička, Calderon–Zygmund operators on generalized Lebesgue spaces $L^{p(\cdot)}$ and problems related to fluid dynamics. *J. Reine Angew. Math.* **563**(2003), 197–220.

[3] A.Yu. Karlovich, Algebras of singular integral operators on Nakano spaces with Khvedelidze weights over Carleson curves with logarithmic whirl points. In: *Pseudodifferential Equations and Some Problems of Mathematical Physics.* (Russian) *The special issues of "Izv. VUZ, Severo-Kavkazski Region. Estestvennye Nauki", dedicated to I.B. Simonenko's* 70*th birthday, Rostov-on-Don, Russia,* 2005, 135–142.

[4] G. Khuskivadze, V. Kokilashvili, and V. Paatashvili, Boundary value problems for analytic and harmonic functions in domains with nonsmooth boundaries. Applications to conformal mappings. *Mem. Differential Equations Math. Phys.* **14**(1998), 195 pp.

[5] B.V. Khvedelidze, Linear discontinuous boundary problems in the theory of functions, singular integral equations and some of their applications. (Russian) *Trudy Tbiliss. Mat. Inst. Razmadze* **23**(1956), 3–158.

[6] D.E. Edmunds, V. Kokilashvili, and A. Meskhi, Two-weight estimates in $L^{p(x)}$ spaces for classical integral operators and applications to the norm summability of Fourier trigonometric series. *Houston J. Math.* **35**(2009), No. 2, 665–689.

[7] V. Kokilashvili and V. Paatashvili, The Dirichlet problem for harmonic functions in the Smirnov class with variable exponent. *Georgian Math. J.* **14**(2007), No. 2, 289–299.

[8] V. Kokilashvili and V. Paatashvili, The Riemann–Hilbert problem in weighted classes of Cauchy type integrals with density from $L^{p(\cdot)}(\Gamma)$. *Complex Anal. Oper. Theory* **2**(2008), No. 4, 569–591.

[9] V. Kokilashvili, V. Paatashvili, and S. Samko, Boundedness in Lebesgue spaces with variable exponent of the Cauchy singular operator on Carleson curves. *Modern operator theory and applications,* 167–186, *Oper. Theory Adv. Appl.,* 170, *Birkhäuser, Basel,* 2007.

[10] V. Kokilashvili, V. Paatashvili, and S. Samko, Boundary value problems for analytic functions in the class of Cauchy-type integrals with density in $L^{p(\cdot)}(\Gamma)$. *Bound. Value Probl.* **2005**, No. 1, 43–71.

[11] V. Kokilashvili and S. Samko, Maximal and fractional operators in weighted $L^{p(x)}$ spaces. *Rev. Mat. Iberoamericana* **20**(2004), No. 2, 493–515.

[12] V. Kokilashvili and S. Samko, Singular integral equations in the Lebesgue spaces with variable exponent. *Proc. A. Razmadze Math. Inst.* **131**(2003), 61–78.

[13] V. Kokilashvili and S. Samko, Singular integrals in weighted Lebesgue spaces with variable exponent. *Georgian Math. J.* **10**(2003), No. 1, 145–156.

[14] V. Kokilashvili and S. Samko, Weighted boundedness in Lebesgue spaces with variable exponents of classical operators on Carleson curves. *Proc. A. Razmadze Math. Inst.* **138**(2005), 106–110.

[15] V. Kokilashvili and V. Paatashvili, The Riemann–Hilbert problem in a domain with piecewise smooth boundaries in weight classes of Cauchy type integrals with a density from variable exponent Lebesgue spaces. *Georgian Math. J.* **16**(2009), No. 4, 737–755.

[16] V. Kokilashvili and V. Paatashvili, Generalizations of I. Vekua's integral representations of holomorphic functions and their application to the Riemann–Hilbert–Poincaré problem. *Function Spaces and Applications* (to appear).

[17] Z. Meshveliani, The Riemann–Hilbert problem in weighted Smirnov classes of analytic functions. *Proc. A. Razmadze Math. Inst.* **137**(2005), 65–86.

[18] N.I. Muskhelishvili, Singular integral equations. Boundary value problems in the theory of function and some applications of them to mathematical physics. (Russian) Third, corrected and augmented edition. With an appendix by B. Bojarski. *Nauka, Moscow,* 1968; English transl.: *Noordhoff International Publishing, Leyden,* 1977.

[19] I. Vekua, On a new integral representation of analytic functions and its applications. (Russian) *Bull. Acad. Sci. Georgian SSR,* 2(1941), 477–484.

[20] I. Vekua, On a linear boundary value problem of Riemann. (Russian) *Trudy Tbiliss. Mat. Inst. Razmadze,* **XI** (1942), 109–139.

Vakhtang Kokilashvili
A. Razmadze Mathematical Institute
1, M. Aleksidze St.
Tbilisi 0193, Georgia
e-mail: kokil@rmi.acnet.ge

Operator Theory:
Advances and Applications, Vol. 216, 41–64

Edge-degenerate Operators at Conical Exits to Infinity

B.-Wolfgang Schulze

Abstract. We develop elements of a calculus of pseudo-differential operators on an infinite cylinder $B^{\asymp} := \mathbb{R} \times B \ni (t, \cdot)$ where the cross section B is a compact manifold with smooth edge Y. The space $B^{\asymp}$ is regarded as a manifold with edge $Y^{\asymp}$ with conical exits to infinity $t \to \pm\infty$. The amplitude functions are families of operators in the edge algebra on B depending on parameters $(t, \tau, \zeta), \zeta \neq 0$. We impose a special degenerate behaviour for $|t| \to \infty$, motivated by the structure of principal edge symbols of the next higher corner calculus, consisting of operators on an infinite singular cone with base B and axial variable t. In this framework we study ellipticity and parametrices.

Mathematics Subject Classification (2000). Primary 35S35; Secondary 35J70.

Keywords. Edge pseudo-differential operators, conical exits to infinity, parameter-dependent symbols, ellipticity and parametrices.

Introduction

This investigation is motivated by the task to understand the structure of parametrices of elliptic operators on a corner manifold M in terms of the symbolic structure adapted to the nature of the underlying singular space M. A topological space M (under some reasonable assumptions on the topology, e.g., paracompact, etc.) belongs to the category $\mathfrak{M}_k$ of manifolds with singularities of order $k \in \mathbb{N}\setminus\{0\}$ if there is a subspace $s_k(M) \in \mathfrak{M}_0$ (where 0 indicates the category of C^∞ manifolds), such that $M \setminus s_k(M) \in \mathfrak{M}_{k-1}$, and $s_k(M)$ contains a neighbourhood V which has the structure of a (locally trivial) X^Δ-bundle over $s_k(M)$ for

$$X^\Delta := \overline{\mathbb{R}}_+ \times X/(\{0\} \times X) \quad \text{for some} \quad X \in \mathfrak{M}_{k-1}.$$

In addition for any transition function $\Omega \times X^\Delta \to \tilde{\Omega} \times X^\Delta$ between different trivialisations of V (where $\Omega, \tilde{\Omega} \subseteq \mathbb{R}^q$ correspond to charts on $s_k(M), q = \dim s_k(M)$)

we have a restriction to an $\mathfrak{M}_{k-1}$-isomorphism

$$\Omega \times X^\wedge \to \tilde{\Omega} \times X^\wedge;\ X^\wedge := \mathbb{R}_+ \times X, \tag{0.1}$$

which is asked to be extendible to an $\mathfrak{M}_{k-1}$-isomorphism $\Omega\times(\mathbb{R}\times X) \to \tilde{\Omega}\times(\mathbb{R}\times X)$. In particular, $\mathfrak{M}_1$ is the category of manifolds with edge-singularities $s_1(M) \in \mathfrak{M}_0$, and those for $\dim s_1(M) = 0$ form the subcategory of manifolds with conical singularities.

Let $\mathrm{Diff}^\nu(\cdot)$ denote the space of differential operators of order $\nu \in \mathbb{N}$ with smooth coefficients (in local coordinates) on the respective smooth manifold in parentheses. For $\nu = 0$ this space has the meaning of $C^\infty(\cdot)$.

From $M \in \mathfrak{M}_k$ and $M \setminus s_k(M) \in \mathfrak{M}_{k-1}$ we find an $s_{k-1}(M) := s_{k-1}(M \setminus s_k(M)) \in \mathfrak{M}_0$ such that $(M \setminus s_k(M)) \setminus s_{k-1}(M) \in \mathfrak{M}_{k-2}$, and so on. By iterating this process we obtain a disjoint decomposition

$$M = s_k(M) \cup s_{k-1}(M) \cup \cdots \cup s_1(M) \cup s_0(M) \tag{0.2}$$

of M into strata $s_j(M)$ of different dimensions, in fact, $\dim s_j(M) < \dim s_{j-1}(M)$, $j = 1, \ldots, k$. We call $s_0(M)$ the main stratum of M, and set $\dim M := \dim s_0(M)$. Now on a singular manifold $M \in \mathfrak{M}_k$ we define spaces of differential operators in an iterative manner. The space $\mathrm{Diff}^\mu_{\mathrm{deg}}(M)$ is defined to be the set of all $A \in \mathrm{Diff}^\mu_{\mathrm{deg}}(M \setminus s_k(M))$ which are

(i) in the case $q = \dim s_k(M) > 0$, close to $s_k(M)$ in the splitting of variables $(t, x, z) \in \mathbb{R}_+ \times X \times \Omega$ of the form

$$A = t^{-\mu} \sum_{j+|\alpha| \le \mu} a_{j,\alpha}(t, z)(-t\partial_t)^j (tD_z)^\alpha \tag{0.3}$$

for coefficients $a_{j,\alpha}(t, z) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, \mathrm{Diff}^{\mu-(j+|\alpha|)}_{\mathrm{deg}}(X))$,

(ii) in the case $q = \dim s_k(M) = 0$, close to $s_k(M)$ in the splitting of variables $(t, x) \in \mathbb{R}_+ \times X$ of the form

$$A = t^{-\mu} \sum_{j=0}^{\mu} a_j(t)(-t\partial_t)^j, \tag{0.4}$$

for coefficients $a_j(t) \in C^\infty(\overline{\mathbb{R}}_+, \mathrm{Diff}^{\mu-j}_{\mathrm{deg}}(X))$.

Let us briefly recall from [22] the idea of formulating a principal symbolic hierarchy

$$\sigma(A) := \{\sigma_0(A), \sigma_1(A), \ldots, \sigma_k(A)\} \tag{0.5}$$

of operators $A \in \mathrm{Diff}^\mu_{\mathrm{deg}}(M)$, also defined in an iterative manner. For $M \in \mathfrak{M}_0$ we define $\sigma_0(A)$ as the homogeneous principal symbol of A in the standard sense, an invariantly defined function in $C^\infty(T^* s_0(M) \setminus 0)$, (positively) homogeneous of order μ in the fibre variables. Since $A \in \mathrm{Diff}^\mu_{\mathrm{deg}}(M)$ for $M \in \mathfrak{M}_k$ for general $k \in \mathbb{N}$, induces an operator $A|_{s_0(M)} \in \mathrm{Diff}^\mu(s_0(M))$ we define $\sigma_0(A) := \sigma_0(A|_{s_0(M)})$

for every k. More generally, an $A \in \mathrm{Diff}^{\mu}_{\mathrm{deg}}(M), M \in \mathfrak{M}_k$, induces an operator $A|_{M\setminus s_k(M)} \in \mathrm{Diff}^{\mu}_{\mathrm{deg}}(M \setminus s_k(M))$. Assuming by induction that we already defined

$$\sigma(A|_{M\setminus s_k(M)}) := \{\sigma_0(A|_{M\setminus s_k(M)}), \sigma_1(A|_{M\setminus s_k(M)}), \dots, \sigma_{k-1}(A|_{M\setminus s_k(M)})\}, \quad (0.6)$$

for $k > 0$ we set

$$\sigma(A) := \{\sigma(A|_{M\setminus s_k(M)}), \sigma_k(A)\} \quad (0.7)$$

for

$$\sigma_k(A)(z,\zeta) := t^{-\mu} \sum_{j+|\alpha|\le\mu} a_{j,\alpha}(0,z)(-t\partial_t)^j (t\zeta)^\alpha \text{ when } \dim s_k(M) > 0, \quad (0.8)$$

$(z,\zeta) \in T^* s_k(M) \setminus 0$, and

$$\sigma_k(A)(z,v) := \sum_{j=0}^{\mu} a_j(0) v^j \text{ when } \dim s_k(M) = 0, \quad (0.9)$$

$v \in \mathbb{C}$. Both (0.8) and (0.9) are operator-valued, namely, with values in $\mathrm{Diff}^{\mu}_{\mathrm{deg}}(X^\wedge)$ for $\dim s_k(M) > 0$ and $\mathrm{Diff}^{\mu}_{\mathrm{deg}}(X)$ for $\dim s_k(M) = 0$. As such they first define families of mappings

$$\sigma_k(A)(z,\zeta) : C^\infty(s_0(X^\wedge)) \to C^\infty(s_0(X^\wedge)), \quad (0.10)$$

and

$$\sigma_k(A)(z,v) : C^\infty(s_0(X)) \to C^\infty(s_0(X)), \quad (0.11)$$

respectively. In the context of ellipticity the spaces in (0.10) and (0.11) should be replaced by suitable weighted Sobolev spaces over $s_0(X^\wedge)$ and $s_0(X)$, respectively. In addition the complex variable v is to be restricted to a suitable weight line

$$\Gamma_\beta := \{v \in \mathbb{C} : \mathrm{Re}\, v = \beta\} \quad (0.12)$$

for some real β. We do not discuss such questions here but focus on some new effects on the mapping properties of $\sigma_k(A)(z,\zeta)$ over $X^\wedge$ for $t \to \infty$ in the case $k = 2$. Compared with $k = 1$ which corresponds to the situation of edge symbols on a manifold with smooth edge, from $k = 2$ on there appear new structures that are connected with the interpretation of $X^\wedge$ as a singular manifold with conical exit to ∞. What concerns the behaviour of $\sigma_2(A)(z,\zeta)$ for $t \to 0$ we refer to the article [17]. Therefore, in order to simplify some formulations we consider the case $X^\asymp := \mathbb{R} \times X$ rather than $X^\wedge$ for $X \in \mathfrak{M}_1$.

1. Edge-degenerate operators

1.1. Conical exits for smooth cross section

In the singular analysis there are many reasons to study operators on manifolds with conical exit to infinity. For instance, if X is a smooth compact manifold and

$$A = r^{-\mu} \sum_{j+|\alpha|\le\mu} a_{j,\alpha}(r,y)(-r\partial_r)^j (tD_y)^\alpha \quad (1.1)$$

an edge-degenerate differential operator on an open stretched wedge $\mathbb{R}_+ \times X \times \Omega \ni (r, x, y)$, with coefficients $a_{j,\alpha}(r, y) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, \mathrm{Diff}^{\mu-(j+|\alpha|)}(X))$, the homogeneous principal edge symbol is a family of operators on $X^\wedge = \mathbb{R}_+ \times X$

$$\sigma_1(A)(y, \eta) := r^{-\mu} \sum_{j+|\alpha| \le \mu} a_{j,\alpha}(0, y)(-r\partial_r)^j (r\eta)^\alpha \tag{1.2}$$

parametrised by $(y, \eta) \in T^*\Omega \setminus 0$,

$$\sigma_1(A)(y, \eta) : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge), \tag{1.3}$$

for the Kegel spaces

$$\mathcal{K}^{s,\gamma}(X^\wedge) := \omega(r)\mathcal{H}^{s,\gamma}(X^\wedge) + (1 - \omega(r))H^s_{\mathrm{cone}}(X^\wedge). \tag{1.4}$$

Here $\omega(r)$ is a cut off function on the half-axis (throughout this paper a cut off function means any element of $C_0^\infty(\overline{\mathbb{R}}_+)$ that is equal to 1 in a neighbourhood of 0). The space $\mathcal{H}^{s,\gamma}(X^\wedge)$ for $s \in \mathbb{N}$ is the defined to be the set of all $u(r, x) \in r^{-n/2+\gamma}L^2(X^\wedge)_{drdx}, n = \dim X$, such that $(r\partial_r)^j D^l u \in r^{-n/2+\gamma}L^2(X^\wedge)$ for every $D^l \in \mathrm{Diff}^l(X)$ and $j, l \in \mathbb{N}, j + l \le s$. For $-s \in \mathbb{N}$ we can define $\mathcal{H}^{s,\gamma}(X^\wedge)$ via duality with respect to the scalar product of

$$\mathcal{H}^{0,0}(X^\wedge) = r^{-n/2}L^2(X^\wedge)$$

and then for arbitrary s by complex interpolation (cf. also [11] concerning interpolation properties of spaces on a cone). The space $H^s_{\mathrm{cone}}(X^\wedge)$ for $X = S^n$ (the unit sphere in $\mathbb{R}^{1+n}_{\tilde{x}}$) may be defined by $\{u|_{X^\wedge} : u \in H^s_{\mathrm{loc}}(\mathbb{R} \times X), (1 - \omega(|\tilde{x}|) \in H^s(\mathbb{R}^{1+n}_{\tilde{x}}))\}$ where $\mathbb{R}^{1+n}_{\tilde{x}} \setminus \{0\}$ is identified with $\mathbb{R}_+ \times X$ via polar coordinates $\mathbb{R}^{1+n}_{\tilde{x}} \setminus \{0\} \to \mathbb{R}_+ \times S^n, \tilde{x} \to (r, x)$. It is now easy to pass to arbitrary X by identifying $U^\wedge$ with a conical set in $\mathbb{R}^{1+n}_{\tilde{x}} \setminus \{0\}$ via a diffeomorphism $\chi_1 : U \to U_1$ for a corresponding open $U_1 \subset S^n$, forming $\chi(U^\wedge) := \{\tilde{x} \in \mathbb{R}^{1+n}_{\tilde{x}} \setminus \{0\} : \tilde{x}/|\tilde{x}| \in \chi_1(U)\}$ and then pulling back $H^s(\mathbb{R}^{1+n}_{\tilde{x}})|_{\chi(U^\wedge)}$ to $U^\wedge$. What concerns local coordinates in U without loss of generality we may identify U with the set $\{(1, x) \in \mathbb{R}^{1+n}_{\tilde{x}} \setminus \{0\} : x \in B_1\}$ for the open unit ball B_1 in $\mathbb{R}^n_x$ (in this case we write $\tilde{x} = (x_0, x_1, \ldots, x_n) = (x_0, x)$). Then the transformation $(r, x) \to (r, rx)$ gives us a difffeomorphism $U^\wedge \to \Gamma$ to an open conical subset of $\mathbb{R}^{1+n}_{\tilde{x}} \setminus \{0\}$ where r in the first component is interpreted as x_0. Then $u \in H^s_{\mathrm{cone}}(X^\wedge)$ is equivalent to $u \in H^s_{\mathrm{loc}}(\mathbb{R} \times X)|_{X^\wedge}$ together with the property

$$(1 - \omega(r))\varphi(x)u(r, rx) \in H^s(\mathbb{R}^{1+n}) \tag{1.5}$$

for every coordinate neighbourhood U on X identified with $B_1 \subset \mathbb{R}^n$, any cut-off function ω, and $\varphi \in C_0^\infty(B_1)$. Recall that

$$(\kappa_\lambda u)(r, x) := \lambda^{(n+1)/2}u(\lambda r, x), \lambda \in \mathbb{R}_+, \tag{1.6}$$

defines a strongly continuous group of isomorphisms

$$\kappa_\lambda : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s,\gamma}(X^\wedge) \tag{1.7}$$

for every $s, \gamma \in \mathbb{R}$, and homogeneity of $\sigma_1(A)(y, \eta)$ means

$$\sigma_1(A)(y, \lambda\eta) = \lambda^\mu \kappa_\lambda \sigma_1(A)(y, \eta)\kappa_\lambda^{-1}, \lambda \in \mathbb{R}_+. \tag{1.8}$$

The ellipticity of operators (1.1) is a condition on the pair of principal symbols $(\sigma_0(A), \sigma_1(A))$. For $\sigma_0(A)$ we ask non-vanishing on $T^*(X^\wedge \times \Omega) \setminus 0$ and close to the edge

$$\tilde{\sigma}_0(A)(r, x, y, \rho, \xi, \eta) \neq 0 \text{ for } (\rho, \xi, \eta) \neq 0, \text{ up to } r = 0 \tag{1.9}$$

where

$$\tilde{\sigma}_0(A)(r, x, y, \rho, \xi, \eta) := r^\mu \sigma_0(A)(r, x, y, r^{-1}\rho, \xi, r^{-1}\eta). \tag{1.10}$$

The condition on $\sigma_1(A)$ as an operator-valued symbolic component should be the bijectivity of (1.3) for some prescribed weight γ, and for all $(y, \eta) \in T^*\Omega \setminus 0$. This may happen, indeed, as a special case. The ellipticity in general, already studied in [18] (and after that in different monographs and applications) admits to pose extra edge conditions, provided that a natural topological obstruction vanishes (which is automatically satisfied when (1.3) itself is bijective). In this case, we ask the bijectivity of a corresponding 2×2 family of block matrix operators with $\sigma_1(A)$ in the upper left corner and other entries of finite rank. The latter property has the consequence that the upper left corner is a family of Fredholm operators. It turns out that from the operator algebra aspect there is no essential difference between the case of bijectivity of $\sigma_1(A)$ or its Fredholm property, provided that the above-mentioned topological condition is satisfied. Therefore, we consider the Fredholm property and ask the existence of a parametrix within a pseudo-differential calculus. It is known from the theory developed in [19], or [20] that a certain (y, η)-dependent version of cone algebra over $X^\wedge$ just contains the parametrices. In particular, for $r \to \infty$ we need what is called the pseudo-differential calculus on a manifold with conical exit to ∞ in the corresponding parameter-dependent form. We do not repeat this material here but only note that the task is partly embedded into the analysis of edge symbols. We need analogous constructions later on for the case that X is replaced by a manifold with edge, and then we have to employ the edge symbols anyway.

1.2. Edge symbols and wedge spaces

Edge symbols in our terminology are specific operator-valued symbols over $\Omega \times \mathbb{R}^q$, where the open set $\Omega \subseteq \mathbb{R}^q$ represents local coordinates on the edge Y. In this section we recall a few necessary notions from this context. A Hilbert space H is said to be endowed with a group action $\kappa = \{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$ if κ is a strongly continuous group of isomorphisms $\kappa_\lambda : H \to H$, and $\kappa_\lambda \kappa_{\lambda'} = \kappa_{\lambda\lambda'}$ for every $\lambda, \lambda' \in \mathbb{R}_+$. Similarly, if E is a Fréchet space, written as a projective limit for Hilbert spaces E^j continuously embedded in E^0 for all j where E^0 is endowed with a group action κ that induces a group action $\kappa|_{E^j}$ in E^j for every j, then we say that κ is a group action in E. Now if H and $\tilde{H}$ are Hilbert spaces with group action κ and $\tilde{\kappa}$, respectively, we have spaces of operator-valued symbols

$$S^\mu(\Omega \times \mathbb{R}^q; H, \tilde{H}) \subseteq C^\infty(\Omega \times \mathbb{R}^q, \mathcal{L}(H, \tilde{H})) \tag{1.11}$$

for any open set $\Omega \subseteq \mathbb{R}^p$, and $\mu \in \mathbb{R}$, defined by the system of symbolic estimates

$$\|\tilde{\kappa}^{-1}_{\langle\eta\rangle}\{D^\alpha_y D^\beta_\eta a(y, \eta)\}\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(H,\tilde{H})} \leq c\,\langle\eta\rangle^{\mu-|\beta|} \tag{1.12}$$

for all $\alpha \in \mathbb{N}^p, \beta \in \mathbb{N}^q$ and $(y,\eta) \in K \times \mathbb{R}^q$, for all $K \subset\subset \Omega$, for constants $c = c(\alpha, \beta, K) > 0$. For instance, if an $a_{(\mu)}(y,\eta) \in C^\infty(\Omega \times (\mathbb{R}^q \setminus \{0\}), \mathcal{L}(H, \tilde{H}))$ is ("twisted") homogeneous in the sense

$$a_{(\mu)}(y, \lambda\eta) = \lambda^\mu \tilde{\kappa}_\lambda a_{(\mu)}(y,\eta)\kappa_\lambda^{-1}, \lambda \in \mathbb{R}_+, \tag{1.13}$$

and $\chi \in C^\infty(\mathbb{R}^q)$ an excision function (i.e., $\equiv 0$ close to 0 and $\equiv 1$ off some other neighbourhood of 0), then $\chi(\eta)a_{(\mu)}(y,\eta) \in S^\mu(\Omega \times \mathbb{R}^q; H, \tilde{H})$. Classical symbols are defined in terms of asymptotic expansions into symbols of the form $\chi a_{(\mu-j)}, j \in \mathbb{N}$, for arbitrary $a_{(\mu-j)}$ that are homogeneous of order $\mu - j$. The corresponding subspace of (1.11) is denoted by

$$S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; H, \tilde{H}).$$

If a consideration is valid for classical and general symbols we write as subscript "(cl)". Analogous notation is used for pairs of Fréchet spaces $E, \tilde{E}$ with group action. In this case $a(y,\eta) \in S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; E, \tilde{E})$ means that for every $k \in \mathbb{N}$ there is an $l = l(k) \in \mathbb{N}$ such that $a(y,\eta) \in S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; E^{l(k)}, \tilde{E}^k)$.

Another crucial notion for our exposition are wedge spaces modelled on spaces with group action. For a Hilbert space H with group action κ the corresponding "abstract" wedge space $\mathcal{W}^s(\mathbb{R}^q, H)$ is defined to be the completion of $\mathcal{S}(\mathbb{R}^q, H)$ with respect to the norm

$$\|u\|_{\mathcal{W}^s(\mathbb{R}^q,H)} := \|\langle\eta\rangle^s \kappa^{-1}_{\langle\eta\rangle} \hat{u}(\eta)\|_{L^2(\mathbb{R}^q_\eta, H)}. \tag{1.14}$$

In the case of a Fréchet space $E = \operatorname{proj\,lim}_{j\in\mathbb{N}} E^j$ space with with group action κ we simply set $\mathcal{W}^s(\mathbb{R}^q, E) := \operatorname{proj\,lim}_{j\in\mathbb{N}} \mathcal{W}^s(\mathbb{R}^q, E^j)$. Examples for spaces with group action are

$$\mathcal{K}^s(X^\wedge) \quad \text{with} \quad \kappa_\lambda u(r,x) := \lambda^{(n+1)/2} u(\lambda r, x)$$

for $n = \dim X$. Examples for operator-valued symbols are edge symbols, furnished by edge-degenerate pseudo-differential families

$$p(r,y,\rho,\eta) := \tilde{p}(r,y,r\rho,r\eta) \text{ for } \tilde{p}(r,y,\tilde{\rho},\tilde{\eta}) \in L^\mu_{\mathrm{cl}}(X, \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}}) \tag{1.15}$$

and associated operator families

$$h(r,y,z,\eta) := \tilde{h}(r,y,z,r\eta) \text{ for } \tilde{h}(r,y,z,\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, M^\mu_{\mathcal{O}}(X; \mathbb{R}^q_{\tilde{\eta}})). \tag{1.16}$$

Here $L^\mu_{\mathrm{cl}}(X; \mathbb{R}^l)$ is the space of classical parameter-dependent pseudo-differential operators of order μ (with parameters in $\mathbb{R}^l$) over the C^∞ manifold X. Moreover, $M^\mu_{\mathcal{O}}(X; \mathbb{R}^q)$ is the subspace of all $h(z,\eta) \in \mathcal{A}(\mathbb{C}, L^\mu_{\mathrm{cl}}(X; \mathbb{R}^q))$ such that $h(\beta + i\rho, \eta) \in L^\mu_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\rho,\eta})$ for every $\beta \in \mathbb{R}$, uniformly in compact β-intervals, where $\mathcal{A}(U, E)$ for an open set $U \subseteq \mathbb{C}$ and a Fréchet space E means the space of holomorphic functions on U with values in E. Given any $f(r,z) \in C^\infty(\mathbb{R}_+, L^\mu_{\mathrm{cl}}(X; \Gamma_{1/2-\delta}))$ for $\Gamma_\beta := \{z \in \mathbb{C} : \operatorname{Re} z = \beta\}$ for any real β where the parameter in $\Gamma_{1/2-\delta} \ni z$ is interpreted as $\operatorname{Im} z$, we can form the weighted Mellin pseudo-differential operator

$$\mathrm{op}^\delta_M(f)u(r) := \int\int_0^\infty (r/r')^{1/2-\delta+i\rho} f(r, 1/2 - \delta + i\rho) u(r') dr'/r' \,đ\rho \tag{1.17}$$

for $đ\rho := (2\pi)^{-1}d\rho$, first on $C_0^\infty(\mathbb{R}_+, C^\infty(X))$, and later on extended to larger distribution spaces.

In the following we assume that the operator functions p and h in (1.15) and (1.16), respectively, are connected via a Mellin quantisation, namely, the relation

$$\mathrm{Op}_r(p)(y,\eta) = \mathrm{op}_M^{\gamma-n/2}(h)(y,\eta) \bmod C^\infty(\Omega, L^{-\infty}(X^\wedge;\mathbb{R}^q)), \tag{1.18}$$

as operator families $C_0^\infty(X^\wedge) \to C^\infty(X^\wedge)$.
For any $\varphi \in C^\infty(\mathbb{R}_+)$ we set

$$\varphi_\eta(r) := \varphi(r[\eta]) \text{ for any fixed } [\eta] \in C^\infty(\mathbb{R}^q), [\eta] > 0, [\eta] = |\eta| \text{ when } |\eta| \geq 1. \tag{1.19}$$

Moreover, for $\varphi, \varphi' \in C^\infty(\mathbb{R}_+)$ we write

$$\varphi \prec \varphi' \quad \text{if } \varphi \equiv 1 \quad \text{on } \operatorname{supp}\varphi. \tag{1.20}$$

Now let $\omega, \omega', \omega''$ be cut-off functions with $\omega'' \prec \omega \prec \omega'$, and form the operator families

$$a_M(y,\eta) := r^{-\mu}\omega_\eta \mathrm{op}_M^{\gamma-n/2}(h)(y,\eta)\omega'_\eta, \tag{1.21}$$

and

$$a_\psi(y,\eta) := r^{-\mu}\chi_\eta \mathrm{Op}_r(p)(y,\eta)\chi'_\eta \text{ for } \chi := 1-\omega, \chi' := 1-\omega''. \tag{1.22}$$

Let ϵ and ϵ' be cut-off functions. Then we have

$$\epsilon\{a_M(y,\eta) + a_\psi(y,\eta)\}\epsilon' \in S^\mu(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)) \tag{1.23}$$

for every $s \in \mathbb{R}$. Other examples of operator-valued symbols are families

$$\sigma \mathrm{Op}_r(p_{\mathrm{int}})(y,\eta)\sigma' \in S^\mu(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\infty}(X^\wedge)) \tag{1.24}$$

for any $\sigma, \sigma' \in C_0^\infty(\mathbb{R}_+), s, \gamma \in \mathbb{R}$, and $p_{\mathrm{int}}(r,y,\rho,\eta) \in C^\infty(\mathbb{R}_+ \times \Omega, L^\mu_{\mathrm{cl}}(X, \mathbb{R}^q))$. Both (1.23) and (1.24) belong to the amplitude functions of the edge algebra. Other ingredients are what we call smoothing Mellin and Green symbols. We content ourselves here with a minimal asymptotic information of the edge calculus that could be encoded by the latter symbols. For the definition of Green operators we set $\mathcal{S}^\gamma(X^\wedge) := \operatorname{proj\,lim}_{N\in\mathbb{N}}\langle r\rangle^{-N}\mathcal{K}^{N,\gamma}(X^\wedge)$. By a Green symbol of the edge calculus of order ν associated with the weight data $(\gamma,\delta) \in \mathbb{R}^2$ we understand a symbol

$$g(y,\eta) \in \bigcap_{s\in\mathbb{R}} S^\nu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{S}^{\delta+\varepsilon}(X^\wedge)) \tag{1.25}$$

for some $\varepsilon = \varepsilon(g) > 0$, such that

$$g^*(y,\eta) \in \bigcap_{s\in\mathbb{R}} S^\nu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,-\delta}(X^\wedge), \mathcal{S}^{-\gamma+\varepsilon}(X^\wedge)). \tag{1.26}$$

The " $*$ " in (1.26) means the (y,η)-wise formal adjoint with respect to the non-degenerate sesquilinear pairings

$$\mathcal{K}^{s,\gamma}(X^\wedge) \times \mathcal{K}^{-s,-\gamma}(X^\wedge) \to \mathbb{C}$$

induced by $(.\,,.)_{\mathcal{K}^{0,0}(X^\wedge)} : C_0^\infty(X^\wedge) \times C_0^\infty(X^\wedge) \to \mathbb{C}$.

In order to define smoothing Mellin symbols of the edge calculus we first consider an $f(y,z) \in C^\infty(\Omega_y, L^{-\infty}(X;\Gamma_{(n+1)/2-\gamma}))$. Then for any two cut-off functions ω, ω' the operator family

$$m(y,\eta) := \omega_\eta r^{-\mu} \mathrm{op}_M^{\gamma-n/2}(f)(y)\omega'_\eta \tag{1.27}$$

belongs to $S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{S}^{\gamma-\mu}(X^\wedge))$ for every $s \in \mathbb{R}$. For our calculus we impose a minimal "asymptotic" information on f, namely $f \in C^\infty(\Omega_y, M_\gamma^{-\infty}(X))$ where $M_\gamma^{-\infty}(X) := \bigcup_{\varepsilon>0} M_\gamma^{-\infty}(X)_\varepsilon$, and

$$\begin{aligned} M_\gamma^{-\infty}(X)_\varepsilon := \{ & f(z) \in \mathcal{A}((n+1)/2-\gamma-\varepsilon < \mathrm{Re}\, z < (n+1)/2-\gamma+\varepsilon), f(\beta+i\rho) \\ & \in L^{-\infty}(X;\Gamma_\beta) \text{ for every } (n+1)/2-\gamma-\varepsilon < \beta < (n+1)/2-\gamma+\varepsilon, \\ & \text{uniformly in compact subintervals}\}. \end{aligned} \tag{1.28}$$

We are now in the position to formulate the spaces of edge amplitude functions with respect to the weight data $\mathbf{g} := (\gamma, \gamma-\mu)$ for a weight $\gamma \in \mathbb{R}$ and a weight shift $\mu \in \mathbb{R}$. The latter comes from the order of operators in the context of ellipticity. First let $R^\mu_{M+G}(\Omega \times \mathbb{R}^q, \mathbf{g})$ denote the set of all $m(y,\eta)+g(y,\eta)$ where $m(y,\eta)$ is as (1.27) for an arbitrary $f(y,z) \in C^\infty(\Omega_y, M_\gamma^{-\infty}(X))$, and an arbitrary Green symbol of order μ associated with the weight data $(\gamma, \gamma-\mu)$. Moreover, let $R^\mu_G(\Omega \times \mathbb{R}^q, \mathbf{g})$ denote the subspace of Green symbols (i.e., for vanishing m).

Definition 1.1. By $R^\mu(\Omega \times \mathbb{R}^q, \mathbf{g})$ for $\mathbf{g} := (\gamma, \gamma-\mu)$ we denote the space of all operator functions of the form

$$a(y,\eta) := a_{\mathrm{hol}}(y,\eta) + a_{\mathrm{int}}(y,\eta) + m(y,\eta) + g(y,\eta) \tag{1.29}$$

where

$$a_{\mathrm{hol}}(y,\eta) = \epsilon\{a_M(y,\eta) + a_\psi(y,\eta)\}\epsilon' \tag{1.30}$$

for some cut-off functions ϵ, ϵ', cf. (1.23),

$$a_{\mathrm{int}}(y,\eta) = \sigma \mathrm{Op}_r(p_{\mathrm{int}})(y,\eta)\sigma' \tag{1.31}$$

for some $\sigma, \sigma' \in C_0^\infty(\mathbb{R}_+)$, cf. (1.24), and $m(y,\eta)+g(y,\eta) \in R^\mu_{M+G}(\Omega \times \mathbb{R}^q, \mathbf{g})$.

Let us now briefly recall the principal symbolic structure of the operators $A = \mathrm{Op}_y(a), a \in R^\mu(\Omega \times \mathbb{R}^q, \mathbf{g})$. First observe that $A \in L^\mu_{\mathrm{cl}}(X^\wedge \times \Omega)$; thus there is the homogeneous principal symbol as a function over $T^*X^\wedge \times \Omega \setminus 0$, namely, $\sigma_0(A)(r,x,y,\rho,\xi,\eta)$. Here x refers to local coordinates on X, and $\sigma_0(A)$ is (positively) homogeneous of order μ in $(\rho,\xi,\eta) \neq 0$. We also observe the reduced symbol $\tilde{\sigma}_0(A)(r,x,y,\rho,\xi,\eta) := r^\mu \sigma_0(A)(r,x,y,r^{-1}\rho,\xi,r^{-1}\eta)$ which is smooth up to $r=0$. In addition we have the so-called homogeneous principal edge symbol which is a family of operators

$$\sigma_1(A)(y,\eta) : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \tag{1.32}$$

defined for $(y,\eta) \in T^*\Omega \setminus 0$ and homogeneous in the sense

$$\sigma_1(A)(y,\lambda\eta) = \lambda^\mu \kappa_\lambda \sigma_1(A)(y,\eta)\kappa_\lambda^{-1} \tag{1.33}$$

for all $\lambda \in \mathbb{R}_+$. For $A = \mathrm{Op}_y(a), a \in R^\mu(\Omega \times \mathbb{R}^q, \mathbf{g})$, we set $\sigma_1(A)(y,\eta) := \sigma_1(a)(y,\eta)$ where

$$\sigma_1(a)(y,\eta) := \sigma_1(a_{\mathrm{hol}})(y,\eta) + \sigma_1(m)(y,\eta) + \sigma_1(g)(y,\eta).$$

The summands are as follows. For any $\varphi \in C^\infty(\mathbb{R}_+)$ we set $\varphi_{|\eta|}(r) := \varphi(r|\eta|)$. Then

$$\sigma_1(a_{\mathrm{hol}})(y,\eta) := r^{-\mu}\{\omega_{|\eta|}\mathrm{op}_M^{\gamma-n/2}(h_0)(y,\eta)\omega'_{|\eta|} + \chi_{|\eta|}\mathrm{Op}_r(p_0)(y,\eta)\chi'_{|\eta|}\} \quad (1.34)$$

for

$$h_0(r,y,z,\eta) := \tilde{h}(0,y,z,r\eta),\ p_0(r,y,\rho,\eta) := \tilde{p}(0,y,r\rho,r\eta).$$

Moreover,

$$\sigma_1(m)(y,\eta) := r^{-\mu}\omega_{|\eta|}\mathrm{op}_M^{\gamma-n/2}(f)(y)\omega'_{|\eta|},\ \sigma_1(g)(y,\eta) := g_{(\mu)}(y,\eta) \quad (1.35)$$

with $g_{(\mu)}(y,\eta)$ being the (twisted) homogeneous principal part of the classical symbol g.

Summing up, for edge amplitude functions $a(y,\eta) \in R^\mu(\Omega \times \mathbb{R}^q, \mathbf{g}), \mathbf{g} := (\gamma, \gamma-\mu)$, we formulated the principal symbolic hierarchy

$$\sigma(a) := (\sigma_0(a), \sigma_1(a)). \quad (1.36)$$

Setting for the moment $\sigma^0(a) := \sigma(a)$ we define the subspace $R^{\mu-1}(\Omega \times \mathbb{R}^q, \mathbf{g})$ of all $a \in R^\mu(\Omega \times \mathbb{R}^q, \mathbf{g})$ such that $\sigma^0(a) = 0$. In $R^{\mu-1}(\Omega \times \mathbb{R}^q, \mathbf{g})$ we have again a two-component principal symbolic hierarchy $\sigma^{-1}(\cdot)$. Inductively we define

$$R^{\mu-j-1}(\Omega \times \mathbb{R}^q, \mathbf{g}) \ \text{ for } \mathbf{g} = (\gamma, \gamma-\mu), j \in \mathbb{N},$$

by $\{a \in R^{\mu-j}(\Omega \times \mathbb{R}^q, \mathbf{g}) : \sigma^j(a) = 0\}$.

Remark 1.2. There is an equivalent way of defining the symbol spaces of Definition 1.1 which employs a result of [7]. The space $R^m(\Omega \times \mathbb{R}^q, \mathbf{g})$ for $\mathbf{g} := (\gamma, \gamma-\mu), m = \mu - j, j \in \mathbb{N}$ is equal to the space of all operator functions of the form

$$a(y,\eta) := a_{\mathrm{hol}}(y,\eta) + a_{\mathrm{int}}(y,\eta) + m(y,\eta) + g(y,\eta) \quad (1.37)$$

where

$$a_{\mathrm{hol}}(y,\eta) := r^{-m}\omega\mathrm{op}_M^{\gamma-n/2}(h)(y,\eta)\omega', \quad (1.38)$$

for arbitrary cut-off functions $\omega(r), \omega'(r)$, and $h(r,y,z,\eta) = \tilde{h}(r,y,z,r\eta)$ for an $\tilde{h}(r,y,z,\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, M_{\mathcal{O}}^m(X;\mathbb{R}^q_{\tilde{\eta}}))$, moreover, $a_{\mathrm{int}}(y,\eta) = \sigma\mathrm{Op}_r(p_{\mathrm{int}})(y,\eta)\sigma'$ for any $\sigma, \sigma' \in C_0^\infty(\mathbb{R}_+), s, \gamma \in \mathbb{R}$, and $p_{\mathrm{int}}(r,y,\rho,\eta) \in C^\infty(\mathbb{R}_+ \times \Omega, L^m_{\mathrm{cl}}(X, \mathbb{R}^q))$, and $m(y,\eta) + g(y,\eta) \in R^m_{M+G}(\Omega \times \mathbb{R}^q, \mathbf{g})$.

1.3. The parameter-dependent edge calculus

Let B be a compact manifold with smooth edge Y. Recall from the general definitions of the Introduction we have $B \setminus Y \in \mathfrak{M}_0$, and every point on Y has a neighbourhood V with the structure of a trivial cone bundle over $V \cap Y$ with fibre X^Δ. This admits local variables in $V \setminus Y$ of the form $(r, x, y) \in \mathbb{R}_+ \times X \times \Omega$ for some open set $\Omega \subseteq \mathbb{R}^q$. We often take $y \in \mathbb{R}^q$ as local coordinates on $V \cap Y$. Since we assume B to be compact there are finitely many such neighbourhoods $V_1, \dots, V_N$ such that $V_1 \cap Y, \dots, V_N \cap Y$ form an open covering of Y. Let $\varphi_1, \dots, \varphi_N$ be a subordinate partition of unity, and $\psi_1, \dots, \psi_N$ be another system of functions $\psi_j \in C_0^\infty(V_j \cap Y)$ such that $\varphi_j \prec \psi_j$ for all $j = 1, \dots, N$, cf. the notation (1.20). The amplitude functions $a(y, \eta)$ of Definition 1.1 also make sense in the variant $a(y, \eta, \lambda)$ where $\eta \in \mathbb{R}^q$ is formally replaced by $(\eta, \lambda) \in \mathbb{R}^{q+l}$ for some $l \in \mathbb{N}$. In other words we have the spaces $R^m(\Omega \times \mathbb{R}^{q+l}, \mathbf{g})$ for $\mathbf{g} := (\gamma, \gamma - \mu), \mu - m \in \mathbb{N}$, and the corresponding subspaces R^m_{M+G} and R^m_G. The above-mentioned symbols now depend also on the parameters λ. Based on a system of charts on B and a subordinate partition of unity we have weighted spaces $H^{s,\gamma}(B)$ consisting of all $u \in H^s_{\mathrm{loc}}(B \setminus Y)$ that are locally near Y in the splitting of variables $(r, x, y) \in X^\wedge \times \mathbb{R}^q$ modelled on $\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge))$. (In the transition maps close to the edge we assume, for simplicity, independence of r for small r.) Let $L^{-\infty}(B, \mathbf{g})$ defined to be the set of all $C \in \bigcap_{s \in \mathbb{R}} \mathcal{L}(H^{s,\gamma}(B), H^{\infty,\gamma-\mu}(B))$ that induce continuous operators

$$C : H^{s,\gamma}(B) \to H^{\infty,\gamma-\mu+\varepsilon}(B), \; C^* : H^{s,-\gamma+\mu}(B) \to H^{\infty,-\gamma+\varepsilon}(B)$$

for all $s \in \mathbb{R}$ and some $\varepsilon = \varepsilon(C) > 0$. The " $*$ " in the latter relation means the formal adjoint with respect to the non-degenerate sesquilinear pairings

$$H^{s,\gamma}(B) \times H^{-s,-\gamma}(B) \to \mathbb{C}$$

induced by $(.,.)_{H^{0,0}(B)} : C_0^\infty(B \setminus Y) \times C_0^\infty(B \setminus Y) \to \mathbb{C}$. Moreover, let $L^{-\infty}(B, \mathbf{g}; \mathbb{R}^l) := \mathcal{S}(\mathbb{R}^l, L^{-\infty}(B, \mathbf{g}))$.

Definition 1.3. The space $L^m(B, \mathbf{g}; \mathbb{R}^l)$ for $\mathbf{g} := (\gamma, \gamma - \mu), \mu - m \in \mathbb{N}$, is defined to be the set of all operator families

$$A(\lambda) = \sum_{j=1}^{N} \phi_j \{\chi_{j,*}^{-1} \omega \mathrm{Op}_y(a_j)(\lambda) \omega'\} \psi_j + A_{\mathrm{int}}(\lambda) + C(\lambda) \tag{1.39}$$

for arbitrary $a_j(y, \eta, \lambda) \in R^m(\mathbb{R}^q \times \mathbb{R}^{q+l}, \mathbf{g})$, singular charts $\chi_j : V_j \setminus Y \to X^\wedge \times \mathbb{R}^q$, cut-off functions ω, ω', moreover, $A_{\mathrm{int}}(\lambda) \in L^m_{\mathrm{cl}}(B \setminus Y; \mathbb{R}^l)$ the distributional kernel of which is compactly supported in $(B \setminus Y) \times (B \setminus Y)$, and $C(\lambda) \in L^{-\infty}(B, \mathbf{g}; \mathbb{R}^l)$.

Other elements of the edge calculus without parameters are valid in analogous form also in the parameter-dependent case. In particular, we have a corresponding parameter-dependent principal symbolic structure. From now on we more or less freely use the tools of the parameter-dependent edge calculus, in particular continuity properties in edge spaces, and the behaviour of edge operators under compositions and formal adjoints. Details may be found in a number of systematic monographs, especially, in [5], [20], [11], or [8].

Ellipticity of elements $A \in L^m(B, \mathbf{g}; \mathbb{R}^l)$ for $\mathbf{g} := (\gamma, \gamma - \mu), \mu - m \in \mathbb{N}$, refers to the case $m = \mu$.

1.4. Norm growth estimates

In (abstract) edge spaces $\mathcal{W}^s(\mathbb{R}^q, H)$ for some Hilbert space with group action $\kappa = \{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$ we can define parameter-dependent norms. Let

$$\mathcal{W}^s_\lambda(\mathbb{R}^q, H)$$

denote the space $\mathcal{W}^s(\mathbb{R}^q, H)$ equipped with the family of norms

$$\|u\|_{\mathcal{W}^s_\lambda(\mathbb{R}^q,H)} := \Big\{ \int \langle\eta\rangle^{2s} \|\kappa^{-1}_{\langle\eta,\lambda\rangle} \hat{u}(\eta)\|^2_H d\eta \Big\}^{1/2}. \tag{1.40}$$

Clearly (1.40) is equivalent to $\|u\|_{\mathcal{W}^s(\mathbb{R}^q,H)} = \big\{ \int \langle\eta\rangle^{2s} \|\kappa^{-1}_{\langle\eta\rangle} \hat{u}(\eta)\|^2_H d\eta \big\}^{1/2}$ for every fixed λ.

Proposition 1.4. *Let $a(\eta, \lambda) \in S^\mu(\mathbb{R}^q \times (\mathbb{R}^l \setminus \{0\}); H, \tilde{H})$. Then $\mathrm{Op}_y(a)(\lambda)$, regarded as a family of continuous operators*

$$\mathrm{Op}_y(a)(\lambda) : \mathcal{W}^s_\lambda(\mathbb{R}^q, H) \to \mathcal{W}^{s-\nu}_\lambda(\mathbb{R}^q, \tilde{H}), \tag{1.41}$$

$s \in \mathbb{R}$, for any $\nu \geq \mu$ satisfies the estimate

$$\|\mathrm{Op}_y(a)(\lambda)\|_{\mathcal{L}(\mathcal{W}^s_\lambda(\mathbb{R}^q,H),\mathcal{W}^{s-\nu}_\lambda(\mathbb{R}^q,\tilde{H}))} \leq c\langle\lambda\rangle^{\max\{\mu,\mu-\nu\}} \tag{1.42}$$

for some $c > 0$, independent of λ.

Proof. We have

$$\|\mathrm{Op}_y(a)(\lambda)u\|^2_{\mathcal{W}^{s-\nu}_\lambda(\mathbb{R}^q,\tilde{H})} = \int \langle\eta\rangle^{2(s-\nu)} \|\tilde{\kappa}^{-1}_{\langle\eta,\lambda\rangle} a(\eta,\lambda)\hat{u}(\eta)\|^2_{\tilde{H}} d\eta. \tag{1.43}$$

Using the symbolic estimate

$$\|\tilde{\kappa}^{-1}_{\langle\eta,\lambda\rangle} a(\eta,\lambda)\kappa_{\langle\eta,\lambda\rangle}\|_{\mathcal{L}(H,\tilde{H})} \leq c\langle\eta,\lambda\rangle^\mu$$

it follows that (1.43) can be estimated by

$$\begin{aligned} &\int \langle\eta\rangle^{2(s-\nu)} \|\tilde{\kappa}^{-1}_{\langle\eta,\lambda\rangle} a(\eta,\lambda)\kappa_{\langle\eta,\lambda\rangle}\|^2_{\mathcal{L}(H,\tilde{H})} \|\kappa^{-1}_{\langle\eta,\lambda\rangle}\hat{u}(\eta)\|^2_H d\eta \\ &\leq c \sup_{\eta\in\mathbb{R}^q} \langle\eta\rangle^{-2\nu} \langle\eta,\lambda\rangle^{2\mu} \int \langle\eta\rangle^{2s} \|\kappa^{-1}_{\langle\eta,\lambda\rangle}\hat{u}(\eta)\|^2_H d\eta \\ &\leq c\langle\lambda\rangle^{\max\{\mu,\mu-\nu\}} \|u\|^2_{\mathcal{W}^s_\lambda(\mathbb{R}^q,H)}. \end{aligned} \tag{1.44}$$

This yields the asserted continuity together with the estimate (1.42). □

Proposition 1.5. *The operator $\mathcal{M}_\varphi$ of multiplication by a function $\varphi \in \mathcal{S}(\mathbb{R}^q)$ induces continuous operators $\mathcal{M}_\varphi : \mathcal{W}^s_\lambda(\mathbb{R}^q, H) \to \mathcal{W}^s_\lambda(\mathbb{R}^q, H)$ for every $s \in \mathbb{R}$, and we have $\|\mathcal{M}_\varphi\|_{\mathcal{L}(\mathcal{W}^s_\lambda(\mathbb{R}^q,H))} \leq c(\varphi)$ for some λ-independent constant $c(\varphi)$ that tends to zero as $\varphi \to 0$ in $\mathcal{S}(\mathbb{R}^q)$.*

Proof. We apply Peetre's inequality

$$\langle\eta\rangle^s \le \langle\eta-\xi\rangle^{|s|}\langle\xi\rangle^s, \tag{1.45}$$

and the estimate

$$\|\kappa^{-1}_{\langle\eta,\lambda\rangle/\langle\xi,\lambda\rangle}\|_{\mathcal{L}(H)} \le c\langle\eta-\xi\rangle^M \tag{1.46}$$

for some $M\in\mathbb{R}$, depending on the group κ. First we have

$$\|\mathcal{M}_\varphi\|^2_{\mathcal{W}^s_\lambda(\mathbb{R}^q,H)} = \int\langle\eta\rangle^{2s}\|\kappa^{-1}_{\langle\eta,\lambda\rangle}F(\varphi u)(\eta)\|^2_H d\eta.$$

Setting

$$m(\eta,\lambda) := \|\langle\eta\rangle^s\kappa^{-1}_{\langle\eta,\lambda\rangle}\int(F\varphi)(\eta-\xi)Fu(\xi)đ\xi\|_H$$

it follows that

$$\|\mathcal{M}_\varphi\|_{\mathcal{W}^s_\lambda(\mathbb{R}^q,H)} = \|m(\eta,\lambda)\|_{L^2(\mathbb{R}^q)}. \tag{1.47}$$

From (1.45) and (1.46) we obtain

$$\begin{aligned} m(\eta,\lambda) &= \|\int\langle\eta\rangle^s\kappa^{-1}_{\langle\eta,\lambda\rangle}\hat\varphi(\eta-\xi)\hat u(\xi)đ\xi\|_H \\ &= \|\int\langle\eta\rangle^s\kappa^{-1}_{\langle\eta,\lambda\rangle/\langle\xi,\lambda\rangle}\hat\varphi(\eta-\xi)\kappa^{-1}_{\langle\xi,\lambda\rangle}\hat u(\xi)đ\xi\|_H \\ &\le \int\langle\eta\rangle^s\|\kappa^{-1}_{\langle\eta,\lambda\rangle/\langle\xi,\lambda\rangle}\hat\varphi(\eta-\xi)\kappa^{-1}_{\langle\xi,\lambda\rangle}\hat u(\xi)\|_H đ\xi \\ &\le c\int\langle\eta-\xi\rangle^{M+|s|-N}\langle\eta-\xi\rangle^N|\hat\varphi(\eta-\xi)|\,\|\langle\xi\rangle^s\kappa^{-1}_{\langle\xi,\lambda\rangle}\hat u(\xi)\|_H đ\xi \\ &\le c\,c_\varphi\int\langle\eta-\xi\rangle^{M+|s|-N}\|\langle\xi\rangle^s\kappa^{-1}_{\langle\xi,\lambda\rangle}\hat u(\xi)\|_H đ\xi \end{aligned} \tag{1.48}$$

for $c_\varphi = \sup_{\xi\in\mathbb{R}^q}\langle\xi\rangle^N|\hat\varphi(\xi)| < \infty$. Now we choose N large enough, set $h(\xi) := \langle\eta-\xi\rangle^{M+|s|-N}$, $g(\xi) := \|\langle\xi\rangle^s\kappa^{-1}_{\langle\xi,\lambda\rangle}\hat u(\xi)\|_H$, and apply Young's inequality. Then

$$\begin{aligned} \|m(\eta,\lambda)\|_{L^2(\mathbb{R}^q)} &\le c\,c_\varphi\|h*g\|_{L^2(\mathbb{R}^q)} \\ &\le c\,c_\varphi\|h\|_{L^1(\mathbb{R}^q)}\|g\|_{L^2(\mathbb{R}^q)} = c(\varphi)\|u\|_{\mathcal{W}^s_\lambda(\mathbb{R}^q,H)} \end{aligned} \tag{1.49}$$

for $c(\varphi) = c\,c_\varphi\|h\|_{L^1(\mathbb{R}^q)}$. □

Corollary 1.6. *Let $a(y,\eta,\lambda)\in S^\mu(\mathbb{R}^q_y\times\mathbb{R}^q_\eta\times(\mathbb{R}^l\setminus\{0\});H,\tilde H)$ be a symbol that is independent of y for large $|y|$. Then* $\mathrm{Op}(a)(\lambda)$ *induces a family of continuous operators* (1.41), *and we have the estimate* (1.42) *for a constant* $c>0$, *independent of* λ.

For purposes below we formulate a parameter-dependent variant of a version of the Calderón-Vaillancourt Theorem for operators with operator-valued symbols in the set-up with group actions. To this end we fix Hilbert spaces H and $\tilde H$ with group actions $\kappa=\{\kappa_\delta\}_{\delta\in\mathbb{R}_+}$ and $\tilde\kappa=\{\tilde\kappa_\delta\}_{\delta\in\mathbb{R}_+}$, respectively. Recall that there are constants $\tilde c$ and $\tilde M$ such that $\|\tilde\kappa_\delta\|_{\mathcal{L}(\tilde H)} \le \tilde c\max(\delta,\delta^{-1})^{\tilde M}$ for all $\delta\in\mathbb{R}_+$.

Theorem 1.7. *Let* $a(y,\eta,\lambda) \in C^\infty(\mathbb{R}^q_y \times \mathbb{R}^q_\eta \times (\mathbb{R}^l_\lambda \setminus \{0\}), \mathcal{L}(H,\tilde{H}))$, *and assume*

$$\pi(a)(\lambda) := \sup_{(y,\eta)\in\mathbb{R}^{2q},\alpha\le A,\beta\le B} \|\tilde{\kappa}^{-1}_{\langle\eta,\lambda\rangle}\{D^\alpha_y D^\beta_\eta a(y,\eta,\lambda)\}\kappa_{\langle\eta,\lambda\rangle}\|_{\mathcal{L}(H,\tilde{H})} < \infty \tag{1.50}$$

for $A := (\tilde{M}+1,\dots,\tilde{M}+1), B := (1,\dots,1)$. *Then* $\mathrm{Op}(a)(\lambda)$ *induces continuous operators*

$$\mathrm{Op}(a)(\lambda) : \mathcal{W}^0_\lambda(\mathbb{R}^q,H) \to \mathcal{W}^0_\lambda(\mathbb{R}^q,\tilde{H}) \tag{1.51}$$

and we have $\|\mathrm{Op}(a)(\lambda)\|_{\mathcal{L}(\mathcal{W}^0_\lambda(\mathbb{R}^q,H),\mathcal{W}^0_\lambda(\mathbb{R}^q,\tilde{H}))} \le c\,\pi(a)(\lambda)$ *for all* $\lambda \in \mathbb{R}^l \setminus 0$ *and a constant* $c > 0$ *independent of* a *and* λ.

Theorem 1.7 can be proved in a similar manner as a corresponding result of the article of Seiler [23] who extended a proof of the Calderón-Vaillancourt theorem of Hwang [10] from the scalar case to the case of operator-valued symbols with group action. An inspection of the details shows that we may admit a dependence on parameters as assumed in the theorem.

Corollary 1.8. *Let* $a(y,\eta,\lambda) \in C^\infty(\mathbb{R}^q_y \times \mathbb{R}^q_\eta \times (\mathbb{R}^l_\lambda \setminus \{0\}), \mathcal{L}(H,\tilde{H}))$, *and let*

$$\|\tilde{\kappa}^{-1}_{\langle\eta,\lambda\rangle}\{D^\alpha_y D^\beta_\eta a(y,\eta,\lambda)\}\kappa_{\langle\eta,\lambda\rangle}\|_{\mathcal{L}(H,\tilde{H})} \le c_{\alpha,\beta}\,\langle\eta,\lambda\rangle^{-j} \tag{1.52}$$

for all $(y,\eta,\lambda) \in \mathbb{R}^q_y \times \mathbb{R}^q_\eta \times (\mathbb{R}^l_\lambda \setminus \{0\}), |\lambda| \ge \varepsilon > 0$, *for all* α,β *as in Theorem* 1.7 *for constants* $c_{\alpha,\beta} = c_{\alpha,\beta}(\varepsilon) > 0$. *Then*

$$\|\mathrm{Op}(a)(\lambda)\|_{\mathcal{L}(\mathcal{W}^0_\lambda(\mathbb{R}^q,H),\mathcal{W}^0_\lambda(\mathbb{R}^q,\tilde{H}))} \le c\,\langle\lambda\rangle^{-j} \tag{1.53}$$

for all $\lambda \in \mathbb{R}^l \setminus \{0\}, |\lambda| \ge \varepsilon$ *for a constant* $c = c(\varepsilon) > 0$.

Theorem 1.9. *Let* $a(y,\eta,\lambda) \in R^m(\mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^l, \mathbf{g})$ *for* $\mathbf{g} = (\gamma,\gamma-\mu), \mu - m \in \mathbb{N}$, *and* $m \le \nu$, *and assume that* a *is independent of* y *for large* $|y|$. *Then we have*

$$\|\mathrm{Op}(a)(\lambda)\|_{\mathcal{L}(\mathcal{W}^s_\lambda(\mathbb{R}^q,\mathcal{K}^{s,\gamma}(X^\wedge)),\mathcal{W}^{s-\nu}_\lambda(\mathbb{R}^q,\mathcal{K}^{s-\nu,\gamma-\mu}(X^\wedge)))} \le c\langle\lambda\rangle^{\max\{m,m-\nu\}} \tag{1.54}$$

for all $\lambda \in \mathbb{R}^l$, *for a constant* $c > 0$. *In the case* $\mu = \nu = 0$ *it follows that*

$$\|\mathrm{Op}(a)(\lambda)\|_{\mathcal{L}(\mathcal{W}^s_\lambda(\mathbb{R}^q,\mathcal{K}^{s,\gamma}(X^\wedge)),\mathcal{W}^s_\lambda(\mathbb{R}^q,\mathcal{K}^{s,\gamma}(X^\wedge)))} \le c\langle\lambda\rangle^m, \tag{1.55}$$

for all $\lambda \in \mathbb{R}^l$.

A result about the growth of operator norms between spaces with norms without parameters can also be deduced.

Theorem 1.10. *Let* $a(y,\eta,\lambda) \in R^m(\mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^l, \mathbf{g})$ *for* $\mathbf{g} = (\gamma,\gamma-\mu), \mu - m \in \mathbb{N}$, *and* $m \le \nu$, *and assume that* a *is independent of* y *for large* $|y|$. *Then we have*

$$\|\mathrm{Op}(a)(\lambda)\|_{\mathcal{L}(\mathcal{W}^s(\mathbb{R}^q,\mathcal{K}^{s,\gamma}(X^\wedge)),\mathcal{W}^{s-\nu}(\mathbb{R}^q,\mathcal{K}^{s-\nu,\gamma-\mu}(X^\wedge)))} \le c\langle\lambda\rangle^{\max\{m,m-\nu\}+M} \tag{1.56}$$

for all $\lambda \in \mathbb{R}^l$, *for a certain constants* $c > 0$, *and* $M = M(s,\nu) > 0$. *In the case* $\mu = \nu = 0$ *and* $s = \gamma = 0$ *we have*

$$\|\mathrm{Op}(a)(\lambda)\|_{\mathcal{L}(\mathcal{W}^0(\mathbb{R}^q,\mathcal{K}^{0,0}(X^\wedge)),\mathcal{W}^0(\mathbb{R}^q,\mathcal{K}^{0,0}(X^\wedge)))} \le c\langle\lambda\rangle^m, \tag{1.57}$$

for all $\lambda \in \mathbb{R}^l$.

After the above considerations the proof is straightforward and left to the reader. Another useful norm growth result for parameter-dependent edge operators is the following theorem.

Theorem 1.11. *Let* $\mathbf{g} := (\gamma, \gamma - \mu)$*; for every* $s', s'' \in \mathbb{R}$ *and every* $N \in \mathbb{N}$ *there exists an* m *with* $\mu - m \in \mathbb{N}$ *such that for every* $A(\lambda) \in L^m(B, \mathbf{g}; \mathbb{R}^l)$

$$\|A(\lambda)\|_{\mathcal{L}(H^{s',\gamma}(B), H^{s'',\gamma-\mu}(B))} \leq c\,\langle\lambda\rangle^{-N} \tag{1.58}$$

for all $\lambda \in \mathbb{R}^l$ *and some constant* $c > 0$*. Moreover, for every* $A(\lambda) \in L^m(B, \mathbf{g}; \mathbb{R}^l)$ *and* $s', s'' \in \mathbb{R}$ *with* $s' - m \geq s''$ *there exists an* $L \in \mathbb{R}$ *such that*

$$\|A(\lambda)\|_{\mathcal{L}(H^{s',\gamma}(B), H^{s'',\gamma-\mu}(B))} \leq c\,\langle\lambda\rangle^{L} \tag{1.59}$$

for all $\lambda \in \mathbb{R}^l$ *and some constant* $c > 0$.

The proof is simple as well and dropped here.

2. Operators on singular manifolds with conical exits

2.1. Edge operator-valued amplitude functions

We establish spaces of parameter-dependent edge operators over $B^{\asymp} := \mathbb{R} \times B$ for a compact manifold B with smooth edge Y. First we fix weight data $\mathbf{g} = (\gamma, \gamma - \mu)$ and an $m \leq \nu$ such that $\mu - m \in \mathbb{N}$, and an exit order $\nu \in \mathbb{R}$ in the variable $t \in \mathbb{R}$.

Definition 2.1. We define

$$\mathbf{S}^{m;\nu}(\mathbf{g}) := \{\tilde{a}(t, [t]\tau, [t]\zeta) : \tilde{a}(t, \tilde{\tau}, \tilde{\zeta}) \in S^{\nu}(\mathbb{R}, L^m(B, \mathbf{g}; \mathbb{R}^{1+d}_{\tilde{\tau},\tilde{\zeta}}))\}. \tag{2.1}$$

Let us first observe that $\mathrm{Op}_t(a)(\zeta)$ induces a continuous operator

$$\mathrm{Op}_t(a)(\zeta) : C_0^{\infty}(\mathbb{R}, H^{s,\gamma}(B)) \to C^{\infty}(\mathbb{R}, H^{s-m,\gamma-\mu}(B)) \tag{2.2}$$

for every $s \in \mathbb{R}$ and $\zeta \neq 0$. The proof is straightforward. We will show below that

$$\mathrm{Op}_t(a)(\zeta) : \mathcal{S}(\mathbb{R}, H^{s,\gamma}(B)) \to \mathcal{S}(\mathbb{R}, H^{s-m,\gamma-\mu}(B)) \tag{2.3}$$

is continuous for every $s \in \mathbb{R}$ and $\zeta \neq 0$.

Remark 2.2. We have $\mathbf{S}^{m;\nu'}(\mathbf{g}) \supseteq \mathbf{S}^{m;\nu}(\mathbf{g})$ for $\nu' \geq \nu$.

Proposition 2.3.

(i) $\varphi(t) \in S^{\sigma}(\mathbb{R})$, $a(t, \tau, \zeta) \in \mathbf{S}^{m;\nu}(\mathbf{g})$ *implies* $\varphi(t)a(t, \tau, \zeta) \in \mathbf{S}^{m;\sigma+\nu}(\mathbf{g})$*. Moreover,* $a(t, \tau, \zeta) \in \mathbf{S}^{m;\nu}(\mathbf{g}), b(t, \tau, \zeta) \in \mathbf{S}^{\tilde{m};\tilde{\nu}}(\tilde{\mathbf{g}})$ *for* $\mathbf{g} = (\gamma, \gamma - \mu, \Theta)$, $\tilde{\mathbf{g}} = (\tilde{\gamma}, \tilde{\gamma} - \tilde{\mu}, \Theta)$, $\gamma = \tilde{\gamma} - \tilde{\mu}$, *implies* $(ab)(t, \tau, \zeta) \in \mathbf{S}^{m+\tilde{m};\nu+\tilde{\nu}}(\mathbf{g} \circ \tilde{\mathbf{g}})$ *for* $\mathbf{g} \circ \tilde{\mathbf{g}} = (\tilde{\gamma}, \tilde{\gamma} - (\mu + \tilde{\mu}), \Theta)$.

(ii) *For every* $a(t, \tau, \zeta) \in \mathbf{S}^{m;\nu}(\mathbf{g})$ *we have*

$$\partial_t^l a \in \mathbf{S}^{m;\nu-l}(\mathbf{g}), \partial_\tau^k a \in \mathbf{S}^{m-k;\nu+k}(\mathbf{g}), \partial_\zeta^\alpha a \in \mathbf{S}^{m-|\alpha|;\nu+|\alpha|}(\mathbf{g})$$

for every $k, l \in \mathbb{N}$, $\alpha \in \mathbb{N}^d$.

Proof. (i) is evident. For (ii) for simplicity we assume $q = 1$ and compute

$$\partial_t \tilde{a}(t, [t]\tau, [t]\zeta) = \big((\partial_t + [t]'\tau\partial_{\tilde{\tau}} + [t]'\zeta\partial_{\tilde{\zeta}})\tilde{a}\big)(t, [t]\tau, [t]\zeta)$$

where $[t]' := \partial_t[t]$. Since $\tilde{\tau}\tilde{a}(t, \tilde{\tau}, \tilde{\zeta})$, $\tilde{\zeta}\tilde{a}(t, \tilde{\tau}, \tilde{\zeta}) \in S^\nu(\mathbb{R}, L^{m+1}(B, \mathbf{g}; \mathbb{R}^{1+d}_{\tilde{\tau},\tilde{\zeta}}))$, and $\partial_{\tilde{\tau}}\tilde{a}, \partial_{\tilde{\zeta}}\tilde{a} \in S^\nu(\mathbb{R}, L^{m-1}(B, \mathbf{g}; \mathbb{R}^{1+d}))$, we obtain

$$\partial_t \tilde{a}(t, [t]\tau, [t]\zeta) = \big((\partial_t + ([t]'/[t])[t]\tau\partial_{\tilde{\tau}} + ([t]'/[t])[t]\zeta\partial_{\tilde{\zeta}})\tilde{a}\big)(t, [t]\tau, [t]\zeta) \in \mathbf{S}^{m;\nu-1}.$$

It follows that $\partial_t^l a \in \mathbf{S}^{m;\nu-l}(\mathbf{g})$ for all $l \in \mathbb{N}$. Moreover, we have $\partial_\tau \tilde{a}(t, [t]\tau, [t]\zeta) = [t](\partial_{\tilde{\tau}}\tilde{a})(t, [t]\tau, [t]\zeta)$ which gives us $\partial_\tau a \in \mathbf{S}^{m-1;\nu+1}$, and, by iteration, $\partial_\tau^k a \in \mathbf{S}^{m-k;\nu+k}$. In a similar manner we can argue for the η-derivatives. □

Proposition 2.4. *Let $\varphi_1, \varphi_2 \in C^\infty(\mathbb{R})$ be strictly positive functions such that $\varphi_j(t) = |t|$ for $|t| \geq c_j$ for some $c_j > 0$, $j = 2$. Then we have*

$$\mathbf{S}^{m;\nu}(\mathbf{g}) = \{a(t, \varphi_1(t)\tau, \varphi_2(t)\zeta) : a(t, \tilde{\tau}, \tilde{\zeta}) \in S^\nu(\mathbb{R}, L^m(B, \mathbf{g}; \mathbb{R}^{1+d}_{\tilde{\tau},\tilde{\zeta}}))\}.$$

Proof. We can write $a(r, \varphi_1(r)\varrho, \varphi_2(r)\eta) = a(r, \psi_1(r)[r]\varrho, \psi_2(r)[r]\eta)$ for $\psi_j(r) \in C^\infty(\mathbb{R})$, $\psi_j(r) = 1$ for $|r| > c$ for some $c > 0$, $j = 2$. Then it suffices to verify that $a(r, \psi_1(r)\tilde{\varrho}, \psi_2(r)\tilde{\eta}) \in S^\nu(\mathbb{R}, L^m_{(\mathrm{cl})}(X; \mathbb{R}^{1+q}_{\tilde{\varrho},\tilde{\eta}}))$; however, this is straightforward. □

Proposition 2.5. $a(r, \varrho, \eta) \in \mathbf{S}^{m;\nu}(\mathbf{g})$ *implies* $a(\lambda t, \varrho, \zeta) \in \mathbf{S}^{m;\nu}(\mathbf{g})$ *for every* $\lambda \in \mathbb{R}_+$.

Proof. It is evident that $\tilde{a}(t, \tilde{\tau}, \tilde{\zeta}) \in S^\nu(\mathbb{R}, L^m(B, \mathbf{g}; \mathbb{R}^{1+d}_{\tilde{\tau},\tilde{\zeta}}))$ implies $\tilde{a}(\lambda t, \tilde{\tau}, \tilde{\zeta}) \in S^\nu(\mathbb{R}, L^m(B, \mathbf{g}; \mathbb{R}^{1+d}_{\tilde{\tau},\tilde{\zeta}}))$. Therefore, it suffices to show $\tilde{a}(t, [\lambda t]\tau, [\lambda t]\zeta) \in \mathbf{S}^{m;\nu}(\mathbf{g})$. Let us write $\tilde{a}(t, [\lambda t]\tau, [\lambda t]\zeta) = \tilde{a}(t, \varphi_\lambda(t)[t]\tau, \varphi_\lambda(t)[t]\zeta)$ for $\varphi_\lambda(t) := [\lambda t]/[t]$. We have $\varphi_\lambda(r) = \lambda$ for $|t| > c$ for a constant $c > 0$, i.e., $\varphi_\lambda(t) - \lambda \in C_0^\infty(\mathbb{R})$. Thus there is an r-excision function $\chi(t)$ (i.e., $\chi \in C^\infty(\mathbb{R})$, $\chi(t) = 0$ for $|t| \leq c_0$, $\chi(t) = 1$ for $|t| \geq c$, for certain $0 < c_0 < c_1$) such that $\chi(t)\tilde{a}(t, [\lambda t]\tau, [\lambda t]\zeta) = \chi(t)\tilde{a}(t, [t]\lambda\tau, [t]\lambda\zeta)$, and this function certainly belongs to $\mathbf{S}^{m;\nu}(\mathbf{g})$. It remains to characterise $(1 - \chi(t))\tilde{a}(t, \varphi_\lambda(t)[t]\tau, \varphi_\lambda(t)[t]\zeta)$ which vanishes for $|t| \leq c_0$, and a simple calculation shows $(1 - \chi(t))\tilde{a}(t, \varphi_\lambda(t)\tilde{\tau}, \varphi_\lambda(t)\tilde{\zeta}) \in C_0^\infty(\mathbb{R}, L^m(B, \mathbf{g}; \mathbb{R}^{1+d}_{\tilde{\tau},\tilde{\zeta}}))$, which is contained in $\mathbf{S}^{m;-\infty}(\mathbf{g})$. □

Lemma 2.6. *For every $a(t, \tau, \zeta)$ and $b(t, \tau, \zeta)$ as in* Proposition 2.3(i) *we have*

$$(\partial_\tau^k a D_t^k b)(t, \tau, \zeta) = c(t, \tau, \zeta) \in \mathbf{S}^{m+\tilde{m}-k;\nu+\tilde{\nu}}(\mathbf{g} \circ \tilde{\mathbf{g}}).$$

Proof. The assertion is a direct consequence of Proposition 2.3. □

Proposition 2.7. *Let $\tilde{a}_j(t, \tilde{\tau}, \tilde{\zeta}) \in S^\nu(\mathbb{R}, L^{\mu-j}(B, \mathbf{g}; \mathbb{R}^{1+d}))$, $j \in \mathbb{N}$, be an arbitrary sequence, $\mu, \nu \in \mathbb{R}$ fixed, and assume that the asymptotic types in the involved Green symbols are independent of j. Then there is an $\tilde{a}(t, \tilde{\tau}, \tilde{\zeta}) \in S^\nu(\mathbb{R}, L^\mu(B, \mathbf{g};$ $\mathbb{R}^{1+d}))$ such that $\tilde{a} - \sum_{j=0}^N \tilde{a}_j \in S^\nu(\mathbb{R}, L^{\mu-(N+1)}(B, \mathbf{g}; \mathbb{R}^{1+d}))$ for every $N \in \mathbb{N}$, and $\tilde{a}$ is unique* $\bmod S^\nu(\mathbb{R}, L^{-\infty}(B, \mathbf{g}; \mathbb{R}^{1+d}))$.

Proof. The proof is similar to the standard one on asymptotic summation of symbols. We can find an asymptotic sum as a convergent series $\sum_{j=0}^{\infty}\chi\big((\tilde\tau,\tilde\zeta)/c_j\big)\tilde a_j(t,\tilde\tau,\tilde\zeta)$ for some excision function χ in $\mathbb{R}^{1+d}$, with a sequence $c_j>0$, $c_j\to\infty$ as $j\to\infty$ so fast, that $\sum_{j=N+1}^{\infty}\chi\big((\tilde\tau,\tilde\zeta)/c_j\big)\tilde a(r,\tilde\tau,\tilde\zeta)$ converges in $S^{\nu}(\mathbb{R},L^{m-(N+1)}(B,\mathbf{g};\mathbb{R}^{1+d}))$ for every N. □

2.2. Compositions

Our next objective is to study compositions $\mathrm{Op}_t(a)(\zeta)\mathrm{Op}_t(b)(\zeta)$ for $a(t,\tau,\zeta)\in\mathbf{S}^{m;\nu}(\mathbf{g}), b(t,\tau,\zeta)\in\mathbf{S}^{\tilde m;\tilde\nu}(\tilde{\mathbf{g}})$ for fixed $\zeta\neq 0$.

Theorem 2.8. *For every fixed $\zeta\neq 0$ we have*

$$\mathrm{Op}_t(a)(\zeta)\mathrm{Op}_t(b)(\zeta)=\mathrm{Op}_t(a\#b)(\zeta), \tag{2.4}$$

$a\#b(t,\tau,\zeta)=\iint e^{-ir\varrho}a(t,\tau+\varrho,\zeta)b(t+r,\tau,\zeta)dr đ\varrho,$

$$a\#b(t,\tau,\zeta)=\sum_{k=0}^{N}\frac{1}{k!}\partial_\tau^k a(t,\tau,\zeta)D_t^k b(t,\tau,\zeta)+r_N(t,\tau,\zeta), \tag{2.5}$$

for

$$r_N(t,\tau,\zeta)=\frac{1}{N!}\iint e^{-ir\varrho}\Big\{\int_0^1(1-\theta)^N(\partial_\tau^{N+1}a)(t,\tau+\theta\varrho,\zeta)d\theta\Big\}(D_t^{N+1}b)(t+r,\tau,\zeta)dr đ\varrho. \tag{2.6}$$

Proof. Theorem 2.8 is formally of the same structure as the composition result in Kumano-go's formalism. The only point is to verify that the involved oscillatory integrals make sense, i.e., that the standard regularising process gives rise to convergent integrals. We shall see the details when we characterise the remainder term according to the following Lemma 2.9. □

Lemma 2.9. *Let $a(t,\tau,\zeta)\in\mathbf{S}^{m;\nu}(\mathbf{g}), b(t,\tau,\zeta)\in\mathbf{S}^{\tilde m;\tilde\nu}(\tilde{\mathbf{g}})$, respectively, with the weight data as in Lemma 2.6. Then for every $s',s''\in\mathbb{R}$, and $I,J,A,k,l,n\in\mathbb{N}$ there is an $N\in\mathbb{N}$ such that*

$$\|D_t^iD_\tau^jD_\zeta^\alpha r_N(t,\tau,\zeta)\|_{\mathcal{L}(H^{s',\gamma+\tilde\mu}(B),H^{s'',\gamma-\mu}(B))}\le c\langle\tau\rangle^{-k}\langle t\rangle^{-l}\langle\zeta\rangle^{-n} \tag{2.7}$$

for all $(t,\tau)\in\mathbb{R}^2$, $|\zeta|\ge\varepsilon>0$, $i,j\in\mathbb{N}$, $\alpha\in\mathbb{N}^d, 0\le i\le I, 0\le j\le J, |\alpha|\le A$, for some $c=c(s',s'',I,J,A,k,l,n,N,\varepsilon)>0$.

Proof. For abbreviation we set

$$\|\cdot\|_{s',s''}:=\|\cdot\|_{\mathcal{L}(H^{s',\gamma+\tilde\mu}(B),H^{s'',\gamma-\mu}(B))},$$
$$\|\cdot\|_{s',s_0}:=\|\cdot\|_{\mathcal{L}(H^{s',\gamma+\tilde\mu}(B),H^{s_0,\gamma}(B))},\ \|\cdot\|_{s_0,s''}:=\|\cdot\|_{\mathcal{L}(H^{s_0,\gamma}(B),H^{s'',\gamma-\mu}(B))}.$$

Moreover, let

$$\tilde a_{N+1}(t,[t]\tau+[t]\theta\varrho,[t]\zeta):=\big(\partial_\tau^{N+1}a\big)(t,\tau+\theta\varrho,\zeta),$$
$$\tilde b_{N+1}(t+r,[t+r]\tau,[t+r]\zeta):=\big(D_t^{N+1}b\big)(t+r,\tau,\zeta).$$

Then

$$r_N(t,\tau,\zeta)=\frac{1}{N!}\iint e^{-ir\varrho}\Big\{\int_0^1(1-\theta)^N\tilde{a}_{N+1}$$
$$(t,[t]\tau+[t]\theta\varrho,[t]\zeta)d\theta\Big\}\tilde{b}_{N+1}(t+r,[t+r]\tau,[t+r]\zeta)dr đ\varrho. \tag{2.8}$$

Since $\tilde{b}_{N+1}(t,\tilde{\tau},\tilde{\zeta})$ takes values in $L^{\tilde{m}}(B,\tilde{\mathbf{g}})$ it can be interpreted as an operator function with values in $\mathcal{L}(H^{s',\gamma+\tilde{\mu}}(B),H^{s'-\tilde{m},\gamma}(B))$. Similarly $\tilde{a}_{N+1}(t,\tilde{\tau},\tilde{\zeta})$ takes values in $L^{m-(N+1)}(B,\mathbf{g})$, i.e., in $\mathcal{L}(H^{s'-\tilde{m},\gamma}(B),H^{s'-\tilde{m}-m+(N+1),\gamma-\mu}(B))$, for any given s'. Thus, when also s'' is prescribed we take N so large that $s'-m-\tilde{m}+(N+1)\geq s''$.

By virtue of Proposition 2.3(ii) we have

$$\tilde{b}_{N+1}(t,\tilde{\tau},\tilde{\zeta})\in\mathbf{S}^{\tilde{m};\tilde{\nu}-(N+1)}(\tilde{\mathbf{g}}),\ \ \tilde{a}_{N+1}(t,\tilde{\tau},\tilde{\zeta})\in\mathbf{S}^{m-(N+1);\nu+(N+1)}(\mathbf{g}),$$

and it follows that

$$\|\tilde{b}_{N+1}(t,\tilde{\tau},\tilde{\zeta})\|_{s',s_0}\leq c\langle t\rangle^{\tilde{\nu}-(N+1)}\langle\tilde{\tau},\tilde{\zeta}\rangle^{B} \tag{2.9}$$

for some $B\geq 0$. Moreover, for every $M\geq 0$ the number N can be chosen so large that

$$\|(\partial_{\tilde{\tau}}^{G}\tilde{a}_{N+1})(t,\tilde{\tau},\tilde{\zeta})\|_{s_0,s''}\leq c\langle t\rangle^{\nu+(N+1)}\langle\tilde{\tau},\tilde{\zeta}\rangle^{-M-G} \tag{2.10}$$

for every G and all $t,\tilde{\tau},\tilde{\zeta}$. Those properties alone, regardless of the concrete nature of the operator functions (here belonging to the parameter-dependent edge calculus), will imply the desired estimates (2.7). The regularised oscillatory integral (2.8) has the form

$$r_N(t,\tau,\zeta)=\frac{1}{N!}\iint e^{-ir\varrho}\langle r\rangle^{-2L}(1-\partial_\varrho^2)^L\langle\varrho\rangle^{-2K}(1-\partial_r^2)^K\Big\{\int_0^1(1-\theta)^N$$
$$\tilde{a}_{N+1}(t,[t]\tau+[t]\theta\varrho,[t]\zeta)d\theta\Big\}\tilde{b}_{N+1}(t+r,[t+r]\tau,[t+r]\zeta)dr đ\varrho$$

for sufficiently large L,K. For simplicity we assume that ζ is a one-dimensional variable. For $l\leq L,k\leq K$ we have

$$\partial_\varrho^{2l}\tilde{a}_{N+1}(t,[t]\tau+[t]\theta\varrho,[t]\zeta)=\big(\partial_{\tilde{\tau}}^{2l}\tilde{a}_{N+1}\big)(t,[t]\tau+[t]\theta\varrho,[t]\zeta)([t]\theta)^{2l}, \tag{2.11}$$

$$\partial_r^{2k}\tilde{b}_{N+1}\ (t+r,[t+r]\tau,[t+r]\zeta)=\Big((\partial_r^{2k}+(\tau\partial_r[t+r])^{2k}+(\zeta\partial_r[t+r])^{2k}\big)$$
$$\tilde{b}_{N+1}\Big)(t+r,[t+r]\tau,[t+r]\zeta)+R \tag{2.12}$$

where R denotes several mixed derivatives of a similar ('better') behaviour. The estimates concerning R are left to the reader. From (2.10), (2.11) we obtain

$$\|\partial_\varrho^{2l}\tilde{a}_{N+1}(t,[t]\tau+[t]\theta\varrho,[t]\zeta)\|_{s_0,s''}\leq c\langle t\rangle^{\nu+(N+1)}\langle[t]\tau+[t]\theta\varrho,[t]\zeta\rangle^{-M-2l}([t]\theta)^{2l}, \tag{2.13}$$

and (2.9), (2.12) yield

$$\|\partial_r^{2k}\tilde{b}_{N+1}(t+r,[t+r]\tau,[t+r]\zeta)\|_{s',s_0}\leq c\langle t+r\rangle^{\tilde{\nu}-(N+1)}\langle[t+r]\tau,[t+r]\zeta\rangle^{B} \tag{2.14}$$

where we take N so large that $\tilde{\nu} - (N+1) \le 0$, and

$$\|\big(\partial_{\tilde{\tau}}^{2k}\ \tilde{b}_{N+1}\big)(t+r,[t+r]\tau,[t+r]\zeta)(\tau\partial_r[t+r])^{2k}\|_{s',s_0} \le c\langle t+r\rangle^{\tilde{\nu}-(N+1)}\langle[t+r]\tau,[t+r]\zeta\rangle^{B-2k}|\tau\partial_r[t+r]|^{2k}, \tag{2.15}$$

$$\big(\partial_{\tilde{\zeta}}^{2k}\tilde{b}_{N+1}\big)(t+r,[t+r]\tau,[t+r]\zeta)(\zeta\partial_r[t+r]^{2k}\|_{s',s_0} \le c\langle t+r\rangle^{\tilde{\nu}-(N+1)}\langle[t+r]\tau,[t+r]\zeta\rangle^{B-2k}|\zeta\partial_r[t+r]|^{2k}. \tag{2.16}$$

The above-mentioned mixed derivatives admit similar estimates (even better ones; so we content ourselves with (2.13), (2.14), (2.15), (2.16)). Let us now establish an estimate for $\|r_N(t,\tau,\zeta)\|_{s',s''}$. First we have

$$\|r_N(t,\tau,\zeta)\|_{s',s''} \le \iint\int_0^1 \|\langle r\rangle^{-2L}(1-\partial_\varrho^2)^L\langle\varrho\rangle^{-2K}(1-\partial_r^2)^K(1-\theta)^N\tilde{a}_{N+1} (t,[t]\tau+[t]\theta\varrho,[t]\zeta)\tilde{b}_{N+1}(t+r,[t+r]\tau,[t+r]\zeta)\|_{s',s''}d\theta dr đ\varrho.$$

The operator norm under the integral can be estimated by expressions of the kind

$$I := \langle t\rangle^{\nu+(N+1)}\langle t+r\rangle^{\tilde{\nu}-(N+1)}\langle r\rangle^{-2L}\langle\varrho\rangle^{-2K}\langle[t]\tau+[t]\theta\varrho,[t]\zeta\rangle^{-M-2l}([t]\theta)^{2l} \langle[t+r]\rangle\tau,\ [t+r]\zeta\rangle^{B}\{1+\langle[t+r]\tau,[t+r]\zeta\rangle^{-2k}(|\tau|+|\zeta|)^{2k}|\partial_r[t+r]|^{2k}\},$$

$l \le L$, $k \le K$, plus terms from R of a similar character, containing several mixed derivatives. Peetre's inequality gives us

$$\langle t\rangle^{\nu+(N+1)}\langle t+r\rangle^{\tilde{\nu}-(N+1)} \le \langle t\rangle^{\nu+\tilde{\nu}}\langle r\rangle^{|\tilde{\nu}-(N+1)|}.$$

Moreover, we have $\langle[t]\tau+[t]\theta\varrho,[t]\zeta\rangle^{-2l}([t]\theta)^{2l} \le c\langle[t]\zeta\rangle^{-2l}[t]^{2l} \le c$ for fixed $\zeta \ne 0$, and

$$\langle[t+r]\tau,[t+r]\zeta\rangle^{-2k}(|\tau|+|\zeta|)^{2k}|\partial_r[t+r]|^{2k} \le \{\langle[t+r]\tau\rangle^{-2k}([t+r]|\tau|)^{2k}+\langle[t+r]\zeta\rangle^{-2k}([t+r]|\zeta|)^{2k}\}[t+r]^{-2k} \le c,$$

using $|\partial_r[t+r]|^{2k} \le c$, $[t+r]^{-2k} \le c$ for all $t,r \in \mathbb{R}$, and $|\eta| \le c\langle\eta\rangle$ for every $\eta \in \mathbb{R}^d$. This yields

$$I \le\ c\langle t\rangle^{\nu+\tilde{\nu}}\langle r\rangle^{|\tilde{\nu}-(N+1)|}\langle r\rangle^{-2L}\langle\varrho\rangle^{-2k}\langle[t]\tau+[t]\theta\varrho,[t]\zeta\rangle^{-M}\langle[t+r]\tau,[t+r]\zeta\rangle^{B}.$$

Writing $M = M' + M''$ for suitable $M', M'' \ge 0$, $B \le M''$, to be fixed later on, we have

$$\langle[t]\tau+[t]\theta\varrho,[t]\zeta\rangle^{-M} = \langle[t]\tau+[t]\theta\varrho,[t]\zeta\rangle^{M'}\langle[t]\tau+[t]\theta\varrho,[t]\zeta\rangle^{M''} \le c\langle[t]\zeta\rangle^{-M'}\langle[t]\tau,[t]\zeta\rangle^{-M''}\langle[t]\theta\varrho\rangle^{M''} \le c\langle[t]\zeta\rangle^{-M'}\langle[t]\tau\rangle^{-M''}\langle[t]\theta\varrho\rangle^{M''}.$$

Here we applied once again Peetre's inequality which gives us also

$$\langle[t+r]\tau,[t+r]\zeta\rangle^{B} \le c\langle[t+r]\tau\rangle^{B}\langle[t+r]\zeta\rangle^{B}$$

since $B \ge 0$. It follows that

$$I \le c\langle t\rangle^{\nu+\tilde{\nu}}\langle r\rangle^{|\tilde{\nu}-(N+1)|-2L}\langle[t]\zeta\rangle^{M''-M'}\langle\varrho\rangle^{-2K}\langle[t]\theta\varrho\rangle^{M''}\langle[t+r]\tau\rangle^{B} \langle[t]\tau\rangle^{-M''}\langle[t+r]\zeta\rangle^{B}\langle[t]\zeta\rangle^{-M}.$$

A straightforward consideration gives us

$$\langle r\rangle^{-B}\langle[t+r]\tau\rangle^{B}\langle[t]\tau\rangle^{-B}\leq c$$

for all r, t, τ, which yields

$$I\leq c\langle t\rangle^{\nu+\tilde{\nu}}\langle r\rangle^{|\tilde{\nu}-(N+1)|-2L+2B}\langle[t]\zeta\rangle^{B-M'}\langle[t]\tau\rangle^{B}\langle\varrho\rangle^{-2K}\langle[t]\theta\varrho\rangle^{B}.$$

Finally, using $\langle\varrho\rangle^{-B}\langle t\rangle^{-B}\langle[t]\theta\varrho\rangle^{B}\leq c$ for all $0\leq\theta\leq 1$ and all t,ϱ we obtain for $|\zeta|\geq\varepsilon>0$

$$I\leq c\langle t\rangle^{\nu+\tilde{\nu}+2B}\langle r\rangle^{|\tilde{\nu}-(N+1)|-2L+2B}\langle\varrho\rangle^{-2K+B}\langle[t]\tau\rangle^{B-M''}\langle[t]\zeta\rangle^{-M}$$

for all $t,r\in\mathbb{R}$, $\tau,\varrho\in\mathbb{R}$, $0\leq\theta\leq 1$. Choosing K, L sufficiently large it follows that

$$\|r_N(t,\tau,\zeta)\|_{s',s''}\leq c\langle t\rangle^{\nu+\tilde{\nu}+2B-M}\langle\tau\rangle^{B-M''}\langle\zeta\rangle^{-M},$$

using $\langle[t]\tau\rangle^{B-M''}\leq c\langle\tau\rangle^{B-M''}$ for $B-M''\leq 0$. We have $\langle[t]\zeta\rangle^{-M}\leq c[t]^{-M}\langle\zeta\rangle^{-M}$ for $|\zeta|\geq\varepsilon>0$.

Now B is fixed, but M, M'' can be chosen independently as large as necessary. Therefore, we proved that for every $s',s''\in\mathbb{R}$ and $k,l,n\in\mathbb{N}$ there is an $N\in\mathbb{N}$ such that

$$\|r_N(t,\tau,\zeta)\|_{s',s''}\leq c\langle\tau\rangle^{-k}\langle t\rangle^{-l}\langle\zeta\rangle^{-n}$$

for all $(t,\tau)\in\mathbb{R}^2$, $|\zeta|\geq\varepsilon>0$, for some $c=c(\varepsilon)>0$. In an analogous manner we can show that

$$\|D_t^iD_\tau^jr_N(t,\tau,\zeta)\|_{s',s''}\leq c\langle\tau\rangle^{-k}\langle t\rangle^{-l}\langle\zeta\rangle^{-n}$$

for all $i,j\in\mathbb{N}$ and all $(t,\tau)\in\mathbb{R}^2$, $|\zeta|\neq 0$, for constants $c=c(\varepsilon,i,j)>0$. □

Remark 2.10. Analogously as (2.4) we can study the composition

$$\mathrm{Op}_t(a)(\zeta)\mathrm{Op}_t(b)(\tilde{\zeta})=\mathrm{Op}_t(a\#b)(\zeta,\tilde{\zeta})$$

where in this case

$$(a\#b)(t,\tau,\zeta,\tilde{\zeta})=\sum_{k=0}^{N}\frac{1}{k!}\partial_\tau^ka(t,\tau,\zeta)D_t^kb(t,\tau,\tilde{\zeta})+r_N(t,\tau,\zeta,\tilde{\zeta})$$

for every $N\in\mathbb{N}$. The remainder is of analogous form as that in Lemma 2.9.

Lemma 2.11. *For every $s',s''\in\mathbb{R}$, $I,J,A,k,l,n\in\mathbb{N}$ there exists an $N\in\mathbb{N}$ such that*

$$\|D_t^iD_\tau^jD_\zeta^\alpha D_{\tilde{\zeta}}^\beta r_N(t,\tau,\zeta,\tilde{\zeta})\|_{\mathcal{L}(H^{s',\gamma_1+\tilde{\mu}}(B),H^{s'',\gamma_1-\mu}(B))}\leq c\langle\tau\rangle^{-k}\langle t\rangle^{-l}\langle\zeta\rangle^{-n}\langle\tilde{\zeta}\rangle^{B}$$

for all $(t,\tau)\in\mathbb{R}^2, |\zeta|,|\tilde{\zeta}|\geq\varepsilon>0, i,j\in\mathbb{N},\alpha,\beta\in\mathbb{N}^d=\leq i\leq I, 0\leq j\leq J, |\alpha|,|\beta|\leq A$, for some constant $c=c(s',s'',I,J,A,k,l,n,N,\varepsilon)>0$ and some $B\geq 0$.

The proof follows by a simple modification of the proof Lemma 2.9.

Proposition 2.12. *Let $\mathbf{g}=(\gamma,\gamma-\mu,\Theta), m,\nu\in\mathbb{R},\mu-m\in\mathbb{N}$. Then for every $s',s''\in\mathbb{R}$ and $I,J,A,k,l,n\in\mathbb{N}$ there exists an $N\in\mathbb{N}$ such that $a(t,\tau,\zeta)\in$*

$S^{m-(N+1);\nu}(\mathbf{g})$ *implies*

$$\|D_t^i D_\tau^j D_\zeta^\alpha r_N(t,\tau,\zeta)\|_{\mathcal{L}(H^{s',\gamma}(B),H^{s'',\gamma-\mu}(B))} \le c\langle\tau\rangle^{-k}\langle t\rangle^{-l}\langle\zeta\rangle^{-n} \tag{2.17}$$

for all $(t,\tau)\in\mathbb{R}^2$, $|\zeta|\ge\varepsilon>0$, $i,j\in\mathbb{N}$, $\alpha\in\mathbb{N}^d, 0\le i\le I, 0\le j\le J, |\alpha|\le A$, *for some* $c=c(s',s'',I,J,A,k,l,n,N,\varepsilon)>0$.

Proposition 2.12 can be proved in a similar manner as Lemma 2.9.

Theorem 2.13. *Let* $a(t,\tau,\zeta)\in\mathbf{S}^{m;\nu}(\mathbf{g})$*; then for every* $\zeta\in\mathbb{R}^d\setminus\{0\}$ *the operator* $\mathrm{Op}(a)(\zeta)$ *extends to a continuous operator* (2.3) *for every* $s\in\mathbb{R}$.

Proof. Theorem 2.13 can be obtained in a similar manner as Theorem 2.8. We only have employed a special case of when the second factor in (2.4) is only the multiplication by a function in $S^{\tilde{\nu}}(\mathbb{R}_t)$; then the conclusions to obtain Lemma 2.9 do not employ the continuity of (2.3) but only (2.2). □

Note that the composition in Theorem 2.8 refers to the continuity of operators in Schwartz spaces which makes the composition formally possible.

2.3. Edge calculus up to infinity

Definition 2.14. We define $\mathbf{L}^{m;\nu}(B^{\asymp},\mathbf{g};\mathbb{R}^d\setminus\{0\})$ for $\mathbf{g}=(\gamma,\gamma-\mu,\Theta), \mu-m\in\mathbb{N}$, to be space of all

$$A(\zeta)=\mathrm{Op}_t(a)(\zeta)+C(\zeta),$$

$\zeta\in\mathbb{R}^d\setminus\{0\}$, for arbitrary $a(t,\tau,\zeta)\in\mathbf{S}^{m;\nu}(\mathbf{g}), C(\zeta)\in\mathbf{L}^{-\infty;-\infty}(B^{\asymp},\mathbf{g};\,\mathbb{R}^d\setminus\{0\})$. Here $\mathbf{L}^{-\infty;-\infty}(B^{\asymp},\mathbf{g};\mathbb{R}^d\setminus\{0\})$ is the space of all operator families

$$C(\zeta)u(t)=\int_{\mathbb{R}} c(t,t',\zeta)u(t')dt' \tag{2.18}$$

for kernels $c(t,t',\zeta)\in\mathcal{S}(\mathbb{R}^d_\zeta\setminus\{0\},\,\mathcal{S}(\mathbb{R}\times\mathbb{R},L^{-\infty}(B,\mathbf{g}))$.

Remark 2.15. It can be verified that the space

$$\mathbf{L}^{-\infty;-\infty}(B^{\asymp},\mathbf{g};\mathbb{R}^d\setminus\{0\}) \tag{2.19}$$

coincides with $\bigcap_{j\in\mathbb{N}}\mathbf{L}^{\mu-j;\nu}(B^{\asymp},\mathbf{g};\mathbb{R}^d\setminus\{0\})$ for any fixed $\nu\in\mathbb{R}$. Moreover, every $C\in\mathbf{L}^{-\infty;-\infty}(B^{\asymp},\mathbf{g};\mathbb{R}^d\setminus\{0\})$ can be represented in the form $C(\zeta)=\mathrm{Op}_t(c)(\zeta)$, $c(t,\tau,\zeta)=\tilde{c}(t,[t]\tau,[t]\zeta)$ for a $\tilde{c}(t,\tilde{\tau},\tilde{\zeta})\in\mathbf{S}^{-\infty;\nu}(\mathbf{g})$. Here we may take $\nu=-\infty$.

Theorem 2.16. $A(\zeta)\in\mathbf{L}^{m;\nu}(B^{\asymp},\mathbf{g};\mathbb{R}^d\setminus\{0\})$, $B(\zeta)\in\mathbf{L}^{\tilde{m};\tilde{\nu}}(B^{\asymp},\tilde{\mathbf{g}});\mathbb{R}^d\setminus\{0\})$ *for* $\mathbf{g}=(\gamma,\gamma-\mu,\Theta)$, $\tilde{\mathbf{g}}=(\tilde{\gamma},\tilde{\gamma}-\tilde{\mu},\Theta)$, $\gamma=\tilde{\gamma}-\tilde{\mu}$, *implies*

$$A(\zeta)B(\zeta)\in\mathbf{L}^{m+\tilde{m};\nu+\tilde{\nu}}(B^{\asymp},\mathbf{g}\circ\tilde{\mathbf{g}};\mathbb{R}^d\setminus\{0\}).$$

For $A(\zeta)=\mathrm{Op}(a)(\zeta)+C(\zeta), B(\zeta)=\mathrm{Op}(b)(\zeta)+D(\zeta)$ *for* $a(t,\tau,\zeta)\in\mathbf{S}^{m;\nu}(\mathbf{g})$, $b(t,\tau,\zeta)\in\mathbf{S}^{\tilde{m},\tilde{\nu}}(\tilde{\mathbf{g}})$ *it follows that*

$$A(\zeta)B(\zeta)=\mathrm{Op}(c)(\zeta) \bmod \mathbf{L}^{-\infty;-\infty}(B^{\asymp},\mathbf{g}\circ\tilde{\mathbf{g}};\mathbb{R}^d\setminus\{0\})$$

for a $c(t,\tau,\zeta)\in\mathbf{S}^{m+\tilde{m};\nu+\tilde{\nu}}(\mathbf{g}\circ\tilde{\mathbf{g}})$. *In particular,* $A(\zeta)\in\mathbf{L}^{-\infty;-\infty}(B^{\asymp},\mathbf{g};\mathbb{R}^d\setminus\{0\})$ *or* $B(\zeta)\in\mathbf{L}^{-\infty;-\infty}(B^{\asymp},\,\tilde{\mathbf{g}};\mathbb{R}^d\setminus\{0\})$ *implies* $A(\zeta)B(\zeta)\in\mathbf{L}^{-\infty;-\infty}(B^{\asymp},\mathbf{g}\circ\tilde{\mathbf{g}};\mathbb{R}^d\setminus\{0\})$.

Proof. Let us first assume $A = \mathrm{Op}(a), B = A = \mathrm{Op}(b)$. Theorem 2.8 gives us $AB = \mathrm{Op}(c_N) + \mathrm{Op}(r_N)$ for $c_N(t,\tau,\zeta) = \sum_{k=0}^{N} 1/k!\, \partial_\tau^k a(t,\tau,\zeta) D_t^k b(t,\tau,\zeta) \in \mathbf{S}^{m+\tilde{m};\nu+\tilde{\nu}}(\mathbf{g} \circ \tilde{\mathbf{g}})$, cf. Lemma 2.6, with r_N as in (2.6). The symbols $a^{(k)}(t,\tau,\zeta) := \partial_\tau^k a(t,\tau,\zeta)$, $b_{(k)}(t,\tau,\zeta) := D_t^k b(t,\tau,\zeta)$, have the form $\tilde{a}^{(k)}(t,[t]\tau,[t]\zeta)$ and $\tilde{b}_{(k)}(t,[t]\tau,[t]\zeta)$ for corresponding $\tilde{a}^{(k)}(t,\tilde{\tau},\tilde{\zeta}) \in S^{\nu+k}(L^{m-k}(B,\mathbf{g};\mathbb{R}^{1+d}))$, and $\tilde{b}_{(k)}(t,\tilde{\tau},\tilde{\zeta}) \in S^{\nu-k}(L^{\tilde{m}}(B,\tilde{\mathbf{g}};\mathbb{R}^{1+d}))$, respectively. By virtue of Proposition 2.7 we have an asymptotic sum $\tilde{p}(t,\tilde{\tau},\tilde{\zeta}) \sim \sum_{k=0}^{\infty} \tilde{a}^{(k)}(t,\tilde{\tau},\tilde{\zeta})\tilde{b}_{(k)}(t,\tilde{\tau},\tilde{\zeta})$ in the space $S^\nu(\mathbb{R}, L^{m+\tilde{m}}(B, \mathbf{g} \circ \tilde{\mathbf{g}}; \mathbb{R}^{1+d}))$. In particular, we have

$$\tilde{p}(t,\tilde{\tau},\tilde{\zeta}) - \sum_{k=0}^{N} \tilde{a}^{(k)}(t,\tilde{\tau},\tilde{\zeta})\tilde{b}_{(k)}(t,\tilde{\tau},\tilde{\zeta}) \in S^\nu(\mathbb{R}, L^{m+\tilde{m}-(N+1)}(B, \mathbf{g} \circ \tilde{\mathbf{g}}; \mathbb{R}^{1+d})).$$

Setting $p(t,\tau,\zeta) := \tilde{p}(t,[t]\tau,[t]\zeta)$ it follows that

$$\mathrm{Op}(a)\mathrm{Op}(b) = \mathrm{Op}(p) - \mathrm{Op}\big(p - \sum_{k=0}^{N} a^{(k)} b_{(k)}\big) + \mathrm{Op}(r_N)$$

for every $N \in \mathbb{N}$. A similar identity holds on the level of amplitude functions. From (2.5) it follows that $a\sharp b = p - p + \sum_{k=0}^{N} a^{(k)} b_{(k)} + r_N$, and we see that $-p + \sum_{k=0}^{N} a^{(k)} b_{(k)} + r_N := l$ is independent of N. By virtue of Lemma 2.9 and Proposition 2.12 the function $l(t,\tau,\zeta)$ satisfies the estimates (2.7) for every pair (s', s'') and all i, j, α. Such an l can be represented by a kernel $c(t,t',\zeta)$ as in the second part of Definition 2.14, such that $\mathrm{Op}_t(l(\zeta)) = C(\zeta)$. It follows altogether $\mathrm{Op}(a)\mathrm{Op}(b) \in \mathbf{L}^{m+\tilde{m};\nu+\tilde{\nu}}(B^{\asymp}, \mathbf{g} \circ \tilde{\mathbf{g}}; \mathbb{R}^d \setminus \{0\})$. From the above norm growth characterisations of $\mathbf{L}^{-\infty;-\infty}$ we can also easily deduce the second part of Theorem 2.16. □

Let us define the formal adjoint $A^*(\zeta)$ an operator family $A(\zeta) \in \mathbf{L}^{m;\nu}(B^{\asymp}, \tilde{\mathbf{g}}; \mathbb{R}^d \setminus \{0\})$ by

$$(A(\zeta)u, v)_{L^2(\mathbb{R}, H^{0,0}(B))} = (u, A^*(\zeta)v)_{L^2(\mathbb{R}, H^{0,0}(B))}$$

for all $u, v \in \mathcal{S}(\mathbb{R}, H^{\infty,\infty}(B))$. Then for $A(\zeta) = \mathrm{Op}_t(a)(\zeta) + C(\zeta)$, $a(t,\tau,\zeta) = \tilde{a}(t,[t]\tau, t[\zeta]) \in \mathbf{S}^{m;\nu}(\mathbf{g})$, $C(\zeta) \in \mathbf{L}^{-\infty;-\infty}(B^{\asymp}, \mathbf{g}; \mathbb{R}^d \setminus \{0\})$ we obtain

$$A^*(\zeta) = \mathrm{Op}_t(a^*)(\zeta) + C^*(\zeta)$$

for $a^*(t',\tau,\zeta) = \tilde{a}^{(*)}(t',[t']\tau,[t']\zeta)$ with $(*)$ indicating the pointwise formal adjoint in the edge calculus over B and $a^*(t',\tau,\zeta)$ being treated as a right symbol. Moreover, if the smoothing operator is given in the form (2.18), then

$$C^*(\zeta)v(t') = \int_{\mathbb{R}} c^{(*)}(t,t',\zeta)v(t)dt$$

where $c^{(*)}$ means the pointwise formal adjoint in the space of smoothing operators over B.

Theorem 2.17. *$A(\zeta) \in \mathbf{L}^{m;\nu}(B^{\asymp}, \mathbf{g}; \mathbb{R}^d \setminus \{0\})$ for $\mathbf{g} = (\gamma, \gamma - \mu, \Theta)$ implies $A^*(\zeta) \in \mathbf{L}^{m;\nu}(B^{\asymp}, \mathbf{g}^*; \mathbb{R}^d \setminus \{0\})$ for $\mathbf{g}^* = (-\gamma + \mu, -\gamma, \Theta)$. In particular, $A(\zeta) = \mathrm{Op}_t(a)(\zeta)$, $a(t, \tau, \zeta) \in \mathbf{S}^{m;\nu}(\mathbf{g})$, entails*

$$A^*(\zeta) = \mathrm{Op}(a^*)(\zeta) \mod \mathbf{L}^{-\infty;-\infty}(B^{\asymp}, \mathbf{g}^*; \mathbb{R}^d \setminus \{0\}), \text{ for an } a^*(t, \tau, \zeta) \in \mathbf{S}^{m;\nu}(\mathbf{g}^*).$$

The proof is straightforward and left to the reader.

2.4. Ellipticity

Definition 2.18. An element $A(\zeta) \in \mathbf{L}^{\mu;\nu}(B^{\asymp}, \mathbf{g}; \mathbb{R}^d \setminus \{0\})$ for $\mathbf{g} = (\gamma, \gamma - \mu, \Theta)$ is called elliptic if in a representation

$$A(\zeta) = \mathrm{Op}_t(a)(\zeta) \mod \mathbf{L}^{-\infty;-\infty}(B^{\asymp}, \mathbf{g}; \mathbb{R}^d \setminus \{0\})$$

for an $a(t, \tau, \zeta) \in \mathbf{S}^{\mu;\nu}(\mathbf{g})$ there is an element $b(t, \tau, \zeta) \in \mathbf{S}^{-\mu;-\nu}(\mathbf{g}^{-1})$ for $\mathbf{g}^{-1} = (\gamma - \mu, \gamma, \Theta)$, such that $1 - ba \in \mathbf{S}^{-1;0}(\mathbf{g}_l)$, $1 - ab \in \mathbf{S}^{-1;0}(\mathbf{g}_r)$ for $\mathbf{g}_l := (\gamma, \gamma, \Theta)$, $\mathbf{g}_r := (\gamma - \mu, \gamma - \mu, \Theta)$.

Remark 2.19. Let $A(\zeta) \in \mathbf{L}^{\mu;\nu}(B^{\asymp}, \mathbf{g}, \mathbb{R}^d \setminus \{0\})$, $B(\zeta) \in \mathbf{L}^{\tilde{\mu};\tilde{\nu}}(B^{\asymp}, \tilde{\mathbf{g}}; \mathbb{R}^d \setminus \{0\})$ for $\mathbf{g} = (\gamma, \gamma - \mu, \Theta)$, $\tilde{\mathbf{g}} = (\tilde{\gamma}, \tilde{\gamma} - \tilde{\mu}, \Theta)$, be elliptic, where $\gamma = \tilde{\gamma} - \tilde{\mu}$. Then $A(\zeta)B(\zeta) \in \mathbf{L}^{\mu+\tilde{\mu};\nu+\tilde{\nu}}(B^{\asymp}, \mathbf{g} \circ \tilde{\mathbf{g}}; \mathbb{R}^d \setminus \{0\})$ is also elliptic.

Theorem 2.20. *Let $A(\zeta) \in \mathbf{L}^{\mu;\nu}(B^{\asymp}, \mathbf{g}; \mathbb{R}^d \setminus \{0\})$ for $\mathbf{g} = (\gamma, \gamma - \mu, \Theta)$ be elliptic. Then there exists a parametrix $B(\zeta) \in \mathbf{L}^{-\mu;-\nu}(B^{\asymp}, \mathbf{g}^{-1}; \mathbb{R}^d \setminus \{0\})$ in the sense*

$$1 - B(\zeta)A(\zeta) \in \mathbf{L}^{-\infty;-\infty}(B^{\asymp}, \mathbf{g}_l; \mathbb{R}^d \setminus \{0\}),$$
$$1 - A(\zeta)B(\zeta) \in \mathbf{L}^{-\infty;-\infty}(B^{\asymp}, \mathbf{g}_r; \mathbb{R}^d \setminus \{0\})$$

(*cf. the notation in Definition* 2.18).

Proof. Let us construct a $B(\zeta)$ such that $1 - B(\zeta)A(\zeta)$ has the asserted property. The construction from the right is similar; the a standard algebraic argument shows that both operators coincide modulo $\mathbf{L}^{-\infty;-\infty}$. From Definition 2.18 we have $c := 1 - ba \in \mathbf{S}^{-1;0}(\mathbf{g}_l)$ for a corresponding symbol b. From now on the proof is straightforward after the prepared tools. We represent $b\sharp a$ by an asymptotic sum $p \in \mathbf{S}^{0;0}(\mathbf{g}_l)$ modulo a smoothing family (cf. also the proof of Theorem 2.16). It follows that $p = 1 - d$ for a $d \in \mathbf{S}^{-1;0}(\mathbf{g}_l)$. By a formal Neumann series argument we find an $f \in \mathbf{S}^{-1;0}(\mathbf{g}_l)$ such that $\mathrm{Op}(1 - f)\mathrm{Op}(p) = \mathrm{Op}((1 - f)\sharp p) = 1$ modulo $\mathbf{L}^{-\infty;-\infty}$. This allows us to set $B := \mathrm{Op}(1 - f)\sharp b$. □

Remark 2.21. Let $A(\zeta) \in \mathbf{L}^{\mu;\nu}(B^{\asymp}, \mathbf{g}; \mathbb{R}^d \setminus \{0\})$ be elliptic, and let $B(\zeta), \widetilde{B}(\zeta) \in \mathbf{L}^{-\mu;-\nu}(B^{\asymp}, \mathbf{g}^{-1}; \mathbb{R}^d \setminus \{0\})$ be two parametrices of $A(\zeta)$. Then we have $B(\zeta) = \widetilde{B}(\zeta) \mod \mathbf{L}^{-\infty;-\infty}(B^{\asymp}, \mathbf{g}^{-1}; \mathbb{R}^d \setminus \{0\})$. Moreover, also $B(\zeta) + C(\zeta)$ for any $C(\zeta) \in \mathbf{L}^{-\infty;-\infty}(B^{\asymp}\mathbf{g}^{-1}; \mathbb{R}^d \setminus \{0\})$ is a parametrix of $A(\zeta)$.

References

[1] J. Abed and B.-W. Schulze, *Edge-degenerate families of pseudo-differential operators on an infinite cylinder*, Oper. Theory: Adv. Appl. **205**, "Pseudo-Differential Operators: Complex Analysis and Partial Differential Equations", Birkhäuser Verlag, Basel, 2009, pp. 59–80.

[2] D. Calvo, C.-I. Martin, and B.-W. Schulze, *Symbolic structures on corner manifolds*, RIMS Conf. dedicated to L. Boutet de Monvel on "Microlocal Analysis and Asymptotic Analysis", Kyoto, August 2004, Keio University, Tokyo, 2005, pp. 22–35.

[3] D. Calvo and B.-W. Schulze, *Edge symbolic structure of second generation*, Math. Nachr. **282** (2009), 348–367.

[4] N. Dines, *Elliptic operators on corner manifolds*, Ph.D. thesis, University of Potsdam, 2006.

[5] Ju.V. Egorov and B.-W. Schulze, *Pseudo-differential operators, singularities, applications*, Oper. Theory: Adv. Appl. **93**, Birkhäuser Verlag, Basel, 1997.

[6] G.I. Eskin, *Boundary value problems for elliptic pseudodifferential equations*, Transl. of Nauka, Moskva, 1973, Math. Monographs, Amer. Math. Soc. **52**, Providence, Rhode Island 1980.

[7] J.B. Gil, B.-W. Schulze, and J. Seiler, *Cone pseudodifferential operators in the edge symbolic calculus*, Osaka J. Math. **37** (2000), 219–258.

[8] G. Harutjunjan and B.-W. Schulze, *Elliptic mixed, transmission and singular crack problems*, European Mathematical Soc., Zürich, 2008.

[9] L. Hörmander, *The analysis of linear partial differential operators*, vol. 1 and 2, Springer-Verlag, New York, 1983.

[10] I.L. Hwang, *The L^2-boundedness of pseudodifferential operators*, Trans. Amer. Math. Soc. **302** (1987), 55–76.

[11] D. Kapanadze and B.-W. Schulze, *Crack theory and edge singularities*, Kluwer Academic Publ., Dordrecht, 2003.

[12] V.A. Kondratyev, *Boundary value problems for elliptic equations in domains with conical points*, Trudy Mosk. Mat. Obshch. **16**, (1967), 209–292.

[13] X. Liu and B.-W. Schulze, *Boundary value problems in edge representation*, Math. Nachr. **280**, 5-6 (2007), 1–41.

[14] C.-I. Martin and B.-W. Schulze, *Parameter-dependent edge operators*, Ann. Glob. Anal. Geom. **171**, 1 (2010) 1–31.

[15] S. Rempel and B.-W. Schulze, *Index theory of elliptic boundary problems*, Akademie-Verlag, Berlin, 1982.

[16] S. Rempel and B.-W. Schulze, *Parametrices and boundary symbolic calculus for elliptic boundary problems without transmission property*, Math. Nachr. **105**, (1982), 45–149.

[17] B.-W. Schulze, *Operators with symbol hierarchies and iterated asymptotics*, Publications of RIMS, Kyoto University **38**, 4 (2002), 735–802.

[18] B.-W. Schulze, *Pseudo-differential operators on manifolds with edges*, Teubner-Texte zur Mathematik, Symp. "Partial Differential Equations", Holzhau 1988, **112**, Leipzig, 1989, pp. 259–287.

[19] B.-W. Schulze, *Pseudo-differential operators on manifolds with singularities*, North-Holland, Amsterdam, 1991.

[20] B.-W. Schulze, *Boundary value problems and singular pseudo-differential operators*, J. Wiley, Chichester, 1998.

[21] B.-W. Schulze, *Operator algebras with symbol hierarchies on manifolds with singularities*, Oper. Theory: Adv. Appl. **125**, Adv. in Partial Differential Equations "Approaches to Singular Analysis" (J. Gil, D. Grieser, and M. Lesch, eds.), Birkhäuser Verlag, Basel, 2001, pp. 167–207.

[22] B.-W. Schulze, *The iterative structure of corner operators*, arXiv: 0901.1967v1 [math.AP], 2009.

[23] J. Seiler, *Continuity of edge and corner pseudo-differential operators*, Math. Nachr. **205** (1999), 163–182.

[24] M.I. Vishik and G.I. Eskin, *Convolution equations in a bounded region*, Uspekhi Mat. Nauk **20**,3 (1965), 89–152.

[25] I. Witt, *On the factorization of meromorphic Mellin symbols*, Advances in Partial Differential Equations (Approaches to Singular Analysis) (S. Albeverio, M. Demuth, E. Schrohe, and B.-W. Schulze, eds.), Oper. Theory: Adv. Appl., Birkhäuser Verlag, Basel, 2002, pp. 279–306.

B.-Wolfgang Schulze
Am Neuen Palais 10
D-14469 Potsdam, Germany
e-mail: schulze@math.uni-potsdam.de

Operator Theory:
Advances and Applications, Vol. 216, 65–78

On a Method for Solving Boundary Problems for a Third-order Equation with Multiple Characteristics

Yusupjon P. Apakov

Abstract. The first boundary problem $\frac{\partial^3 u}{\partial x^3} - \frac{\partial^2 u}{\partial y^2} = f(x, y)$ is considered in the domain $D = \{(x, y) : \ 0 < x < p, \ 0 < y < l\}$. Uniqueness of the solution is proven with the method of energy integral. The Green function is constructed for the first boundary value problem, through which the explicit solution of the problem is obtained.

Mathematics Subject Classification (2000). 35G15.

Keywords. Third-order partial differential equation, fundamental solutions, Green function, boundary value problem.

1. Introduction

The third-order equation with multiple characteristics

$$L(u) = \frac{\partial^3 u}{\partial x^3} - \frac{\partial^2 u}{\partial y^2} = f(x, y) \tag{1.1}$$

was considered for the first time in [1–4]. Then it has appeared in [5–6] where various boundary problems were studied using the method of potentials.

We note that the equation (1.1) is conjugate to the equation

$$u_{xxx} + u_{yy} = F(x, y)$$

which is the linear part (for $\nu = 0$) of the so-called VT-equation (Viscous Transonic equation)

$$u_{xxx} + u_{yy} - \frac{\nu}{y} u_y = u_x u_{xx}.$$

For $\nu = 1$, the VT-equation expresses an axi-symmetric flow, and for $\nu = 0$, it expresses a plane parallel flow [7–8].

In [9] the fundamental solution of the equation (1.1) is constructed expressed by the degenerate hypergeometric function in the form

$$\begin{aligned} U\left(x,y;\xi,\eta\right) &= |y-\eta|^{\frac{1}{3}}\, f\left(t\right), \quad -\infty<t<\infty, \\ V\left(x,y;\xi,\eta\right) &= |y-\eta|^{\frac{1}{3}}\, \varphi\left(t\right), \qquad t<0, \end{aligned} \tag{1.2}$$

here

$$f(t)=\frac{2\sqrt[3]{2}}{\sqrt{3}\pi}t\Psi\left(\frac{1}{6},\frac{4}{3};\tau\right),\quad \varphi(t)=\frac{36\Gamma\left(1/3\right)}{\sqrt{3}\pi}\,t\Phi\left(\frac{1}{6},\frac{4}{3};\tau\right),$$

$$\tau=\frac{4}{27}t^3, \qquad t=\frac{x-\xi}{|y-\eta|^{\frac{2}{3}}}\ ,$$

where $\Psi(a,b;x)$, $\Phi(a,b;x)$ are degenerate hypergeometric functions (see [10]).

Using estimates of degenerate hypergeometric functions estimates of the fundamental solutions are obtained when the argument approaches infinity. For the function $U(x,y;\xi,\eta)$, the estimate:

$$\left|\frac{\partial^{h+k}U}{\partial x^h dy^k}\right| \leqslant C_{kh}\,|y-\eta|^{\frac{1-(-1)^k}{2}}\,|x-\xi|^{-\frac{1}{2}\left[2h+3k-1+\frac{3}{2}\left(1-(-1)^k\right)\right]}$$

as

$$\left|\frac{x-\xi}{|y-\eta|^{\frac{2}{3}}}\right|\to\infty$$

holds, where C_{kh} – const, $k,h=0,1,2,\dots$ are constants.

For $V(x,y;\xi,\eta)$, there are analogue estimates for $(x-\xi)\,|y-\eta|^{-\frac{2}{3}}\to-\infty$.

In [11, 12] some boundary problems for equation (1.1) are studied in a rectangular domain. In these papers the solution is attained by the Fourier method and, for this, zeros at $y=0$ and $y=l$ were required. In this work the Green function is constructed for the first boundary problem and through it the explicit solution is obtained.

2. Statement of the problem

In the domain $D=\{(x,y):\ 0<x<p,\ 0<y<l\}$ we consider the equation (1.1) where $p>0$, $l>0$ are constant numbers.

The function $u(x,y)$ satisfying the equation (1.1) in D and belonging to the class $C^{3,2}_{x,y}\left(D\right)\cap C^{1,0}_{x,y}\left(\overline{D}\right)$ is said to be *a regular solution of equation* (1.1).

Problem A. *Find a regular solution of the equation* (1.1) *satisfying in D the boundary conditions*

$$u\left(x,0\right)=\varphi_1\left(x\right),\quad u\left(x,l\right)=\varphi_2\left(x\right), \tag{2.1}$$

$$u\left(0,y\right)=\psi_1\left(y\right),\quad u\left(p,y\right)=\psi_2\left(y\right),\quad u_x\left(p,y\right)=\psi_3\left(y\right) \tag{2.2}$$

where

$$\varphi_i\left(x\right)\in C\left[0,p\right],\ i=1,2,\quad \psi_j\left(y\right)\in C\left[0,l\right],\ \ j=\overline{1,3}\ ,\, f\left(x,y\right)\in C^{0,2}_{x,y}\left(\overline{D}\right).$$

Besides, the compatibility conditions

$$\varphi_1(0) = \psi_1(0), \ \varphi_1(p) = \psi_2(0), \ \varphi_1'(p) = \psi_3(0), \quad \varphi_2(0) = \psi_1(l),$$
$$\varphi_2(p) = \psi_2(l), \quad \varphi_2'(p) = \psi_3(l), \quad f(x,0) = f(x,l) = 0.$$

are satisfied.

3. Uniqueness of the solution

Theorem 1. *Problem* A *cannot have more than one solution.*

Proof. Let Problem A have two solutions, say $u_1(x,y)$ and $u_2(x,y)$. Then $u(x,y) = u_1(x,y) - u_2(x,y)$ satisfies the equation $u_{xxx} - u_{yy} = 0$ and homogenous boundary conditions. We will prove that $u(x,y) \equiv 0$ in D. Consider the identity

$$\frac{\partial}{\partial x}\left(uu_{xx} - \frac{1}{2}u_x^2\right) - \frac{\partial}{\partial y}(uu_y) + u_y^2 = 0. \tag{3.1}$$

Integrating identity (3.1) over the domain D and considering homogenous boundary condition, we obtain

$$\frac{1}{2}\int_0^l u_x^2(0,y)\,dy + \iint_D u_y^2(x,y)\,dxdy = 0$$

Hence $u_y(x,y) = 0$, that is $u(x,y) = \phi(x)$. From $u(x,0) = 0$, we get $\phi(x) = 0$, then $u(x,y) \equiv 0$. □

4. Existence of the solution

Let us move prove existence of the solution for Problem A. We consider the conjugated differential operators

$$L \equiv \frac{\partial^3}{\partial \xi^3} - \frac{\partial^2}{\partial \eta^2}, \qquad L^* \equiv -\frac{\partial^3}{\partial \xi^3} - \frac{\partial^2}{\partial \eta^2}.$$

There is the identity:

$$\varphi L[\psi] - \psi L^*[\varphi] \equiv \frac{\partial}{\partial \xi}(\varphi\psi_{\xi\xi} - \varphi_\xi\psi_\xi + \varphi_{\xi\xi}\psi) - \frac{\partial}{\partial \eta}(\varphi\psi_\eta - \varphi_\eta\psi)$$

where φ, ψ are sufficiently smooth functions.

Integrating this identity over the domain D, we obtain

$$\iint_D [\varphi L[\psi] - \psi L^*[\varphi]]\,d\xi d\eta = \iint_D \frac{\partial}{\partial \xi}(\varphi\psi_{\xi\xi} - \varphi_\xi\psi_\xi + \varphi_{\xi\xi}\psi)\,d\xi d\eta - \iint_D \frac{\partial}{\partial \eta}(\varphi\psi_\eta - \varphi_\eta\psi)d\xi d\eta. \tag{4.1}$$

Now we take the fundamental solution $U(x,y;\xi,\eta)$ of equation $u_{xxx}-u_{yy}=0$, as the function φ. As the function of (ξ,η) $U(x,y;\xi,\eta)$ satisfies

$$L^*[U] \equiv -U_{\xi\xi\xi} - U_{\eta\eta} = 0$$

at $(x,y) \neq (\xi,\eta)$. As the function ψ, we take any regular solution $u(\xi,\eta)$ of the equation $u_{xxx} - u_{yy} = f(x,y)$. Observing that $U_\eta(x,y;\xi,\eta)$ has a singularity at $y=\eta$, we divide the domain D into two domains: $D = \lim\limits_{\varepsilon\to 0}(D_1^\varepsilon \cup D_2^\varepsilon)$ where

$$D_1^\varepsilon = \{(\xi,\eta):\ 0<\xi<p,\ 0<\eta<y-\varepsilon\},$$
$$D_2^\varepsilon = \{(\xi,\eta):\ 0<\xi<p,\ ,y+\varepsilon<\eta<l\}.$$

Then the identity (4.1) gets the form

$$\iint\limits_D U(x,y;\xi,\eta)f(\xi,\eta)\,d\xi d\eta$$
$$= \lim_{\varepsilon\to 0+}\int_0^p\int_0^{y-\varepsilon}\frac{\partial}{\partial\xi}(Uu_{\xi\xi} - U_\xi u_\xi + U_{\xi\xi}u)d\xi d\eta$$
$$+ \lim_{\varepsilon\to 0+}\int_0^p\int_{y+\varepsilon}^{l}\frac{\partial}{\partial\xi}(Uu_{\xi\xi} - U_\xi u_\xi + U_{\xi\xi}u)d\xi d\eta$$
$$- \lim_{\varepsilon\to 0+}\int_0^p\int_0^{y-\varepsilon}\frac{\partial}{\partial\eta}(Uu_\eta - U_\eta u)\,d\xi d\eta - \lim_{\varepsilon\to 0+}\int_0^p\int_{y+\varepsilon}^{l}\frac{\partial}{\partial\eta}(Uu_\eta - U_\eta u)\,d\xi d\eta$$
$$= \lim_{\varepsilon\to 0+}\int_0^{y-\varepsilon}(Uu_{\xi\xi} - U_\xi u_\xi + U_{\xi\xi}u)\big|_{\xi=0}^{\xi=p}d\eta + \lim_{\varepsilon\to 0+}\int_{y+\varepsilon}^{l}(Uu_{\xi\xi} - U_\xi u_\xi + U_{\xi\xi}u)\big|_{\xi=0}^{\xi=p}d\eta$$
$$- \lim_{\varepsilon\to 0+}\int_0^p(Uu_\eta - U_\eta u)\Big|_{\eta=0}^{\eta=y-\xi}d\xi - \lim_{\varepsilon\to 0+}\int_0^p(Uu_\eta - U_\eta u)\Big|_{\eta=y+\varepsilon}^{\eta=l}d\xi$$
$$= \int_0^y(Uu_{\xi\xi} - U_\xi u_\xi + U_{\xi\xi}u)\big|_{\xi=0}^{\xi=p}d\eta + \int_y^l(Uu_{\xi\xi} - U_\xi u_\xi + U_{\xi\xi}u)\big|_{\xi=0}^{\xi=p}d\eta$$
$$- \lim_{\varepsilon\to 0+}\int_0^p\left[U(x,y;\xi,y-\varepsilon)u_\eta(\xi,y-\varepsilon) - U(x,y;\xi,0)u_\eta(\xi,0)\right]d\xi$$
$$+ \lim_{\varepsilon\to 0+}\int_0^p\left[U_\eta(x,y;\xi,y-\varepsilon)u(\xi,y-\varepsilon) - U_\eta(x,y;\xi,0)u(\xi,0)\right]d\xi$$

$$- \lim_{\varepsilon \to 0+} \int_0^p \left[U(x,y;\xi,l)\, u_\eta(\xi,l) - U(x,y;\xi,y+\varepsilon)\, u_\eta(\xi,y+\varepsilon) \right] d\xi$$

$$+ \lim_{\varepsilon \to 0+} \int_0^p \left[U_\eta(x,y;\xi,l)\, u(\xi,l) - U_\eta(x,y;\xi,y+\varepsilon)\, u(\xi,y+\varepsilon) \right] d\xi$$

$$= \int_0^l \left(U u_{\xi\xi} - U_\xi u_\xi + U_{\xi\xi} u \right) \Big|_{\xi=0}^{\xi=p} d\eta$$

$$- \int_0^p \left[U(x,y;\xi,l)\, u_\eta(\xi,l) - U(x,y;\xi,0)\, u_\eta(\xi,0) \right] d\xi$$

$$+ \int_0^p \left[U_\eta(x,y;\xi,l)\, u(\xi,l) - U_\eta(x,y;\xi,0)\, u(\xi,0) \right] d\xi$$

$$+ \lim_{\varepsilon \to 0+} \int_0^p U_\eta(x,y;\xi,y-\varepsilon)\, u(\xi,y-\varepsilon)\, d\xi$$

$$- \lim_{\varepsilon \to 0+} \int_0^p U_\eta(x,y;\xi,y+\varepsilon)\, u(\xi,y+\varepsilon)\, d\xi.$$

Simplifying this expression, we obtain

$$\begin{aligned} \iint_D U(x,y;\xi,\eta) f(\xi,\eta)\, d\xi d\eta &= \int_0^l \left[U u_{\xi\xi} - U_\xi u_\xi + U_{\xi\xi} u \right] \Big|_{\xi=0}^{\xi=p} d\eta \\ &\quad - \int_0^p U(x,y;\xi,\eta) u_\eta(\xi,\eta) \Big|_{\eta=0}^{\eta=l} d\xi \\ &\quad + \int_0^p U_\eta(x,y;\xi,\eta) u(\xi,\eta) \Big|_{\eta=0}^{\eta=l} d\xi \\ &\quad + \lim_{\varepsilon \to 0+} \int_0^p U_\eta(x,y;\xi,y-\varepsilon)\, u(\xi,y-\varepsilon)\, d\xi \\ &\quad - \lim_{\varepsilon \to 0+} \int_0^p U_\eta(x,y;\xi,y+\varepsilon)\, u(\xi,y+\varepsilon)\, d\xi. \end{aligned} \tag{4.2}$$

Considering Theorem 3 in [13], we obtain from (4.2)

$$\iint\limits_D U(x,y;\xi,\eta)f(\xi,\eta)\,d\xi d\eta$$

$$= \int\limits_0^l \left(Uu_{\xi\xi} - U_\xi u_\xi + U_{\xi\xi}u\right)|_{\xi=0}^{\xi=p} d\eta$$

$$- \int\limits_0^p U(x,y;\xi,\eta)u_\eta(\xi,\eta)|_{\eta=0}^{\eta=l} d\xi + \int\limits_0^p U_\eta(x,y;\xi,\eta)u(\xi,\eta)|_{\eta=0}^{\eta=l} d\xi - 2u(x,y).$$

Hence, we have finally

$$2u(x,y) = \int\limits_0^l \left(Uu_{\xi\xi} - U_\xi u_\xi + U_{\xi\xi}u\right)|_{\xi=0}^{\xi=p} d\eta - \int\limits_0^p \left(Uu_\eta - U_\eta u\right)|_{\eta=0}^{\eta=l} d\xi$$
$$- \iint\limits_D U(x,y;\xi,\eta)f(\xi,\eta)\,d\xi d\eta. \tag{4.3}$$

Let now $W(x,y,\xi,\eta)$ be any regular solution of the equation $L^*[u]=0$, and $u(x,y)$ be any regular solution of the equation $u_{xxx} - u_{yy} = f(x,y)$. Then assuming in (4.1) $\varphi = W(x,y;\xi,\eta)$, $\psi = u(\xi,\eta)$, we have

$$0 = \int\limits_0^l \left(Wu_{\xi\xi} - W_\xi u_\xi + W_{\xi\xi}u\right)|_{\xi=0}^{\xi=p} d\eta - \int\limits_0^p \left(Wu_\eta - W_\eta u\right)|_{\eta=0}^{\eta=l} d\xi$$
$$- \iint\limits_D W(x,y;\xi,\eta)f(\xi,\eta)\,d\xi d\eta; \tag{4.4}$$

$$2u(x,y) = \int\limits_0^l \left(Gu_{\xi\xi} - G_\xi u_\xi + G_{\xi\xi}u\right)|_{\xi=0}^{\xi=p} d\eta - \int\limits_0^p \left(Gu_\eta - G_\eta u\right)|_{\eta=0}^{\eta=l} d\xi$$
$$- \iint\limits_D G(x,y;\xi,\eta)f(\xi,\eta)\,d\xi d\eta. \tag{4.5}$$

where

$$G(x,y;\xi,\eta) = U(x,y;\xi,\eta) - W(x,y;\xi,\eta).$$

Now we construct the function $G(x,y;\xi,\eta)$ which for (x,y) must have the following properties: as a function of $(x,y) \neq (\xi,\eta)$

$$\begin{cases} L[G] = 0, \\ G(x,0;\xi,\eta) = G(x,l;\xi,\eta) = 0, \\ G(0,y;\xi,\eta) = G(p,y;\xi,\eta) = G_x(p,y;\xi,\eta) = 0, \end{cases} \tag{4.6}$$

as a function of (ξ, η) :

$$\begin{cases} L^*[G] = 0, \\ G(x, y; \xi, 0) = G(x, y; \xi, l) = 0, \\ G(x, y; 0, \eta) = G(x, y; p, \eta) = G_\xi(x, y; 0, \eta) = 0. \end{cases} \tag{4.7}$$

For this purpose we solve the following auxiliary problem.

Problem A$_1$. *Find the regular solution of the equation* (1.1) *satisfying the boundary conditions:*

$$u(x, 0) = 0, \quad u(x, l) = 0, \quad 0 < x < p, \tag{4.8}$$

$$u(0, y) = u(p, y) = u'_x(p, y) = 0, \quad 0 < y < l. \tag{4.9}$$

We will seek the solution of the stated problem in the form of (see [14])

$$u(x, y) = \sum_{k=1}^{\infty} X_k(x) \sin \frac{k\pi}{l} y \tag{4.10}$$

The function $f(x, y)$ can be decomposed with respect to the particular system $\left\{\sin \frac{k\pi}{l} y\right\}$ of trigonometric functions as

$$f(x, y) = \sum_{k=0}^{\infty} f_k(x) \sin \frac{k\pi}{l} y \tag{4.11}$$

where

$$f_k(x) = \frac{2}{l} \int_0^l f(x, y) \sin \frac{k\pi}{l} y dy.$$

Substituting (4.10), (4.11) into (1.1), we obtain

$$\sum_{k=0}^{\infty} \left(X_k'''(x) + \lambda_k^3 X_k(x) - f_k(x) \right) \sin \frac{k\pi}{l} y = 0$$

To find the function $X_k(x)$, we obtain the following problem

$$\begin{cases} L[X_k] = X_k'''(x) + \lambda_k^3 X_k(x) = f_k(x) \\ X_k(0) = X_k(p) = X'_k(p) = 0 \end{cases} \tag{4.12}$$

where

$$\lambda_k^3 = \left(\frac{k\pi}{l} \right)^2.$$

We look for the solution of problem (4.12). Using the method of constructing Green's function [15] which has following characteristic properties:

1. $G_k(x,\xi)$ is continuous and has continuous derivatives for $0 \le x \le p$;
2. Its second-order derivative with respect to x has a jump discontinuity at $x=\xi$ being equal to 1, i.e.,

$$\left.\frac{\partial^2 G_k(x,\xi)}{\partial x^2}\right|_{x=\xi+0} - \left.\frac{\partial^2 G_k(x,\xi)}{\partial x^2}\right|_{x=\xi-0} = 1.$$

3. In each of the intervals $0 \le x \le \xi$ and $\xi \le x \le p$, the following function $G_k(x,\xi)$, considered as a function of x, is the solution of the equation

$$L[G_k] = \frac{\partial^3 G_k}{\partial x^3} + \lambda_k^3 G_k = 0.$$

4. The following $G_k(0,\xi) = G_k(p,\xi) = G_{kx}(p,\xi) = 0.$

We construct the Green function. So linearly independent solutions of the equation

$$X_k''' + \lambda_k^3 X_k = 0$$

have the form

$$X_1 = e^{-\lambda_k x}, \quad X_2 = e^{\frac{\lambda_k}{2}x}\cos\frac{\sqrt{3}}{2}\lambda_k x, \quad X_3 = e^{\frac{\lambda_k}{2}x}\sin\frac{\sqrt{3}}{2}\lambda_k x.$$

We represent the Green function to be sought in the form

$$G_k(x,\xi) = \begin{cases} a_1 e^{-\lambda_k x} + a_2 e^{\frac{\lambda_k}{2}x}\cos\frac{\sqrt{3}}{2}\lambda_k x + a_3 e^{\frac{\lambda_k}{2}x}\sin\frac{\sqrt{3}}{2}\lambda_k x, & 0 \le x \le \xi; \\ b_1 e^{-\lambda_k x} + b_2 e^{\frac{\lambda_k}{2}x}\cos\frac{\sqrt{3}}{2}\lambda_k x + b_3 e^{\frac{\lambda_k}{2}x}\sin\frac{\sqrt{3}}{2}\lambda_k x, & \xi \le x \le p \end{cases} \tag{4.13}$$

where $a_1, a_2, a_3, b_1, b_2, b_3$ are yet unknown functions of ξ.

Using Properties 1) and 2) of the Green function and substituting $c_k(\xi) = b_k(\xi) - a_k(\xi)$, $k = 1,2,3$, we obtain the system of linear equations for finding the function $c_k(\xi)$:

$$\begin{cases} c_1 e^{-\lambda_k \xi} + c_2 e^{\frac{\lambda_k}{2}\xi}\cos\frac{\sqrt{3}}{2}\lambda_k \xi + c_3 e^{\frac{\lambda_k}{2}\xi}\sin\frac{\sqrt{3}}{2}\lambda_k \xi = 0, \\ -c_1 e^{-\lambda_k \xi} + c_2 e^{\frac{\lambda_k}{2}\xi}\cos\left(\frac{\sqrt{3}}{2}\lambda_k \xi + \frac{\pi}{3}\right) + c_3 e^{\frac{\lambda_k}{2}\xi}\sin\left(\frac{\sqrt{3}}{2}\lambda_k \xi + \frac{\pi}{3}\right) = 0, \\ c_1 e^{-\lambda_k \xi} + c_2 e^{\frac{\lambda_k}{2}\xi}\cos\left(\frac{\sqrt{3}}{2}\lambda_k \xi + \frac{2\pi}{3}\right) + c_3 e^{\frac{\lambda_k}{2}\xi}\sin\left(\frac{\sqrt{3}}{2}\lambda_k \xi + \frac{2\pi}{3}\right) = \frac{1}{\lambda_k^2}. \end{cases}$$

The determinant of this system is equal to the value of the Wronskian $W(X_1, X_2, X_3)$ at the point $x = \xi$, therefore it is different from zero and is equal to $\frac{3\sqrt{3}}{2}$. Calculating Δ_{c_1}, $i = 1, 2, 3$, we find:

$$c_1(\xi) = \frac{e^{\lambda_k \xi}}{3\lambda_k^2}, \quad c_2(\xi) = -\frac{2e^{-\frac{\lambda_k}{2}\xi} \sin\left(\frac{\sqrt{3}}{2}\lambda_k \xi + \frac{\pi}{6}\right)}{3\lambda_k^2},$$

$$c_3(\xi) = \frac{2e^{-\frac{\lambda_k}{2}\xi} \cos\left(\frac{\sqrt{3}}{2}\lambda_k \xi + \frac{\pi}{6}\right)}{3\lambda_k^2}.$$

Next, we use Property 4) of Green's function. In our case, these relations will take the form

$$\begin{cases} b_1 + b_2 = \dfrac{1}{3\lambda_k^2}\left(e^{\lambda_k \xi} - 2e^{-\frac{\lambda_k}{2}\xi} \sin\left(\dfrac{\sqrt{3}}{2}\lambda_k \xi + \dfrac{\pi}{6}\right)\right), \\ b_1 e^{-\lambda_k p} + b_2 e^{\frac{\lambda_k}{2}p} \cos \dfrac{\sqrt{3}}{2}\lambda_k p + b_3 e^{\frac{\lambda_k}{2}p} \sin \dfrac{\sqrt{3}}{2}\lambda_k p = 0, \\ -b_1 e^{-\lambda_k p} + b_2 e^{\frac{\lambda_k}{2}p} \cos\left(\dfrac{\sqrt{3}}{2}\lambda_k p + \dfrac{\pi}{3}\right) + b_3 e^{\frac{\lambda_k}{2}p} \sin\left(\dfrac{\sqrt{3}}{2}\lambda_k p + \dfrac{\pi}{3}\right) = 0. \end{cases}$$

Because $X_1(0)$, $X_2(l)$, $X_3'(l)$, are linearly independent the determinant of this system is different from zero:

$$\Delta = \frac{\sqrt{3}}{2}\left(e^{\lambda_k p} - 2e^{-\frac{\lambda_k}{2}p} \sin\left(\frac{\sqrt{3}}{2}\lambda_k p + \frac{\pi}{6}\right)\right) \neq 0.$$

Calculating Δ_{b_i}, $i = 1, 2, 3$, we find

$$b_1 = \frac{e^{\lambda_k \xi} - 2e^{-\frac{\lambda_k}{2}\xi} \sin\left(\frac{\sqrt{3}}{2}\lambda_k \xi + \frac{\pi}{6}\right)}{3\lambda_k^2\left(1 - 2e^{-\frac{3}{2}\lambda_k p} \sin\left(\frac{\sqrt{3}}{2}\lambda_k p + \frac{\pi}{6}\right)\right)},$$

$$b_2 = -\frac{2e^{-\frac{3}{2}\lambda_k p}\left(e^{\lambda_k \xi} - 2e^{-\frac{\lambda_k}{2}\xi} \sin\left(\frac{\sqrt{3}}{2}\lambda_k \xi + \frac{\pi}{6}\right)\right) \sin\left(\frac{\sqrt{3}}{2}\lambda_k p + \frac{\pi}{6}\right)}{3\lambda_k^2\left(1 - 2e^{-\frac{3}{2}\lambda_k p} \sin\left(\frac{\sqrt{3}}{2}\lambda_k p + \frac{\pi}{6}\right)\right)},$$

$$b_3 = \frac{2e^{-\frac{3}{2}\lambda_k p}\left(e^{\lambda_k \xi} - 2e^{-\frac{\lambda_k}{2}\xi} \sin\left(\frac{\sqrt{3}}{2}\lambda_k \xi + \frac{\pi}{6}\right)\right) \cos\left(\frac{\sqrt{3}}{2}\lambda_k p + \frac{\pi}{6}\right)}{3\lambda_k^2\left(1 - 2e^{-\frac{3}{2}\lambda_k p} \sin\left(\frac{\sqrt{3}}{2}\lambda_k p + \frac{\pi}{6}\right)\right)}.$$

Taking into account that $a_k(\xi) = b_k(\xi) - c_k(\xi)$, $k = 1, 2, 3$, we find a_k, $k = 1, 2, 3$:

$$a_1 = \frac{2e^{-\lambda_k\left(\frac{3}{2}p-\xi\right)} \sin\left(\frac{\sqrt{3}}{2}\lambda_k p + \frac{\pi}{6}\right) - 2e^{-\frac{\lambda_k}{2}\xi} \sin\left(\frac{\sqrt{3}}{2}\lambda_k \xi + \frac{\pi}{6}\right)}{3\lambda_k^2\left(1 - 2e^{-\frac{3}{2}\lambda_k p} \sin\left(\frac{\sqrt{3}}{2}\lambda_k p + \frac{\pi}{6}\right)\right)},$$

$$a_2 = -a_1 = -\frac{2e^{-\lambda_k\left(\frac{3}{2}p-\xi\right)} \sin\left(\frac{\sqrt{3}}{2}\lambda_k p + \frac{\pi}{6}\right) - 2e^{-\frac{\lambda_k}{2}\xi} \sin\left(\frac{\sqrt{3}}{2}\lambda_k \xi + \frac{\pi}{6}\right)}{3\lambda_k^2\left(1 - 2e^{-\frac{3}{2}\lambda_k p} \sin\left(\frac{\sqrt{3}}{2}\lambda_k p + \frac{\pi}{6}\right)\right)},$$

$$a_3 = \frac{2e^{-\lambda_k\left(\frac{3}{2}p-\xi\right)} \cos\left(\frac{\sqrt{3}}{2}\lambda_k p + \frac{\pi}{6}\right) - 2e^{-\frac{\lambda_k}{2}\xi} cos\left(\frac{\sqrt{3}}{2}\lambda_k \xi + \frac{\pi}{6}\right)}{3\lambda_k^2\left(1 - 2e^{-\frac{3}{2}\lambda_k p} \sin\left(\frac{\sqrt{3}}{2}\lambda_k p + \frac{\pi}{6}\right)\right)}$$
$$+ \frac{4e^{-\frac{\lambda_k}{2}(3p+\xi)} \sin\frac{\sqrt{3}}{2}\lambda_k(p-\xi)}{3\lambda_k^2\left(1 - 2e^{-\frac{3}{2}\lambda_k p} \sin\left(\frac{\sqrt{3}}{2}\lambda_k p + \frac{\pi}{6}\right)\right)}.$$

Substituting the obtained values into (4.13), we obtain the function $G_k(x, \xi)$ in the form:

$$\begin{aligned} G_k(x,\xi) = \frac{1}{\Delta}\Bigg\{ & 2e^{-\lambda_k\left(\frac{3}{2}p+x-\xi\right)} \sin\left(\frac{\sqrt{3}}{2}\lambda_k p + \frac{\pi}{6}\right) \\ & - 2e^{-\frac{\lambda_k}{2}(2x+\xi)} \sin\left(\frac{\sqrt{3}}{2}\lambda_k \xi + \frac{\pi}{6}\right) \\ & - 2e^{-\lambda_k\left(\frac{3}{2}p-\xi-\frac{x}{2}\right)} \sin\left[\frac{\sqrt{3}}{2}\lambda_k(p-x) + \frac{\pi}{6}\right] \\ & + 2e^{-\frac{\lambda_k}{2}(\xi-x)} \sin\left[\frac{\sqrt{3}}{2}\lambda_k(\xi-x) + \frac{\pi}{6}\right] \\ & + 4e^{-\frac{\lambda_k}{2}(3p+\xi-x)} \sin\left[\frac{\sqrt{3}}{2}\lambda_k(p-\xi)\right] \sin\frac{\sqrt{3}}{2}\lambda_k x \Bigg\}, \\ & 0 \leqslant x \leqslant \xi; \end{aligned} \tag{4.14a}$$

$$G_k(x,\xi) = \frac{1}{\overline{\Delta}}\left\{-2e^{-\frac{\lambda_k}{2}(2x+\xi)}\sin\left(\frac{\sqrt{3}}{2}\lambda_k\xi+\frac{\pi}{6}\right)\right. \tag{4.14b}$$

$$-2e^{-\lambda_k\left(\frac{3}{2}p-\xi-\frac{x}{2}\right)}\sin\left[\frac{\sqrt{3}}{2}\lambda_k(p-x)+\frac{\pi}{6}\right]+e^{-\lambda_k(x-\xi)}$$

$$\left.+4e^{-\frac{\lambda_k}{2}(3p+\xi-x)}\sin\left[\frac{\sqrt{3}}{2}\lambda_k(p-x)+\frac{\pi}{6}\right]\sin\left(\frac{\sqrt{3}}{2}\lambda_k\xi+\frac{\pi}{6}\right)\right\},$$

$$\xi \leqslant x \leqslant p$$

where

$$\overline{\Delta} = 3\lambda_k^2\left(1-2e^{-\frac{3}{2}\lambda_k p}\sin\left(\frac{\sqrt{3}}{2}\lambda_k p+\frac{\pi}{6}\right)\right).$$

It is easy to verify that the function determined by formula (4.14) possesses all the properties formulated for the Green function. Thus the Green function has been constructed, hence, the solution of the problem A_1 has the form

$$X_k(x) = \int_0^p G_k(x,\xi)\, f_k(\xi)d\xi, \tag{4.15}$$

Then by the formula (4.10), taking into account (4.15), the solution of problem A_1 becomes the form

$$u(x,y) = \sum_{k=1}^{\infty}\int_0^p G_k(x,\xi)f_k(\xi)\,d\xi\sin\frac{\pi k}{l}y = \int_0^p\sum_{k=1}^{\infty}G_k(x,\xi)\sin\frac{\pi k y}{l}f_k(\xi)\,d\xi. \tag{4.16}$$

If the function $u(x,y)$ and its derivations u_{xxx}, u_{yy} converge uniformly in D, then the function $u(x,y)$ gives the solution of problem A_1. We estimate the function (4.16) as:

$$|u(x,y)| \leqslant \left|\int_0^p\sum_{k=1}^{\infty}G_k(x,\xi)\sin\frac{\pi k y}{l}f_k(\xi)\,d\xi\right| \tag{4.17}$$

$$\leqslant \int_0^p\sum_{k=1}^{\infty}|G_k(x,\xi)|\left|\sin\frac{\pi k}{l}y\right||f_k(\xi)|\,d\xi \leqslant \int_0^p\sum_{k=1}^{\infty}|G_k(x,\xi)|\,|f_k(\xi)|\,d\xi.$$

Under the assumptions stated above, the following inequality [16] is valid for the function $f(x,y)$:

$$|f_k(\xi)| \leqslant \frac{M_1}{k^2}, \quad M_1 = const > 0,$$

since $f_k(\xi)$ are the Fourier coefficients of $f(x,y)$ in the segment $(0,l)$.

Taking this into account, (4.17) can be rewritten as:

$$|u(x,y)| \leqslant \int_0^p \sum_{k=1}^{\infty} |G_k(x,\xi)|\,|f_k(\xi)|\,d\xi \leqslant M_1 \int_0^p \sum_{k=1}^{\infty} \frac{1}{k^2}\ |G_k(x,\xi)|\,d\xi. \tag{4.18}$$

Estimating $G_k(x,\xi)$, we find from (4.14):

$$|G_k(x,\xi)| \leqslant \begin{cases} \dfrac{10}{3}\dfrac{e^{-\frac{3}{2}\lambda_k p}}{\lambda_k^2} + \dfrac{2}{3}\dfrac{e^{-\frac{1}{2}\lambda_k \delta_1}}{\lambda_k^2}, & 0 \leqslant x < \xi, \quad 0 < \delta_1 < \xi - x, \\ \dfrac{8}{3}\dfrac{e^{-\frac{3}{2}\lambda_k p}}{\lambda_k^2} + \dfrac{1}{3}\dfrac{e^{-\frac{1}{2}\lambda_k \delta_2}}{\lambda_k^2}, & \xi < x \leqslant l, \quad 0 < \delta_2 < x - \xi, \end{cases}$$

or

$$|G_k(x,\xi)| \leqslant \frac{10}{3}\frac{e^{-\frac{3}{2}\lambda_k p}}{\lambda_k^2} + \frac{2}{3}\frac{e^{-\frac{1}{2}\lambda_k \delta}}{\lambda_k^2} = M_2 k^{-\frac{4}{3}}. \tag{4.19}$$

Then we obtain from (4.18)

$$|u(x,y)| \leqslant\ M_3 k^{-\frac{10}{3}}.$$

Hence the series (4.16) converges uniformly. Next we show that the series of the derivatives u_{xxx} converges uniformly. We have

$$\frac{\partial^3 u(x,y)}{\partial x^3} = \int_0^p \sum_{k=1}^{\infty} \frac{\partial^3}{\partial x^3} G_k(x,\xi) f_k(\xi)\,d\xi = \int_0^p \sum_{k=1}^{\infty} \lambda_k^3 G_k(x,\xi)\, f_k(\xi)\ d\xi, \tag{4.20}$$

$$\left|\frac{\partial^3 u(x,y)}{\partial x^3}\right| \leqslant \int_0^p \sum_{k=1}^{\infty} \left|\lambda_k^3 G_k(x,\xi)\right| |f_k(\xi)|\,d\xi \leqslant M_4 \int_0^p \sum_{k=1}^{\infty} \frac{1}{k^2}\ \left|\lambda_k^3 G_k(x,\xi)\right| d\xi, \tag{4.21}$$

hence,

$$\left|\lambda_k^3 G_k(x,\xi)\right| \leqslant \frac{10}{3}\lambda_k e^{-\frac{3}{2}\lambda_k p} + \frac{2}{3}\lambda_k e^{-\frac{1}{2}\lambda_k \delta} \leqslant M_5 k^{\frac{2}{3}}$$

and we have from (4.21)

$$\left|\frac{\partial^3 u(x,y)}{\partial x^3}\right| \leq M_6 k^{-\frac{4}{3}}, \qquad M_i = \text{const} \geq 0, \quad i = \overline{1,6}.$$

We obtain that the series (4.20) converges uniformly. Since

$$\frac{\partial^2 u(x,y)}{\partial y^2} = \frac{\partial^3 u(x,y)}{\partial x^3},$$

the uniform convergence of the derivatives $\dfrac{\partial^2 u}{\partial y^2}$ is also proven.

That is, it is possible to differentiate the series (4.16) term by term, what is necessary to satisfy the equation (1.1). Change of the order of summation and integration is always valid, since the series under the integral (4.16) converges with respect to ξ.

Replacing $f_k(\xi)$ with their values in the solution (4.16), we obtain the final solution of the auxiliary problem A_1 in the form

$$u(x,y) = \int_0^p \sum_{k=1}^{\infty} G_k(x,\xi) \sin \frac{\pi k y}{l} f_k(\xi)\, d\xi$$
$$= \frac{2}{l} \int_0^p \sum_{k=1}^{\infty} G_k(x,\xi) \int_0^l f(\xi,\eta) \sin \frac{\pi k}{l}\eta \sin \frac{\pi k}{l} y d\eta d\xi$$
$$= \int_0^p \int_0^l f(\xi,\eta) \frac{2}{l} \sum_{k=1}^{\infty} G_k(x,\xi) \sin \frac{\pi k}{l}\eta \, \sin \frac{\pi k}{l} y \, d\xi d\eta$$
$$= \int_0^p \int_0^l G(x,\xi,y,\eta) f(\xi,\eta)\, d\xi\, d\eta$$

where

$$G(x,\xi,y,\eta) = \frac{2}{l} \sum_{k=1}^{\infty} G_k(x,\xi) \; \sin \frac{\pi k}{l}\eta \; \sin \frac{\pi k}{l} y. \tag{4.22}$$

It is easy to be sure that the function $G(x,\xi,y,\eta)$ satisfies all the conditions of the problems (4.6) and (4.7).

The function (4.22) is the Green function of the first boundary problem for the domain D. Convergence of the series (4.22) follows from the estimate (4.17) for the function $G_k(x,\xi)$ at $x \neq \xi$. Taking the boundary conditions (4.6), (4.7) for the function $G(x,\xi,y,\eta)$ and the boundary conditions (2.2), (3.1) into account from (4.4) the solution of problem A is attained in the explicit form:

$$2u(x,y) = \int_0^l G_{\xi\xi}(x,y,p,\eta)\, \psi_2(\eta)\, d\eta - \int_0^l G_{\xi\xi}(x,y,0,\eta)\, \psi_1(\eta)\, d\eta$$
$$- \int_0^l G_{\xi}(x,y,p,\eta)\, \psi_3(\eta)\, d\eta + \int_0^p G_{\eta}(x,y,\xi,l)\, \varphi_2(\xi)\, d\xi$$
$$- \int_0^p G_{\eta}(x,y,\xi,0)\, \varphi_1(\xi)\, d\xi - \iint_D G(x,y,\xi,\eta)\, f(\xi,\eta)\, d\xi d\eta. \tag{4.23}$$

Eventually we have gained the solution in explicit form. Thereby we have proved the following result.

Theorem 2. *Let* $\varphi_i(x) \in C[0,p]$, $i = 1,2$, $\psi_j(y) \in C[0,l]$, $j = \overline{1,3}$, $f(x,y) \in C^{0,2}_{x,y}\left(\overline{D}\right)$, *and the condition of convergence is valid. Then the solution of problem A has the form* (4.23) *where the Green function* $G(x,\xi,y,\eta)$ *is determined by the formula* (4.22).

References

[1] H. Block, *Sur les équations linéaires aux dérivées partielles a caractéristiques multiples*, Note 1, Ark. Mat. Astron. Fys. 1912, **7**, No. 13, 1–34; Note 2, 1913, ibid. 7, No. 21, 1–30; Note 3, 1913, ibid. 8, No. 23, 1–51.

[2] E. Del Vecchio, *Sulle equazioni* $Z_{xxx} - Z_y + \varphi_1(x, y) = 0$, $Z_{xxx} - Z_{yy} + \varphi_2(x, y) = 0$. Memorie R. Accad. Sci. Torino. Ser. 2. 1915, **66**, 1–41.

[3] E. Del Vecchio, *Sur deux problèmes d'intégration pour les équations paraboliques* $Z_{\xi\xi\xi} - Z_\eta = 0$, $Z_{\xi\xi\xi} - Z_{\eta\eta} = 0$. Ark. Mat. Astron. Fys. 1916, **11**, p. 32–43.

[4] L. Cattabriga, *Potenziali di linea e di dominio per equazioni non paraboliche in due variabili a caratteristiche multiple.* Rendiconti del seminario matimatico della univ. di Padova. 1961, vol. 31, 1–45.

[5] S. Abdinazarov, *On one third-order equation.* Izv. AN UzSSR, ser. fiz.-mat. nauk, 1989, **6**, 3–6 (in Russian).

[6] A.R. Khashimov, *On one problem for the equation of the mixed type with multiple characteristics.* Uzbek Math. Journal, 1995, **2**, 93–97 (in Russian).

[7] O.S. Ryzhov, *Asymptotical portrait for flow of rotation bodies with the stream of viscous and head conducting gas.* Prikladnaya matematika i mekhanika, 1952, **2**(6), 1004–1014 (in Russian).

[8] V.N. Diesperov, *On the Green function for the linearized viscous transonic equation.* Jurnal vychislitelnoy matematiki i matematicheskoy fiziki, 1972, **12**(5), 1265–1279 (in Russian).

[9] T.Dj. Djuraev, Yu.P. Apakov, *On the automodel solution of one third-order equation with multiple characteristics.* Herald of the Samara State Technical University, series of Physical and Mathematical Sciences, 2007, **2**(15), 18–26 (in Russian).

[10] H. Bateman, A. Erdelyi, *Higher Transcendental Functions.* Vol. 1. "Nauka", Moscow, 1973, 296 p. (in Russian).

[11] Yu. Irgashev, Yu.P. Apakov, *The first boundary value problem for the third-order equation of pseudoelliptic type.* Uzbek Math. Journal, 2006, **2**, 44–51 (in Russian).

[12] Yu.P. Apakov, *On the solution of one boundary-value problem for a non homogenous equation of the* 3*rd order.* Thesis of the International Conference "Differential Equations, Theory of Functions and Applications", 2007, 28 May–2 June. Novosibirsk, 65–66 (in Russian).

[13] T.Dj. Djurayev, Yu.P. Apakov, *To the theories of the equation of third order with multiple characteristics containing the second derivative on a time.* Ukrainian mathematical journal. Kiev, 2010, **62**, No. 1, 40–51 (in Russian).

[14] A.N. Tikhonov, A.A. Samarskiy, *Equation of Mathematical Physics.* "Nauka", Moskov, 1977, 735 p. (in Russian).

[15] M.L. Krasnov, A.I. Kiselyev, G.I. Makarenko, *Integral Equations.* The 2nd edition. "Nauka", Moscow, 1976, 216 p. (in Russian).

[16] L.D. Kudryavtsev, *Kurs Matematicheskogo Analiza* (*Calculus*). Vol. 3. The 2nd edition. "Visshaya Shkola", Moscow. 1989, 352 p. (in Russian).

Yusupjon P. Apakov
Namangan Engineering-Pedagogical Institute, Namangan, Uzbekistan
e-mail: apakov.1956@mail.ru

Operator Theory:
Advances and Applications, Vol. 216, 79–91

On Stability and Trace Regularity of Solutions to Reissner-Mindlin-Timoshenko Equations

George Avalos and Daniel Toundykov

Communicated by F. Bucci and I. Lasiecka

Abstract. Uniform stability of Reissner-Mindlin-Timoshenko (RMT) plates is addressed. Similarly to waves, Kirchhoff plates, and elastodynamics, boundary stabilization of the RMT model relies on an observability inequality, which in turn necessitates the derivation of certain trace regularity estimates. The exponential stability of RMT plates has been quoted for many years, yet, to the best of our knowledge, a detailed analysis of a requisite trace regularity result does not appear to exist in the literature. The purpose of this note is to provide such details.

Mathematics Subject Classification (2000). Primary: 74K20, 93B07; Secondary: 93D15.

Keywords. Reisser-Mindlin plate, Mindlin-Timoshenko plate, boundary damping, Neumann feedback, stability, trace regularity.

1. Introduction

In the past few years there has been an increased interest in Reissner-Mindlin-Timoshenko (RMT) plate equations, inasmuch as they provide a more accurate description of flexural vibrations of thin elastic plates (vis-a-vis Kirchhoff plate models). In addition to a large body of results on applications of the finite element method to this system – a topic beyond the scope of this article – a number of new analytic developments have emerged.

For modeling and variational framework see the articles by R. Paroni, P. Podio-Guidugli, and G. Tomassetti [PPGT06, PPGT07]; M. Pedersen [Ped07c,

The research of George Avalos was partially supported by the National Science Foundation under Grants DMS-0606776 and DMS-0908476.
The research of Daniel Toundykov was partially supported by the National Science Foundation under Grant DMS-0908270.

Ped08]; V. Rensburg, L. Zietsman and Merve [vRZvdM09]. Coupled PDE dynamics was discussed by C. Giorgi and M. Naso [GN06] (RMT with thermal effects), and M. Grobbelaar [Dal06, Dal08] (coupling with acoustic and thermal dynamics).

I. Chueshov and I. Lasiecka studied global exponential attractor for the system with full interior damping [CL06]. Boundary controllability of RMT plates was addressed by M. Pedersen in [Ped07a, Ped07b]; C. Giorgi, F. Vegni [GV07] proved uniform stability of a viscoelastic model with exponentially decaying memory kernels.

The focus of the discussion below will be on the uniform stability of the RMT system subject to *boundary feedback controls*. In this area the known results are somewhat less comprehensive. S. Fernández and D. Hugo [FS09] proved (strong) non-exponential stability, when boundary feedbacks act only on the filament angles of the state vector. Uniform stability of a (3D) structural acoustics model with an interface on an RMT plate is treated in an upcoming paper by the authors [AT10].

While an analytic approach to exponential stability of linear RMT plates was presented by J. Lagnese back in 1980s [Lag89], we do not believe there exists in the literature a comprehensive analysis of the necessary trace estimates arising in the general problem of boundary stabilization for this model. The associated challenges can be circumvented by imposing additional geometric constraints on the shape of the domain. Thus, the observability result established in [Ped07b] requires the domain to be star-shaped; likewise [FS09] deals with a rectangular boundary. The result in [Lag89] does not explicitly place geometrical conditions on the boundary where the feedback is active, however, it omits the discussion of the necessary trace estimates (for example, [Lag89, equation (3.38)] involves well-defined *traces of the solutions* and cannot be justified by quoting Korn's inequality). The key role played by trace regularity in this context was first noted by I. Lasiecka and R. Triggiani in [LT92] for wave equations; analogous conclusions for plates and linear elasticity were later made by M.A. Horn [Hor98a]; in a more recent paper M. Grobbelaar [Dal06] remarked on the importance of trace regularity estimates for RMT plates. The goal of this note is to address this aspect which, as far as we are aware, has been missing from the literature to date.

1.1. Role of geometry and trace regularity in stabilization of PDE's

Boundary observability of hyperbolic PDE systems is intrinsically linked with the geometric configuration of the underlying physical domain. This phenomenon was first rigorously exhibited for wave equations in the seminal paper by C. Bardos, G. Lebeau and J. Rauch [BLR92], which proved that observability of finite-energy solutions necessarily requires all rays of geometric optics within the domain to interact with the controlled boundary. For a comprehensive overview of geometric aspects of control theory see R. Gulliver, I. Lasiecka, W. Littman, R. Triggiani [GLLT04].

However, when considering the construction of such "reverse" inequalities for second-order hyperbolic PDE's, even Neumann feedbacks acting on the *entire* *boundary* require some form of geometrical restrictions. In fact, dating from the earliest boundary controllability studies, which focused on the wave equation, the analysis relied on the star-shaped property of the underlying domain. In the course of invoking a so-called multiplier method, the key to obtaining the necessary estimates was in finding a smooth vector field $\mathfrak{h}(x)$ whose Hessian was strictly positive definite in the interior of the domain, while on the boundary $\mathfrak{h}(x) \cdot \boldsymbol{\nu}(x) > 0$, with $\boldsymbol{\nu}$ denoting the outward unit normal field. A sufficient geometric assumption would be for the domain to be star-shaped with respect to some fixed interior point x_0. Sufficiency of this condition for control and observation was first conjectured by J. Quinn and D. Russell in [QR77], and subsequently established in the work of G. Chen [Che79]; the corresponding vector field is radial and given by $\mathfrak{h}(x) = x - x_0$. Existence of fields with similar properties was applied by J. Lagnese to study uniform stability of elasticity systems, [Lag83] and thin plates [Lag89].

One would, however, expect that *full boundary* damping of Neumann type – for instance on a wave equation:

$$\begin{aligned} w_{tt}(x,t) - \Delta w(x,t) &= 0, && x \in \Omega \subset \mathbb{R}^n, \quad t \in (0,T) \\ \frac{\partial w}{\partial \boldsymbol{\nu}}(x,t) + w(x,t) &= -g(w_t(x,t)), && x \in \Gamma := \partial\Omega, \quad t \in (0,T), \end{aligned} \tag{1.1}$$

– would not necessitate any geometric restrictions, and, at least for a suitable feedback map g, would suffice to exponentially stabilize the system. However, the proof of stability in this situation requires showing that Neumann feedback also "controls" the tangential derivatives on the boundary – a conjecture whose argument has historically been a highly nontrivial challenge and which became the primary reason why the star-shaped condition was being employed even for full boundary dissipation. The fact that indeed no restrictions are actually necessary was first discovered by I. Lasiecka and R. Triggiani [LT92]. The proof required microlocal analysis and regularity theory for elliptic PDE's. Essentially the theorem showed that the normal component and the velocity feedback of the solution to (1.1) also offered control on tangential derivatives of the solution on the boundary:

$$\int_{\lambda}^{T-\lambda} \int_{\Omega} \left| \frac{\partial w}{\partial \boldsymbol{\tau}} \right|^2 dxdt \leq C_{T,\lambda} \int_0^T \int_{\Omega} \left(\left| \frac{\partial w}{\partial \boldsymbol{\nu}} \right|^2 + w_t^2 \right) dxdt + \text{l.o.t.}(w), \tag{1.2}$$

(where l.o.t. represents "lower-order terms" which essentially correspond to seminorms of w in spaces that compactly embed into the finite-energy space $H^1 \times L^2$). A need for an inequality of this type arises whenever boundary stabilization of a hyperbolic system is considered, however, because of relatively canonical appearance of the microlocal quantities involved, the proof of (1.2) does not immediately carry over to other types of PDE's, especially when a system is comprised of several coupled equations. An extension of the estimate (1.2) to linear elasticity was first carried out by M. Horn [Hor98a]; for an excellent overview of boundary trace regularity and its connection to stability see also another paper by that author [Hor98b]. For another proof and some additional details on this result for elastodynamics see [AT09].

Extensions to Kirchhoff plates were developed by I. Lasiecka and R. Triggiani in [LT93, LT00]; a substantially more challenging analog for shells was later treated by them in [LT02]. To this date, however, no versions of this argument have been available for RMT plates.

1.2. Goals and challenges

The necessity of (1.2)-type inequality for stabilization of plates was pointed out in [Hor98b], and for the RMT model in particular, by M. Grobbelaar in [Dal06]. The difficulty in attempting to follow the program outlined in [LT92] is that the system now consists of three 2nd-order coupled hyperbolic equations which entails special algebraic considerations: namely, the connection between the "damped" co-normal derivative (associated to the divergence-form elliptic part of the system) and the tangential gradient on the boundary must be addressed; a similar challenge is known to arise in the case of the system of dynamic elasticity. The corresponding inequality for the RMT plate is presented in Theorem 3.1; it constitutes the main result of this paper and is a key technical step to the proof of the exponential stability of the associated linear system, as presented in Theorem 2.1.

2. Reissner-Mindlin-Timoshenko (RMT) plate

The RMT equations were introduced by Reissner [Rei45] and Mindlin [Min51]. The origins of the model go back to the theory of flexural vibrations of elastic beams: it had been long known that the classical Euler-Bernoulli (EB) equations offer limited accuracy when it comes to vibrations of higher modes; the EB model is also inapplicable when the cross-sectional dimension of the beam is comparable to the wave-length of flexural motions. Rayleigh [RS45] (first published in 1877–1878) attempted to correct the error by taking into account the effect of rotatory inertia. Subsequently Timoshenko (e.g., see the 1921 and 1937 papers in [Tim53]) included the effect of shear deformations. The RMT system is a 2-dimensional analog of the Timoshenko beam.

Unlike the classical Kirchhoff plate theory, the hypotheses underlying the RMT equations do not assert that the filaments of the plate remain perpendicular to the deformed mid-surface, and shear and rotatory inertia are taken into account. For a summary of equations see, for instance, [Lag89, Ch. 3]. The state vector of the plate is given by a vector $[u, \psi, \phi]$, where u is the deflection of the plate's mid-surface occupying domain $\Omega \subset \mathbb{R}^2$, and ψ, ϕ are rotation angles of the filaments of the plate.

Let the parameters ρ and h stand respectively for the (constant) mass density of the plate and its thickness; α is the *shear modulus*, β: the *modulus of flexural rigidity*, and $0 < \mu < 1$. Functions f_1, f_2, f_3 represent generic forcing terms. The

RMT equations read:

$$(\rho h)\frac{\partial^2 u}{\partial t^2} \quad -\alpha \operatorname{div}\big[\psi + u_x,\ \phi + u_y\big] = f_1 \tag{2.1}$$

$$\left(\tfrac{\rho h^3}{12}\right)\frac{\partial^2}{\partial t^2}\begin{bmatrix}\psi\\ \phi\end{bmatrix} -\beta \operatorname{div}\begin{bmatrix}\psi_x + \mu\phi_y & \frac{1}{2}(1-\mu)(\psi_y + \phi_x)\\ \frac{1}{2}(1-\mu)(\psi_y+\phi_x) & \mu\psi_x + \phi_y\end{bmatrix} + \alpha\begin{bmatrix}\psi + u_x\\ \phi + u_y\end{bmatrix} = \begin{bmatrix}f_2\\ f_3\end{bmatrix}.$$

The div operator in the second equation denotes the divergence applied to each row vector (or divergence of the column vectors). The initial data for this system belongs to the following *finite energy space*:

$$\Big\{\{u(0),\psi(0),\phi(0)\},\{u_t(0),\psi_t(0),\phi_t(0)\}\Big\} \in \mathscr{H} \cong [H^1(\Omega)]^3 \times [L^2(\Omega)]^3.$$

Let us introduce a more compact notation:

$$\mathbf{U} = \begin{bmatrix}u\\ \psi\\ \phi\end{bmatrix},\quad \mathbf{F} = \begin{bmatrix}f_1\\ f_2\\ f_3\end{bmatrix},\quad \mathcal{M} := \begin{bmatrix}\rho h & \\ & \rho h^3/12\end{bmatrix},\quad C_\mu := \tfrac{1}{2}(1-\mu)$$

$$\boldsymbol{\mathcal{S}} = \mathcal{S}(\mathbf{U}) := \begin{bmatrix}\alpha(\psi+u_x) & \alpha(\phi+u_y)\\ \beta(\psi_x+\mu\phi_y) & \beta C_\mu(\psi_y+\phi_x)\\ \beta C_\mu(\psi_y+\phi_x) & \beta(\mu\psi_x+\phi_y)\end{bmatrix},\quad \boldsymbol{\mathcal{Q}} = \mathcal{Q}(\mathbf{U}) := \begin{bmatrix}0\\ \alpha(\psi+u_x)\\ \alpha(\phi+u_y)\end{bmatrix}.$$

Then a generic RMT system with a Neumann boundary data can be written as

$$\mathcal{M}\mathbf{U}_{tt} - \operatorname{div}\boldsymbol{\mathcal{S}} + \boldsymbol{\mathcal{Q}} = \mathbf{F} \quad \text{in} \quad Q_T := \Omega\times(0,T) \tag{2.2}$$

$$\{\mathbf{U}(0),\mathbf{U}_t(0)\} = \{\mathbf{U}_0,\mathbf{U}_1\} \in \mathscr{H} \tag{2.3}$$

$$\boldsymbol{\mathcal{S}}\boldsymbol{\nu} + K\mathbf{U} = -\mathbf{G} \quad \text{in} \quad \Sigma_T := \Gamma\times(0,T);\quad \Gamma := \partial\Omega, \tag{2.4}$$

for a symmetric positive definite matrix K. Here $\boldsymbol{\mathcal{S}}\boldsymbol{\nu}$ stands for the regular matrix-vector multiplication (i.e., the column vector whose entries are the dot-products of $\boldsymbol{\nu}$ with the row-vectors of $\boldsymbol{\mathcal{S}}$).

The energy functional for the above system is equivalent (via a version of Korn's inequality) to the squared norm of the solution $\{\mathbf{U},\mathbf{U}_t\}$ on $\mathscr{H}$, and is given by [AT10]:

$$\begin{aligned}\mathcal{E}(t) &= \mathcal{E}(\mathbf{U}_t(t),\mathbf{U}(t))\\ &:= \tfrac{1}{2}\left(\rho h\|u_t(t)\|^2_{L^2(\Omega)} + \frac{\rho h^3}{12}\|\psi_t(t)\|^2_{L^2(\Omega)} + \frac{\rho h^3}{12}\|\phi_t(t)\|^2_{L^2(\Omega)}\right)\\ &\quad + \frac{\alpha}{2}\|\psi(t)+u_x(t)\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|\phi(t)+u_y(t)\|^2_{L^2(\Omega)}\\ &\quad + \frac{\beta}{2}\left(\|\psi_x(t)\|^2_{L^2(\Omega)} + 2\mu\langle\psi_x(t),\phi_y(t)\rangle_\Omega + \|\phi_y(t)\|^2_{L^2(\Omega)}\right)\\ &\quad + \frac{\beta C_\mu}{2}\|\psi_y(t)+\phi_x(t)\|^2_{L^2(\Omega)}.\end{aligned} \tag{2.5}$$

Well-posedness of (2.2)–(2.4) for linear boundary feedbacks follows the standard semigroup theory assuming the source $\mathbf{F}$ has a suitable structure; in fact, the argument extends to non-linear boundary feedbacks via the theory of m-accretive operators. The reader is referred to [AT10] for details.

Consider a simplified version of this model with $\mathbf{F} = \mathbf{0}$ and with a linear dissipative. Then the following uniform stability result holds:

Theorem 2.1 (Exponential stability of the linear model). *Let* $\mathbf{F} = \mathbf{0}$ *and* $\mathbf{G} = G(\mathbf{U}_t)$ *for a positive definite* 3×3 *matrix* G. *Then the system* (2.2)–(2.4) *is exponentially stable, in the sense that there exists* $T > 0$ (*dependent on the diameter of* Ω), $\gamma > 0$, *and* $C(\mathcal{E}(0))$, *the latter dependent only the initial energy, such that* $\mathcal{E}(t) \leq C(\mathcal{E}(0))e^{-\gamma t}$ *for all* $t > T$.

As was mentioned above, the known versions of this result omit some details of the necessary trace estimates. The complete argument can be found in the upcoming paper by the authors [AT10], which, in addition, addresses a more general setting of coupled wave and plate dynamics, with fully nonlinear feedbacks restricted to portions of the boundary. The argument uses weighted multipliers based on non-radial fields, and relies on the aforementioned trace regularity estimates, which show that the co-normal derivative and the velocity feedback uniformly control the gradient of the solution on the boundary.

3. Trace regularity of solutions to RMT equations

The following inequality is one of the key arguments leading to the proof the uniform boundary stability of (2.2)–(2.4); e.g., as stated in Theorem 2.1 for the linear version of the system.

Theorem 3.1 (Trace estimates for the RMT plate). *Let* $\{\mathbf{U}, \mathbf{U}_t\}$ *be a solution to* (2.2). *Then for any* $T > 0$, $\delta > 0$ *and positive* $\lambda < T/2$, *there exists a constant* $C_{T,\lambda,\delta}$ *such that*

$$\int_\lambda^{T-\lambda} \int_\Gamma |\nabla \mathbf{U}|^2 \, dx\, dt \leq C_{T,\lambda,\delta} \int_0^T \int_\Gamma \left(|\boldsymbol{\mathcal{S}}\, \boldsymbol{\nu}|^2 + |\mathbf{U}_t|^2 \right) \, dx\, dt \\ + C_{T,\lambda,\delta} \|\mathbf{F}\|^2_{\left[H^{-\frac{1}{2}+\delta}(Q_T)\right]^3} \\ + C_{T,\lambda,\delta}\, l.o.t.$$

$$l.o.t. = l.o.t.(\mathbf{U}, \mathbf{U}_t) := \left\|\mathbf{U}\right\|^2_{[H^{1/2+\delta}(Q_T)]^3} + \|\mathbf{U}_t\|^2_{[H^{-1/2+\delta}(Q_T)]^3} + |\mathbf{U}|^2_{[L^2(\Sigma_T)]^3}.$$

For any $\varepsilon > 0$, *l.o.t. satisfies*

$$C_{T,\lambda,\delta}\, l.o.t. \leq \varepsilon \int_0^T \mathcal{E}(t) dt + C_{T,\lambda,\delta,\varepsilon} \int_0^T |\mathbf{U}|^2_{[L^2(\Omega)]^3}. \tag{3.1}$$

In addition,

$$M[u_x, u_y, \psi_x, \psi_y, \phi_x, \phi_y]^t = [\boldsymbol{\mathcal{S}}\, \boldsymbol{\nu},\ \nabla u \cdot \boldsymbol{\tau},\ \nabla \psi \cdot \boldsymbol{\tau},\ \nabla \phi \cdot \boldsymbol{\tau}]^t, \tag{3.2}$$

where $\boldsymbol{\tau}(x) = \{\tau_1(x), \tau_2(x)\}$, $x \in \Gamma$ *is a smooth tangential frame on* Γ *and*

$$M = \begin{bmatrix} \alpha\nu_1 & \alpha\nu_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta\nu_1 & \beta c_\mu \nu_2 & \beta c_\mu \nu_2 & \beta\mu\nu_1 \\ 0 & 0 & \beta\mu\nu_2 & \beta c_\mu \nu_1 & \beta c_\mu \nu_1 & \beta\nu_2 \\ \tau_1 & \tau_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tau_1 & \tau_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tau_1 & \tau_2 \end{bmatrix}, \quad \det M = \text{const} = \alpha\beta^2 c_\mu > 0. \tag{3.3}$$

Remark 3.2 (Regularity of solutions). To derive the result of Theorem 3.1 one needs the trace of the gradient of a solution to be well defined on the boundary; in particular, it is convenient to assume that the solutions are strong (or, more generally, at least possess the regularity $\{\mathbf{U}, \mathbf{U}_t\} \in [H^{3/2+\varepsilon}(\Omega)]^3 \times [H^1(\Omega)]^3$). However, that does not prevent one from utilizing this result when investigating stability of weak solutions. In fact, it is a standard procedure in the multiplier method to carry out all the calculations for smooth initial data, and then extend only the very final estimate to all weak solutions by density. All that is needed for stability analysis of weak solutions is that the RHS of the estimate in Theorem 3.1 depends solely on the finite-energy norms and on the boundary data.

The rest of the paper is devoted to the proof of Theorem 3.1. The argument follows the pioneering strategy of I. Lasiecka and R. Triggiani [LT92], and employs special technical arguments similar to the ones invoked for elastodynamics in [Hor98a, AT09].

Pick $\bar{x}$ on the boundary Γ. Let O be a small open neighborhood of $\bar{x}$; apply a smooth cutoff that localizes $\mathbf{U}$ to $O \cap \Omega$. Then using a change of coordinates pass to a locally equivalent elliptic system on a half-space (see, e.g., [Hör03, Sect. 6.4]). Define

$$\boldsymbol{\Omega} := \mathbb{R}_x^+ \times \mathbb{R}_y,$$

$$\mathbf{Q} := \mathbb{R}_t \times \boldsymbol{\Omega}, \quad \boldsymbol{\Sigma} := \mathbb{R}_t \times \{x = 0, y \in \mathbb{R}\}.$$

Subsequently we will suppress multiplicity when indicating norms, i.e., $H^\alpha(\mathbf{Q})$ will stand for $[H^\alpha(\mathbf{Q})]^3$ etc. Use notation

$$D_x := \frac{1}{i}\frac{\partial}{\partial x}, \quad D_y := \frac{1}{i}\frac{\partial}{\partial y}, \quad D_t := \frac{1}{i}\frac{\partial}{\partial t}$$

to write the equations in general form as

$$\mathcal{P}(x, y, D_x, D_y, D_t)\mathbf{U} = \mathbf{F} \quad \text{in} \quad \mathbf{Q} \tag{3.4a}$$

$$\mathcal{B}(0, y, D_x, D_y)\mathbf{U}\big|_{\{x=0\}} = \mathbf{G} \quad \text{in} \quad \boldsymbol{\Sigma}. \tag{3.4b}$$

where

$$\mathcal{P}(x, y, D_x, D_y, D_t) := -k(x, y)D_t^2 + \mathcal{A}(x, y, D_x, D_y) \tag{3.5}$$

with $k(x, y)$ being is a strictly positive definite diagonal matrix, and $\mathcal{A}$: a strongly elliptic (3×3 matrix) operator of order 2. Under a Fourier-Laplace transform (except in the normal direction) substitute

$$D_x \rightsquigarrow \xi, \quad D_y \to \eta, \quad D_t \to \tau = \sigma - i\gamma \quad (\gamma > 0), \qquad \boldsymbol{\nu} = \{-1, 0\}.$$

The entries of the corresponding principal (3×3 matrix) symbol Symb[$\mathcal{A}$] of the operator $\mathcal{A}$ are homogeneous second-order polynomials in ξ and η. Since $\mathcal{A}$ is strongly elliptic, there exists $a_0 > 0$ such that

$$\Re\,\mathrm{Symb}[\mathcal{A}] \geq a_0(|\eta|^2 + |\xi^2|)\,\mathrm{Id}\,. \tag{3.6}$$

Henceforth we shall restrict the analysis to $\sigma, \eta \geq 0$ since the proof in the other quadrants is analogous. For a constant $c_0 > 0$, to be defined in a moment, split the quadrant into three regions as illustrated in Figure 1:

$$\begin{aligned} \mathfrak{R}_e &:= \big\{ \{\eta, \sigma\} : |\sigma| < c_0|\eta| \big\} \\ \mathfrak{R}_{tr} &:= \big\{ \{\eta, \sigma\} : c_0|\eta| \leq |\sigma| \leq 2c_0|\eta| \big\} \\ \mathfrak{R}_h &:= \big\{ \{\eta, \sigma\} : |\sigma| > 2c_0|\eta| \big\}. \end{aligned} \tag{3.7}$$

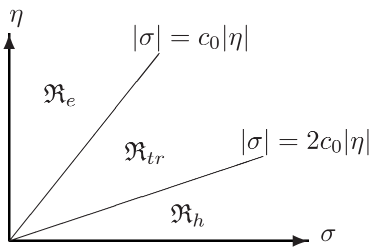


FIGURE 1. Decomposition of the frequency domain into elliptic ($\mathfrak{R}_e$), and non-elliptic regions, the latter consisting of the hyperbolic ($\mathfrak{R}_h$) and the "transitional" ($\mathfrak{R}_{tr}$) sectors.

In order to use the solution $\mathbf{U}$ of the original system (2.2) in the local version (3.4), we must restrict it in time. For $0 < \lambda < T/2$ let $\theta(t) \in C_0^\infty(\mathbb{R})$ be such that

$$\theta(t) = \begin{cases} 1 & t \in]\lambda,\ T - \lambda[\\ 0 & t \in \mathbb{R} \setminus [0, T] \\ \text{ a } C^\infty \text{ function with range }]0,1[& \text{elsewhere}\,. \end{cases}$$

In addition, define an "elliptic cutoff," namely an operator $X(x, y; t)$ with a homogeneous symbol of order 0 in the class $S^0(\mathbb{R}^3_{txy})$, given by a C^∞ function which satisfies

$$\chi(\sigma, \eta) = \begin{cases} 1 & \text{in } \mathfrak{R}_e \\ 0 & \text{in } \mathfrak{R}_h \end{cases}. \tag{3.8}$$

Apply these cutoffs sequentially:

$$\begin{cases} \mathcal{P}(X\theta\mathbf{U}) &= [\mathcal{P}, X]\theta\mathbf{U} + X[\mathcal{P}, \theta]\mathbf{U} + X\theta\mathbf{F} \\ \mathcal{B}(X\theta\mathbf{U}) &= [\mathcal{B}, X]\theta\mathbf{U} + X\theta\mathbf{G} \end{cases}.$$

Define

$$\tilde{\mathbf{U}} := \theta\mathbf{U}, \quad \tilde{\mathbf{F}} := \theta\mathbf{F}, \qquad \tilde{\mathbf{G}} := \theta\mathbf{G},$$

then $\operatorname{supp} X\tilde{\mathbf{U}}$ is restricted to a compact set $\boldsymbol{\Omega} \times [\lambda, T-\lambda] \subset \mathbf{Q} = \mathbb{R}_t \times \mathbb{R}_x \times \mathbb{R}_y$ with boundary $\boldsymbol{\Sigma} = \mathbb{R}_t \times \mathbb{R}_y$, and the system reads:

$$\begin{aligned} (\mathcal{P}X)\tilde{\mathbf{U}} &= [\mathcal{P}, X]\tilde{\mathbf{U}} - Xk(x,y)\Big(\theta''\mathbf{U} + 2\theta'\mathbf{U}_t\Big) + X\tilde{\mathbf{F}} && \text{in} \quad \mathbf{Q}, \\ (\mathcal{B}X)\tilde{\mathbf{U}} &= [\mathcal{B}, X]\tilde{\mathbf{U}} + X\tilde{\mathbf{G}} && \text{in} \quad \boldsymbol{\Sigma}. \end{aligned} \tag{3.9}$$

3.1. Elliptic region $\mathfrak{R}_e$

Due to the strong ellipticity property (3.6) of $\mathcal{A}$ it is possible to choose $c_0 > 0$ from (3.7) sufficiently small, so that in $\mathfrak{R}_e$:

$$\Re \operatorname{Symb}[\mathcal{P}](x, y; \xi, \eta, \tau) \geq \frac{1}{2} \Re \operatorname{Symb}[\mathcal{A}](x, y; \xi, \eta).$$

Moreover, the pair $\{\mathcal{A}, \mathcal{B}\}$ satisfies the L-condition (the Shapiro-Lopatinskii condition, see, e.g., [WRL95, Sect. 9.3]) as follows from the unique solvability of the associated elliptic problem $\{\mathcal{A}\mathbf{U} = 0, \mathcal{B}\mathbf{U} = 0\}$, via, for instance, the Lax-Milgram Theorem. Consequently, $\mathcal{P}X$ is also strongly elliptic, and $\mathcal{B}X$ is elliptic with respect to $\mathcal{P}X$.

Taking into account that X belongs to the operator class $OPS^0_{0,0}(\boldsymbol{\Sigma})$, the standard elliptic estimate holds (see, for example, [LM68, P. 188, Theorem 7.4]):

$$\begin{aligned} &\|X\tilde{\mathbf{U}}\|_{H^{3/2}(\mathbf{Q})} + \|\tilde{\mathbf{U}}\|_{H^1(\boldsymbol{\Sigma})} \\ &\lesssim \|[\mathcal{P}, X]\tilde{\mathbf{U}}\|_{H^{-\frac{1}{2}+\delta}(\mathbf{Q})} \\ &\quad + \|\theta''\mathbf{U}\|_{H^{-\frac{1}{2}+\delta}(\mathbf{Q})} + \|\theta'\mathbf{U}_t\|_{H^{-\frac{1}{2}+\delta}(\mathbf{Q})} + \|\tilde{\mathbf{F}}\|_{H^{-\frac{1}{2}+\delta}(\mathbf{Q})} \\ &\quad + \|[\mathcal{B}, X]\tilde{\mathbf{U}}\|_{L^2(\boldsymbol{\Sigma})} + \|\tilde{\mathbf{G}}\|_{L^2(\boldsymbol{\Sigma})}; \end{aligned} \tag{3.10}$$

where here and henceforth, the notation $a(s) \lesssim b(s)$ will indicate $a \leq Cb(s)$ for some constant C independent of s. In (3.10) since we have measured the $3/2$ fractional power of the elliptic operator, then the interior norms cannot be identified with spaces in the Sobolev scale and must account for the distance of points to the boundary. In order to replace these norms with regular Sobolev norms we have increased the order of terms in the interior by adding $\delta > 0$.

Adjust the elliptic estimate (3.10) appealing to the following facts:

- Operator $\mathcal{B}$ can be decomposed as $\frac{\partial}{\partial \boldsymbol{\nu}} + \mathcal{B}_{\tan}$ where $\mathcal{B}_{\tan} \in OPS^1_{1,0}(\boldsymbol{\Sigma})$. Since X is a 0-order tangential operator, then

$$[\mathcal{B}, X] = [\mathcal{B}_{\tan}, X] \in OPS^0_{1,0},$$

  which follows from the asymptotic representation of composition operators, see, e.g., [Tay81, Theorem 4.4, p. 46].
- Similarly $[\mathcal{P}, X] \in OPS^1_{1,0}$.

Ultimately arrive at:

$$\begin{aligned}&\|\tilde{\mathbf{U}}\|_{H^1(\boldsymbol{\Sigma})} \\ &\lesssim \left\|\mathbf{U}\right\|_{H^{1/2+\delta}(\mathbf{Q})} + \|\mathbf{U}_t\|_{H^{-1/2+\delta}(\mathbf{Q})} + \|\mathbf{U}\|_{L^2(\boldsymbol{\Sigma})} + \|\mathbf{F}\|_{H^{-1/2+\delta}(\mathbf{Q})} + \|\mathbf{G}\|_{L^2(\boldsymbol{\Sigma})}.\end{aligned} \tag{3.11}$$

3.2. Hyperbolic region $\mathfrak{R}_h$

Now, according to the frequency domain decomposition (3.7)

$$|\eta| c_0 \leq |\sigma|. \tag{3.12}$$

Hence it suffices to estimate $\nabla \mathbf{U}$ on the boundary via $\mathbf{U}_t$ and the tangential derivatives $\frac{\partial}{\partial_\tau}\mathbf{U}$, for they are likewise dominated by the velocity component in the hyperbolic sector. Let $\boldsymbol{\nu} = \{\nu_1, \nu_2\}$, $\boldsymbol{\tau} = \{\tau_1, \tau_2\} = \{-\nu_2, \nu_1\}$, and matrix M, as in (3.3), be such that

$$M(\nabla \mathbf{U}) = M[u_x, u_y, \psi_x, \psi_y, \phi_x, \phi_y]^t = [\boldsymbol{\mathcal{S}}\,\boldsymbol{\nu}, \nabla u \cdot \boldsymbol{\tau}, \nabla \psi \cdot \boldsymbol{\tau}, \nabla \phi \cdot \boldsymbol{\tau}]^t.$$

Direct calculation shows:

$$\det M = \det \begin{bmatrix} \alpha\nu_1 & \alpha\nu_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta\nu_1 & \beta c_\mu \nu_2 & \beta c_\mu \nu_2 & \beta\mu\nu_1 \\ 0 & 0 & \beta\mu\nu_2 & \beta c_\mu \nu_1 & \beta c_\mu \nu_1 & \beta\nu_2 \\ \tau_1 & \tau_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tau_1 & \tau_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tau_1 & \tau_2 \end{bmatrix} = \alpha\beta^2 c_\mu > 0$$

since $c_\mu := \frac{1}{2}(1-\mu)$ and $\mu < 1$. Therefore, it is possible to find a constant $C > 0$ so that

$$|\nabla(1-X)\tilde{\mathbf{U}}| \leq C\left(\left|\mathcal{S}[(1-X)\tilde{\mathbf{U}}]\,\boldsymbol{\nu}\right| + |\partial_\tau(1-X)\tilde{\mathbf{U}}|\right).$$

Note that the commutator of the tangential operators ∂_τ and $(1-X)$ has order zero, hence we may commute them modulo lower-order terms; in addition let us bound the tangential derivatives by velocity via (3.12):

$$\|\nabla(1-X)\tilde{\mathbf{U}}\|_{L^2(\boldsymbol{\Sigma})} \lesssim \left\|\mathcal{S}[(1-X)\tilde{\mathbf{U}}]\,\boldsymbol{\nu}\right\|_{L^2(\boldsymbol{\Sigma})} + \left\|(1-X)\tilde{\mathbf{U}}_t\right\|_{L^2(\boldsymbol{\Sigma})} + \|\mathbf{U}\|_{L^2(\boldsymbol{\Sigma})}. \tag{3.13}$$

3.3. Combined estimates

Put together (3.11) and (3.13); after squaring and adjusting the constants get

$$\begin{aligned}\|\nabla \mathbf{U}\|^2_{L^2(\boldsymbol{\Sigma})} &\lesssim \|X\tilde{\mathbf{U}}\|^2_{L^2(\boldsymbol{\Sigma})} + \|(1-X)\tilde{\mathbf{U}}\|^2_{L^2(\boldsymbol{\Sigma})} \\ &\lesssim \left\|\mathbf{U}\right\|^2_{H^{1/2+\delta}(\mathbf{Q})} + \|\mathbf{U}_t\|^2_{H^{-1/2+\delta}(\mathbf{Q})} + \|\mathbf{U}\|^2_{L^2(\boldsymbol{\Sigma})} \\ &\quad + \|\mathbf{U}_t\|^2_{L^2(\boldsymbol{\Sigma})} + \|\mathcal{S}[(1-X)\tilde{\mathbf{U}}]\,\boldsymbol{\nu}\|^2_{L^2(\boldsymbol{\Sigma})} + \left\|(1-X)\tilde{\mathbf{U}}_t\right\|^2_{L^2(\boldsymbol{\Sigma})} \\ &\quad + \|\mathbf{F}\|^2_{H^{-1/2+\delta}(\mathbf{Q})} + \|\mathbf{G}\|^2_{L^2(\Sigma)}.\end{aligned}$$

Next, we readily have the bound

$$\|\mathcal{S}[(1-X)\tilde{\mathbf{U}}]\,\boldsymbol{\nu}\|^2_{L^2(\boldsymbol{\Sigma})} \lesssim \left(\|\boldsymbol{\mathcal{S}}\,\boldsymbol{\nu}\|^2_{L^2(\boldsymbol{\Sigma})} + \|\mathbf{U}\|^2_{L^2(\boldsymbol{\Sigma})}\right).$$

Since the time-direction is tangential in a collar of the boundary, interpolation gives

$$\|\mathbf{U}_t\|^2_{H^{-1/2+\delta}(\mathbf{Q})} \leq \|\mathbf{U}\|^2_{H^{1/2+\delta}(\mathbf{Q})} \lesssim C_\varepsilon \|\mathbf{U}\|^2_{L^2(\mathbf{Q})} + \varepsilon \|\mathbf{U}\|^2_{H^1(\mathbf{Q})}$$

which confirms (locally) the estimate on lower-order norms of velocity within (3.1).

Expressing $\mathbf{G}$ via $\boldsymbol{\mathcal{S}\nu}$ (up to lower-order terms) leads to the result of Theorem 3.1 in local coordinates, which is equivalent to the original statement, up to a perturbation by L^2-norms of $\mathbf{U}$ and its derivatives of non-principal order (below the $H^1(\Omega)$ level). The bound (3.1) on the lower-order terms l.o.t. readily follows via space-time interpolation, Sobolev embeddings, and the fact that the $[H^1(\Omega)]^3$ norm of $\mathbf{U}$ is controlled by the energy $\mathcal{E}(t)$. □

References

[AT10] George Avalos and Daniel Toundykov. Boundary stabilization of nonlinear structural acoustic interactions with interface on a Reissner-Mindlin plate. Preprint, 2010.

[AT09] George Avalos and Roberto Triggiani. Boundary feedback stabilization of a coupled parabolic-hyperbolic Stokes-Lamé PDE system. *J. Evol. Equ.*, 9(2):341–370, 2009.

[BLR92] Claude Bardos, Gilles Lebeau, and Jeffrey Rauch. Sharp sufficient conditions for the observation, control, and. *SIAM J. Control Optim.*, 30(5):1024–1065, 1992.

[Che79] Goong Chen. Energy decay estimates and exact boundary value controllability for the wave equation in a bounded domain. *J. Math. Pures Appl. (9)*, 58(3):249–273, 1979.

[CL06] Igor Chueshov and Irena Lasiecka. Global attractors for Mindlin-Timoshenko plates and for their Kirchhoff limits. *Milan J. Math.*, 74:117–138, 2006.

[Dal06] Marie Grobbelaar-Van Dalsen. On a structural acoustic model with interface a Reissner-Mindlin plate or a Timoshenko beam. *J. Math. Anal. Appl.*, 320:121–144, 2006.

[Dal08] Marié Grobbelaar-Van Dalsen. On a structural acoustic model which incorporates shear and thermal effects in the structural component. *J. Math. Anal. Appl.*, 341(2):1253–1270, 2008.

[FS09] Hugo D. Fernández Sare. On the stability of Mindlin-Timoshenko plates. *Quart. Appl. Math.*, 67(2):249–263, 2009.

[GLLT04] R. Gulliver, Irena Lasiecka, W. Littman, and Roberto Triggiani. *The case for differential geometry in the control of single and coupled PDEs: the structural acoustic chamber*, volume 137 of *Geometric methods in inverse problems and PDE control; IMA Vol. Math. Appl.*, pages 73–181. Springer, 2004.

[GN06] Claudio Giorgi and Maria Grazia Naso. Mathematical models of Reissner-Mindlin thermoviscoelastic plates. *J. Thermal Stresses*, 29(7):699–716, 2006.

[GV07] Claudio Giorgi and Federico M. Vegni. The longtime behavior of a nonlinear Reissner-Mindlin plate with exponentially decreasing memory kernels. *J. Math. Anal. Appl.*, 326(2):754–771, 2007.

[Hor98a] Mary Ann Horn. Implications of sharp trace regularity results on boundary stabilization of the system of linear elasticity. *J. Math. Anal. Appl.*, 223(1):126–150, 1998.

[Hor98b] Mary Ann Horn. A review on the influence of boundary regularity on the question of boundary stabilization. *Math. Models Methods Appl. Sci.*, 8(7):1199–1222, 1998. Special issue on Barrett Lectures in control theory (Knoxville, TN, 1997).

[Hör03] Lars Hörmander. *The analysis of linear partial differential operators. I.* Springer-Verlag, Berlin, 2003. Distribution theory and Fourier analysis, Reprint of the second (1990) edition.

[Lag83] John Lagnese. Boundary stabilization of linear elastodynamic systems. *SIAM J. Control Optim.*, 21(6):968–984, 1983.

[Lag89] John E. Lagnese. *Boundary stabilization of thin plates*, volume 10. Society for Industrial and Applied Mathematics, SIAM; Philadelphia, PA, 1989.

[LM68] Jacques-Louis Lions and E. Magenes. *Problèmes aux limites non homogènes et applications. Vol.* 1. Dunod, Paris, 1968.

[LT92] Irena Lasiecka and Roberto Triggiani. Uniform stabilization of the wave equation with dirichlet or neumann feedback control without geometrical conditions. *Appl. Math. Optim.*, 25(2):189–224, 1992.

[LT93] Irena Lasiecka and Roberto Triggiani. Sharp trace estimates of solutions of Kirchhoff and Euler-Bernoulli equations. *Appl. Math. Optim.*, 28(3):277–306, 1993.

[LT00] Irena Lasiecka and Roberto Triggiani. Sharp regularity theory for elastic and thermoelastic Kirchoff [kirchhoff] equations with free boundary conditions. *Rocky Mountain J. Math.*, 30(3):981–1024, 2000.

[LT02] Irena Lasiecka and Roberto Triggiani. Uniform stabilization of a shallow shell model with nonlinear boundary feedbacks. *J. Math. Anal. Appl.*, 269(2):642–688, 2002.

[Min51] Raymond D. Mindlin. Thickness-shear and flexural vibrations of crystal plates. *J. Appl. Physics*, 22:316–323, 1951.

[Ped07a] Michael Pedersen. The functional analytic setting of HUM. I. General theory. *Int. J. Pure Appl. Math.*, 37(3):297–320, 2007.

[Ped07b] Michael Pedersen. The functional analytic setting of HUM. II. The Mindlin-Timoshenko plate model. *Int. J. Pure Appl. Math.*, 40(3):373–397, 2007.

[Ped07c] Michael Pedersen. Well-posedness and regularity of the controlled Mindlin-Timoshenko plate model. *Int. J. Pure Appl. Math.*, 40(2):273–289, 2007.

[Ped08] Michael Pedersen. A variational formulation of the controlled plate model. *Int. J. Pure Appl. Math.*, 43(1):27–42, 2008.

[PPGT06] Roberto Paroni, Paolo Podio-Guidugli, and Giuseppe Tomassetti. The Reissner-Mindlin plate theory via Γ-convergence. *C. R. Math. Acad. Sci. Paris*, 343(6):437–440, 2006.

[PPGT07] Roberto Paroni, Paolo Podio-Guidugli, and Giuseppe Tomassetti. A justification of the Reissner-Mindline plate theory through variational convergence. *Anal. Appl.* (*Singap.*), 5(2):165–182, 2007.

[QR77] J.P. Quinn and D.L. Russell. Asymptotic stability and energy decay rates for solutions of hyperbolic equations with boundary damping. *Proc. Roy. Soc. Edinburgh Sect. A*, 77(1-2):97–127, 1977.

[Rei45] Eric Reissner. The effect of transverse shear deformation on the bending of elastic plates. *J. Appl. Mech.*, 12:A–69–A–77, 1945.

[RS45] John William Rayleigh and Baron Strutt. *The Theory of Sound*. Dover Publications, New York, N. Y, 1945. 2nd ed.

[Tay81] Michael E. Taylor. *Pseudodifferential operators*, volume 34. Princeton University Press, Princeton, N.J., 1981.

[Tim53] S.P. Timoshenko. *The collected papers of Stephen P. Timoshenko*. McGraw-Hill Book Company, Inc, New York-Toronto-London, 1953.

[vRZvdM09] N.F.J. van Rensburg, L. Zietsman, and A.J. van der Merwe. Solvability of a Reissner-Mindlin-Timoshenko plate-beam vibration model. *IMA J. Appl. Math.*, 74(1):149–162, 2009.

[WRL95] J.T. Wloka, B. Rowley, and B. Lawruk. *Boundary value problems for elliptic systems*. Cambridge University Press, 1995.

George Avalos and Daniel Toundykov
University of Nebraska-Lincoln
Department of Mathematics
Lincoln, NE, 68588, USA
e-mail: gavalos@math.unl.edu
dtoundykov@math.unl.edu

Operator Theory:
Advances and Applications, Vol. 216, 93–120

Linearization of a Coupled System of Nonlinear Elasticity and Viscous Fluid

Lorena Bociu and Jean-Paul Zolésio

Communicated by F. Bucci and I. Lasiecka

Abstract. We model the coupled system formed by an incompressible fluid and a nonlinear elastic body. We work with large displacement, small deformation elasticity (or St. Venant elasticity), which makes the problem very interesting from the physical point of view. The elastic body is three-dimensional $\Omega \in \mathbb{R}^3$, and thus it can not be reduced to its boundary Γ (like in the case of a membrane or a shell). In this paper, we study the static problem, and in view of the stability analysis we derive the linearization of the system, which turns out to be different from the usual coupling of classical linear models. New extra terms (for example those involving the boundary curvatures) play an important role in the final linearized system around some equilibrium.

Mathematics Subject Classification (2000). 76D03, 35Q30, 35Q35.

Keywords. Nonlinear elasticity, Navier-Stokes, potential fluid, linearization, coupled system.

1. Introduction

1.1. The problem and the model

The problem we address is the interaction between an incompressible, viscous fluid and a 3-d nonlinear elastic body. The interaction takes place on the common boundary (interface) and is realized via suitable transmission boundary conditions. We consider the steady regime associated with this coupling, which contrary to common belief, is more subtle than the dynamical one (since in real life, evolution is more plausible than equilibrium).

The research of Lorena Bociu was supported by the National Science Foundation under International Research Fellowship OISE-0802187 .

We assume existence of the steady state fluid-structure interaction ([23]), and in view of the stability analysis, our **goal** is to derive the linearization of the system. We accomplish this by perturbing the steady regime by a parameter of variation s, and then computing the derivatives with respect to s (shape derivatives). In the end, we obtain the linearization of the coupled fluid-structure problem around rest. While there are many models pertaining to this problem (i.e., coupling of linear elasticity and fluid), the one that we obtain after linearization is quite different and reveals **new features**, including the **presence of the curvature** terms on the common boundary. Thus the boundary and its curvatures play a key role in the analysis and can not be neglected. This is particularly important when the boundary oscillates, sending the mean curvature to "infinity". Modelling of this geometrical aspect is critical for a correct physical interpretation of the fluid-structure interaction.

1.2. Notation

For the rest of the paper, we use the repeated index convention for summation whenever the same Latin index appears twice, and the following notation:

- $(Df(a))_{ij} = \partial_j f_i(a) \in \mathbb{M}^3$ is the gradient matrix at $a \in X$ of any vector field $f = (f_i) : X \subset \mathbb{R}^3 \to \mathbb{R}^3$.
- $\operatorname{div} f(a) = \partial_i f_i \in \mathbb{R}$ is the divergence of $f : X \subset \mathbb{R}^3 \to \mathbb{R}^3$ at $a \in X$.
- $\operatorname{Div} T(a) = \partial_j T_{ij} e_i \in \mathbb{R}^3$ is the divergence of any second-order tensor field $T = (T_{ij}) : X \subset \mathbb{R}^3 \to \mathbb{M}^3$ at $a \in X$.

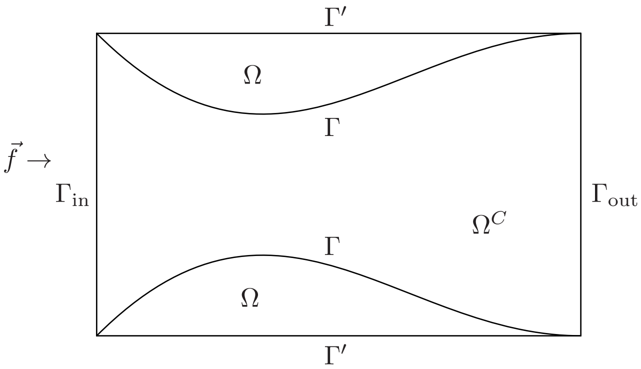
Since we ignore the distinction between covariant and contravariant components, we will identify the set of all second-order tensors with the set $\mathbb{M}^3$ of all square matrices of order three.

- $A^* =$ transpose of A, for any $A \in \mathbb{M}^3$.
- $A..B = \operatorname{tr}(A^*B) \in \mathbb{R}$ is the matrix inner product in $\mathbb{M}^3$.
- $\operatorname{Cof}(A) = \det(A)A^{-*}$ is the cofactor matrix of any invertible matrix $A \in \mathbb{M}^3$.
- $d_\Omega(x) = \begin{cases} \inf_{y\in\Omega} |y-x| & , \Omega \neq \emptyset \\ \infty & , \Omega = \emptyset \end{cases}$ is the distance function from a point x to $\Omega \in \mathbb{R}^n$.
- $b_\Omega(x) = d_\Omega(x) - d_{\Omega^c}(x)$, $\forall x \in \mathbb{R}^n$ is the oriented distance function from x to Ω, for any $\Omega \subset \mathbb{R}^n$.

1.3. PDE model

In what follows, we describe the model under consideration. Let $\mathcal{D} \in \mathbb{R}^3$ be a bounded domain. We assume that $\mathcal{D}$ is comprised of two open domains $\mathcal{D} = \Omega \cup \Omega^C$, and has smooth boundary $\partial\mathcal{D} = \Gamma' \cup \Gamma_{\text{in}} \cup \Gamma_{\text{out}}$ (see the figure on top of the next page).

The elastic body occupies domain Ω with sufficiently smooth boundary $\Gamma \cup \Gamma'$, and is described by a nonlinear elastic equation in terms of the displacement u. We work with large displacement, small deformation elasticity (or St. Venant elasticity [11]), which makes the problem difficult from the mathematical point of view, and very interesting from the physical point of view. The elastic body



is three-dimensional $\Omega \in \mathbb{R}^3$, and thus it can not be reduced to its boundary Γ (like in the case of a membrane or a shell). The fluid occupies domain Ω^C with boundary $\Gamma \cup \Gamma_{\text{in}} \cup \Gamma_{\text{out}}$, and is described by a Navier-Stokes equation in terms of the velocity of the fluid w and the pressure p. $\nu > 0$ represents the viscosity of the fluid. The fluid sticks to the boundary Γ, and thus we are dealing with a homogeneous boundary condition [see (1.1)]. The interaction takes place on the common boundary Γ and is realized via suitable transmission boundary conditions: we require continuity of both the velocities (the velocity of the fluid and the velocity of the boundary) and the normal stress tensors across the interface Γ. We assume that there is a flux $\vec{f}$ coming into $\mathcal{D}$ through Γ_{in}, that will determine the velocity of the fluid w.

The PDE model for the fluid-structure interaction defined by the variables (w, p, u) is given by

$$\begin{cases} -\nu\,\Delta\vec{w} + Dw.w + \nabla p = 0 & \Omega^c \\ \operatorname{div} w = 0 & \Omega^c \\ -\operatorname{Div}\mathcal{T} = 0 & \Omega \\ w = 0 & \Gamma \\ \mathcal{T}.n = p\vec{n} - \epsilon(w).n & \Gamma \\ u = 0 & \Gamma' \end{cases} \tag{1.1}$$

where $2\epsilon(w_s) = Dw_s + Dw_s^*$ and $\mathcal{T} : \bar{\Omega} \to \mathbb{S}^3$ is the Cauchy stress tensor given by:

$$\mathcal{T} = \Big(\frac{1}{\det(D\varphi)} D\varphi \cdot \Sigma(\sigma(u)) \cdot (D\varphi)^*\Big) \circ \varphi^{-1} \tag{1.2}$$

where $\varphi = I + u$ is the deformation of the reference configuration $\overline{\mathcal{O}} \in \mathbb{R}^3 \to \mathbb{R}^3$, $\sigma(u) = \frac{1}{2}(Du^* + Du + Du^*Du)$ is the Green-St Venant strain tensor, and $\Sigma(\sigma(u)) = \lambda(\operatorname{tr}\sigma(u))I + 2\mu\sigma(u)$ defines the second Piola-Kirchhoff stress tensor, with λ and μ being the Lamé constants of the material. For a detailed explanation on the nonlinear elastic component of the coupled system and formula (1.2), please see Appendix A.

With (1.1), we associate the following boundary conditions on Γ_{in} and Γ_{out}. Let $c(x)$ be a given, smooth function defined on Γ_{in} such that

$$\begin{cases} c(x) = 0 & \text{on } \partial\Gamma_{\text{in}}, \\ w.n_{\text{in}} = c(x) & \text{on } \Gamma_{\text{in}}. \end{cases} \tag{1.3}$$

Then it follows that

$$0 = \int_{\Omega^c} \operatorname{div} w \; dx = \int_{\Gamma_{\text{in}}} w.n_{\text{in}} \; d\Gamma_{\text{in}} + \int_{\Gamma_{\text{out}}} w.n_{\text{out}} \; d\Gamma_{\text{out}}. \tag{1.4}$$

At this point, we choose $\alpha \in \mathbb{R}$ verifying

$$\begin{cases} \alpha = w.n_{\text{out}} \quad \text{on } \Gamma_{\text{out}}, \\ \int_{\Gamma_{\text{out}}} \alpha \; d\Gamma_{\text{out}} = -\int_{\Gamma_{\text{in}}} c(x) \; d\Gamma_{\text{in}}. \end{cases} \tag{1.5}$$

The model has a variety of applications in naval and aerospace engineering, as well as cell biology and biomedical engineering. One specific example of the above-mentioned model is a 3D tube with elastic walls through which a fluid is flowing, and is very important in the study of arterial diseases (the tube represents the artery, the elastic body is the wall of the artery and the fluid is the blood).

We model the fluid by a Navier-Stokes equation due to the specific application we have in mind, i.e., the blood flow in an artery. Nevertheless, there are a variety of other applications for the model considered where the fluids have low viscosity or satisfy the Darcy law. These are the cases of potential fluid (incompressible and irrotational fluid, i.e., $v = \nabla\phi$, where v is the velocity of the fluid and ϕ, the velocity potential of the fluid, satisfies the Laplace equation $\Delta\phi = 0$) coupled with nonlinear elasticity.

We will study the problem in both cases, first for potential fluid, and then for the Navier-Stokes flow. The two problems are quite different from the mathematical point of view: in the case of potential fluid, we deal with a Neumann boundary condition (which corresponds to skidding at the boundary), while in the case of a Navier-Stokes fluid, we work with a homogeneous Dirichlet boundary condition (corresponding to the sticking property on the boundary, due to viscosity).

1.4. Special case: Potential fluid

We work with an incompressible and irrotational fluid, i.e., $v = \nabla\phi$, where v is the velocity of the fluid and ϕ (the velocity potential of the fluid) satisfies the Laplace equation $\Delta\phi = 0$ (due to the incompressibility condition which translates into $\nabla \cdot u = 0$). If p represents the pressure of the fluid and ν the viscosity, then the flow is described by the following Navier-Stokes equation

$$-\nu\Delta v + Dv.v + \nabla p = \rho\vec{g} \tag{1.6}$$

where ρ is the density, and $\vec{g}$ is the gravitational acceleration. Due to the fluid being irrotational (vorticity curl $v = 0$), the convective acceleration reduces to

$Dv.v = \nabla\Big(\dfrac{\|v\|^2}{2}\Big)$ and thus (1.6) becomes

$$\nabla(\frac{1}{2}\|\nabla\phi\|^2 + p - \rho g x_3) = 0.$$

This provides us with the formula for the pressure p of the fluid:

$$p = p_0 + \frac{1}{2}\|\nabla\phi\|^2 - \rho g x_3. \tag{1.7}$$

1.5. Parameter of variation s

We perturb the steady regime presented above by assuming that the flux entering the domain $\mathcal{D}$ is dependent on a variation parameter s, i.e., $c(x)$ is a given, smooth function defined on Γ_{in} such that for some constant $a \geq 0$,

$$\begin{cases} c(x) = 0 & \text{on} \;\; \partial\Gamma_{\text{in}}, \\ w_s.n_{\text{in}} = (a+s)c(x) & \text{on} \;\; \Gamma_{\text{in}}. \end{cases} \tag{1.8}$$

Then it follows that

$$0 = \int_{\Omega_s^c} \operatorname{div} w_s dx = \int_{\Gamma_{\text{in}}} w_s.n_{\text{in}} d\Gamma_{\text{in}} + \int_{\Gamma_{\text{out}}} w_s.n_{\text{out}} d\Gamma_{\text{out}}. \tag{1.9}$$

For any $s \geq 0$, we choose $\alpha_s \in \mathbb{R}$ verifying

$$\begin{cases} \alpha_s = w_s.n_{\text{out}} \;\; \text{on} \;\; \Gamma_{\text{out}}, \\ \int_{\Gamma_{\text{out}}} \alpha_s \; d\Gamma_{\text{out}} = -(a+s)\int_{\Gamma_{\text{in}}} c(x)\; d\Gamma_{\text{in}}, \;\; \text{for all} \;\; s \geq 0. \end{cases} \tag{1.10}$$

If the elastic body occupies a reference configuration $\overline{\mathcal{O}} \in \mathbb{R}^3$ with smooth boundary $\mathcal{S} \cup \Gamma'$, then, when subjected to applied forces, it occupies a deformed configuration $\Omega_s = \varphi_s(\overline{\mathcal{O}})$, with smooth boundary $\Gamma_s \cup \Gamma'$ (where Γ' is fixed). The deformation map in this case is dependant on the parameter s: $\varphi_s : \overline{\mathcal{O}} \to \mathbb{R}^3$, but nevertheless is smooth enough, injective, and orientation-preserving. The displacement $u_s : \overline{\mathcal{O}} \to \mathbb{R}^3$ becomes $u_s = \varphi_s - I$, where I is the identity map $I : \overline{\mathcal{O}} \to \mathbb{R}^3$. Similarly, for the fluid present in the system, the velocity and pressure are now functions of s: w_s, and p_s, and thus we have the following coupled system for the interaction:

$$\begin{cases} -\nu\,\Delta\vec{w}_s \;+\; Dw_s.w_s \;+\; \nabla p_s \;=\; 0 & \Omega_s^c \\ \operatorname{div} w_s = 0 & \Omega_s^c \\ -\operatorname{Div}\mathcal{T}_s = 0 & \Omega_s \\ w_s = 0 & \Gamma_s \\ \mathcal{T}_s.n_s = p_s\vec{n}_s - \epsilon(w_s).\vec{n}_s & \Gamma_s \\ u = 0 & \Gamma' \\ \int_{\Gamma_{\text{out}}} \alpha_s \; d\Gamma_{\text{out}} = -(a+s)\int_{\Gamma_{\text{in}}} c(x)\; d\Gamma_{\text{in}}, \;\; \text{for all} \;\; s \geq 0, \end{cases} \tag{1.11}$$

with

$$\begin{cases} c(x) = 0 & \text{on } \partial\Gamma_{\text{in}}, \\ w_s.n_{\text{in}} = (a+s)c(x) & \text{on } \Gamma_{\text{in}} \\ w_s.n_{\text{out}} = \alpha_s & \text{on } \Gamma_{\text{out}}, \end{cases} \tag{1.12}$$

where n_s is the unit outer normal vector along Γ_s, $2\epsilon(w_s) = Dw_s + Dw_s^*$, and $\mathcal{T}_s : \bar{\Omega}_s \to \mathbb{S}^3$ is the Cauchy stress tensor (associated to s), given by

$$\mathcal{T}_s = \Big(\frac{1}{\det(D\varphi_s)} D\varphi_s \cdot \Sigma(\sigma(u_s)) \cdot (D\varphi_s)^*\Big) \circ \varphi_s^{-1}. \tag{1.13}$$

In the particular case of potential fluid, recall that the pressure is given by $p_s = p_0 + \frac{1}{2}\|\nabla\phi_s\|^2 - \rho g x_3$, and thus (1.11) becomes:

$$\begin{cases} \Delta\phi_s = 0 & \Omega_s^c \\ -\operatorname{Div}\mathcal{T}_s = 0 & \Omega_s \\ \nabla\phi_s \cdot n_s = 0 & \Gamma_s \\ \mathcal{T}_s.n_s = (p_0 + \frac{1}{2}\|\nabla\phi_s\|^2 - \rho g x_3)n_s & \Gamma_s \\ u = 0 & \Gamma' \\ \int_{\Gamma_{\text{out}}} \alpha_s \, d\Gamma_{\text{out}} = -(a+s)\int_{\Gamma_{\text{in}}} c(x)\, d\Gamma_{\text{in}}, \quad \text{for all } \ s \geq 0 \end{cases} \tag{1.14}$$

where $c(x)$ is a given, smooth function defined on Γ_{in} such that for some $a \geq 0$,

$$\begin{cases} c(x) = 0 & \text{on } \partial\Gamma_{\text{in}}, \\ \frac{\partial}{\partial n}\phi_s = (a+s)c(x) & \text{on } \Gamma_{\text{in}}. \end{cases} \tag{1.15}$$

Since we have that

$$0 = \int_{\Omega_s^c} \operatorname{div}(\nabla\phi_s)dx = \int_{\Gamma_{\text{in}}} \frac{\partial\phi_s}{\partial n_{\text{in}}} d\Gamma_{\text{in}} + \int_{\Gamma_{\text{out}}} \frac{\partial\phi_s}{\partial n_{\text{out}}} d\Gamma_{\text{out}}, \tag{1.16}$$

then, for any $s \geq 0$, we choose $\alpha_s \in \mathbb{R}$ verifying

$$\begin{cases} \alpha_s = \frac{\partial\phi_s}{\partial n_{\text{out}}} \quad \text{on } \Gamma_{\text{out}}, \\ \int_{\Gamma_{\text{out}}} \alpha_s \, d\Gamma_{\text{out}} = -(a+s)\int_{\Gamma_{\text{in}}} c(x)\, d\Gamma_{\text{in}}, \quad \text{for all } \ s \geq 0. \end{cases} \tag{1.17}$$

2. Main results

Our first two results are for the particular case of potential fluid, meaning the speed of the fluid v derives from a harmonic potential in Ω^c, i.e., $v_s = \nabla\phi_s$. We first obtained the linearization of the coupled fluid-structure problem around some steady regime, then around fluid at rest ($\phi = 0$).

Let $\phi' = \frac{\partial}{\partial s}\phi_s\Big|_{s=0}$ and $u' = \frac{\partial}{\partial s}u_s\Big|_{s=0}$ be the shape derivatives of (ϕ, u).

Theorem 2.1 (Linearization of the coupled potential fluid-structure system around steady regime). *In system* (1.14), *for* $s = 0$, *we assume that the speed of the fluid* v *is steady, but not zero. With the following notation*

$$\mathbf{\Phi} = \phi',$$
$$U = u' \circ (I + u)^{-1}, \quad \text{and}$$
$$p = p_0 + \frac{1}{2}|\nabla_\Gamma \phi|^2 + \rho g x_3,$$

we obtain the following linearized system (*around steady flow*) *for the fluid-structure coupling* $(\mathbf{\Phi}, U)$*:*

$$\Delta \mathbf{\Phi} = 0 \quad \text{in } \Omega^c$$
$$\frac{\partial}{\partial n}\mathbf{\Phi} = -\operatorname{div}_\Gamma(\langle U, n\rangle \nabla_\Gamma \phi\,) \quad \text{on } \Gamma$$
$$\operatorname{Div}(\mathcal{T}') = 0 \quad \text{in } \Omega$$

$$\begin{aligned}\mathcal{T}'.n =& [\mathcal{T} - pI].(D^*_\Gamma U.n \;+\; D^2 b_\Omega.U_\Gamma) + \langle \nabla_\Gamma \mathbf{\Phi}, \nabla_\Gamma \phi\rangle\,\vec{n} \\ &+ (\langle n,\, D^2\phi.\nabla\phi\rangle \;+\; \rho g\, n_3\,)\langle U, n\rangle \vec{n} + \langle U, n\rangle \operatorname{Div}_\Gamma \mathcal{T} \quad \text{on } \Gamma \end{aligned} \tag{2.1}$$

with the boundary conditions

$$\frac{\partial}{\partial n}\mathbf{\Phi} = c(x) \quad \text{on } \Gamma_{\text{in}}$$
$$\frac{\partial}{\partial n}\mathbf{\Phi} = -\frac{\int_{\Gamma_{\text{in}}} c(x)d\Gamma}{\int_{\Gamma_{\text{out}}} d\Gamma} \quad \text{on } \Gamma_{\text{out}},$$

where $\mathcal{T}'$ *and* $\mathcal{T}$ *are given by* (3.15) *and* (1.2), *respectively.*

Theorem 2.1 shows that the linearized model is different from the usual coupling of linear models. More specifically, we note the presence of the curvatures of the common boundary Γ. The same phenomenon will be observed for the linearized models in the case of potential fluid-structure system around "rest" (Theorem 2.2), as well as in the case of Navier-Stokes-elastic structure (Theorem 2.3).

Theorem 2.2 (Linearization of the coupled potential fluid-structure system around rest). *This is the particular situation for* (1.14) *when, considering* $a = 0$, *then at* $s = 0$ *the forcing condition on* Γ_{in} *is zero and thus* $\phi = 0$. *Nevertheless, some pressure term remains* ($p = p_0 + \rho g\, x_3$), *and thus the linearization of the system becomes*

$$\Delta \mathbf{\Phi} = 0 \quad \text{in } \Omega^c$$
$$\frac{\partial}{\partial n}\mathbf{\Phi} = 0 \quad \text{on } \Gamma$$
$$\operatorname{Div}(\mathcal{T}') = 0 \quad \text{in } \Omega$$

$$\begin{aligned}\mathcal{T}'.n =& [\mathcal{T} - pI].(D^*_\Gamma U.n \;+\; D^2 b_\Omega.U_\Gamma) \\ &+ \rho g\, n_3 \langle U, n\rangle \vec{n} + \langle U, n\rangle \operatorname{Div}_\Gamma \mathcal{T} \quad \text{on } \Gamma \end{aligned} \tag{2.2}$$

with the same notation and the same boundary conditions on Γ_{in} *and* Γ_{out} *as in Theorem* 2.1.

Below we present the result that describes the linearization of the Navier-Stokes fluid-elastic structure. Let $w' = \frac{\partial}{\partial s} w_s\big|_{s=0}$, $p' = \frac{\partial}{\partial s} p_s\big|_{s=0}$, and $u' = \frac{\partial}{\partial s} u_s\big|_{s=0}$ be the shape derivatives of (w, p, u).

Theorem 2.3 (Linearization around rest for the coupling viscous fluid-elastic structure). *In system* (1.11)*, we assume that* $a = 0$*, so that at* $s = 0$*, we get the "rest" system with* $\vec{w} = 0$*. Then we obtain the following linearized model around fluid at rest:*

$$\begin{cases} -\Delta w' + \nabla p' = 0 & \Omega^c \\ \operatorname{div} w' = 0 & \Omega^c \\ w' = 0 & \Gamma \\ -\operatorname{Div}(\mathcal{T}') = 0 & \Omega \\ \mathcal{T}'.n = (p'I - \epsilon(w')).n + \langle \nabla p, n\rangle \langle U, n\rangle \vec{n} & \\ \qquad + (pI - \mathcal{T})(D^*_\Gamma U.n + D^2 b_\Omega.U_\Gamma) - \langle U, n\rangle \operatorname{Div}_\Gamma(\mathcal{T}) & \Gamma \\ w'.n_{\text{in}} = c(x) & \Gamma_{\text{in}} \\ w'.n_{\text{out}} = -\int_{\Gamma_{\text{in}}} c(x)\, d\Gamma / \int_{\Gamma_{\text{out}}} d\Gamma & \Gamma_{\text{out}} \end{cases} \tag{2.3}$$

where, as before, $U = u' \circ (I + u)^{-1}$*, and* $\mathcal{T}'$ *and* $\mathcal{T}$ *are given by* (3.15) *and* (1.2)*, respectively.*

Note here again that the coupling that we obtain is more complicated than just the coupling of the linear problems in the variables (u', w', p'). Indeed, the boundary curvatures play an important role in the analysis of the coupled fluid-structure interaction. These terms can not be neglected, since when the boundary has oscillations, the mean curvature H is not bounded.

3. Preliminaries

3.1. The moving boundary Γ_s

Recall that the deformation map φ maps the reference boundary $\mathcal{S}$ to Γ. Similarly, the deformation φ_s maps $\mathcal{S}$ to Γ_s.

At this point it is convenient to introduce the map $T_s : \bar{\Omega} \to \bar{\Omega}_s$ that builds the moving boundary Γ_s:

$$T_s = \varphi_s \circ \varphi^{-1} \tag{3.1}$$

and the speed $V(s, \cdot)$ associated with the flow mapping T_s:

$$V(s, \cdot) = \left(\frac{\partial}{\partial_s} T_s\right) \circ T_s^{-1} = \frac{\partial}{\partial_s} \varphi_s \circ \varphi_s^{-1}. \tag{3.2}$$

This means that $T_s(V) : X \to x(s)$, where $x(s)$ satisfies the following differential equation

$$\begin{cases} \frac{\partial}{\partial_s} x = V(s, x(s)) \\ x(0) = X \end{cases} \tag{3.3}$$

which is equivalent to $x(s) = X + \displaystyle\int_0^s V(t, x(t))dt$.

3.2. Transport of scalar operators

Let $u' = \frac{\partial}{\partial s} u_s|_{s=0}$. Then from (3.2) we have

$$\frac{d}{ds}T_s = V(s) \circ T_s \Rightarrow \frac{d}{ds}T_s\Big|_{s=0} = \frac{d}{ds}u_s|_{s=0} \circ \varphi^{-1} = u' \circ (u+I)^{-1} = V(0).$$

Moreover, from [15], we have the following identities:

$$\frac{d}{ds}DT_s(X) = DV(s, T_s(X))DT_s(X), \quad DT_0(X) = I$$

$$\Rightarrow \frac{d}{ds}DT_s\Big|_{s=0} = DV(0) \quad \text{and} \quad \frac{d}{ds}(DT_s)^{-1}\Big|_{s=0} = -DV(0).$$

$$\frac{d}{ds}\det DT_s(X) = \operatorname{tr} DV(s, T_S(X)) \det DT_s(X) = \operatorname{div} V(s, T_s(X)) \det DT_s(X),$$

$$\Rightarrow \frac{d}{ds}\det(DT_S)\Big|_{s=0} = \operatorname{div} V(0).$$

Now let $\vec{E}$ be a C^1 vector field defined over D. Then we have the following proposition, proved in Appendix B.

Proposition 3.1.

$$(\operatorname{div} E) \circ T = \det(DT)^{-1} \ \operatorname{div}(\det(DT)\,(DT)^{-1}.(E \circ T)). \tag{3.4}$$

Similarly, we can prove the following proposition:

Proposition 3.2. *For any $\phi \in H^1(D)$, we have the following identity:*

$$\Delta\phi \circ T = \det(DT)^{-1} \operatorname{div}(\det(DT)(DT)^{-1}(DT)^{-*}\nabla(\phi \circ T)). \tag{3.5}$$

These identities will be used later in the proofs of our theorems.

3.3. Transport of vector operators

Let $\mathcal{T}$ be a $N \times N$ matrix function defined on D. We consider the vector Divergence operator $\operatorname{Div}\vec{\mathcal{T}}$ being defined as the vector whose ith component is the (scalar) divergence of the vector composed of the ith line of the matrix $\mathcal{T}$:

$$(\operatorname{Div}\vec{\mathcal{T}})_i = \operatorname{div}(\mathcal{T}_{i,.}) = \Sigma_{j=1,\dots,N} \ \frac{\partial}{\partial x_j}\mathcal{T}_{i,j}.$$

From the previous section we obtain that

$$(\,(\operatorname{Div}\vec{\mathcal{T}}) \circ T)_i = (\operatorname{Div}\vec{\mathcal{T}})_i \circ T = \det(DT)^{-1} \ \operatorname{div}(\det(DT)\,(DT)^{-1}.(\mathcal{T}_{i,.}) \circ T\ ).$$

It turns out that $(\mathcal{T}_{i,.}) \circ T$ is the ith column vector of the matrix $\mathcal{T}^* \circ T$ so that $(DT)^{-1}.(\mathcal{T}_{i,.}) \circ T$ is the ith column of the matrix $(DT)^{-1}.\mathcal{T}^* \circ T$, and thus the ith line of the matrix $(DT)^{-1}.\mathcal{T} \circ T$. Therefore, using Proposition 3.1, we obtain the following identity:

Proposition 3.3.

$$(\operatorname{Div}\vec{\mathcal{T}}) \circ T = \det(DT)^{-1} \ \operatorname{Div}(\det(DT)\,(DT)^{-1}.(\mathcal{T} \circ T)\,). \tag{3.6}$$

3.4. Boundary change of variable

Let $\Gamma = \partial\Omega$ be a C^1 manifold. Then there exists a covering of open subsets $\bar{\Omega} \subset \cup_{i=1}^{m} \mathcal{O}_i$, and charts $c_i : \mathcal{O}_i \to B$ (the open unit ball in $\mathbb{R}^n$) such that $c_i(\Gamma \cap \mathcal{O}_i) \subset B_0 = \{x = (x', 0) \in B\}$ and $c_i(\Omega \cap \mathcal{O}_i) \subset B_+ = \{x = (x', z) \in B \text{ s.t. } z > 0\ \}$. We use the notation $x = (x', x_n)$ for a point $(x_1, \dots, x_n) \in \mathbb{R}^n$, where $x' = (x_1, \dots, x_{n-1})$. Let r_i be a partition of unity for the family of open sets $\{\mathcal{O}_i\}_{i=1}^{m}$, i.e., $r_i \in C_c^\infty(\mathcal{O}_i)$, $0 \le r_i \le 1$, and $\Sigma_{i=1}^{m} r_i = 1$ in a neighborhood of the boundary Γ. For any $f \in L^1(\Gamma)$ we have

$$\int_\Gamma f\, d\Gamma = \Sigma_{i=1}^{m} \int_{\Gamma \cap \mathcal{O}_i} r_i f\, d\Gamma$$
$$= \Sigma_{i=1}^{m} \int_{B_0} r_i o c_i^{-1} f o c_i^{-1}\, ||\mathrm{cof}(D(c_i^{-1})).e_n||\, dx'$$

Recall that for any square matrix A, the cofactor matrix is

$$\mathrm{cof}(A) = (\det A) A^{-*} \ \Rightarrow\ \mathrm{cof}(A^{-1}) = \frac{1}{\det A}\ A^*$$

Since we have that $D(c_i^{-1}) = (Dc_i)^{-1} \circ c_i^{-1}$, then we obtain the following identity:

$$\mathrm{cof}(D(c_i^{-1})) = \mathrm{cof}((Dc_i)^{-1}) \circ c_i^{-1} = \left(\frac{1}{\det Dc_i}\, (Dc_i)^* \right) \circ c_i^{-1}$$

It can be easily verified that if T is a smooth enough transformation we have, with $\Sigma = T(\Gamma)$,

$$\int_{T(\Gamma)} f\, d\Sigma = \int_\Gamma f \circ T\ \omega\ d\Gamma$$

where $\omega = ||\mathrm{cof}(DT).n|| = |\det(DT)|\, ||(DT)^{-*}.n||$, and n is the unitary normal field on Γ.

Moreover, we have the following lemma ([15]):

Lemma 3.1. *If the mapping $s \to T_s(V)$ is in $C^1([0,\tau]; C^k(\overline{D}, \mathbb{R}^n))$, then*

$$s \to n_s \circ T_s = \frac{DT_s^{-*} n}{||DT_s^{-*} n||} \quad \text{is in} \quad C^1([0,\tau]; C^k(\Gamma))$$

where n and n_s are the outward normal fields respectively to Ω and Ω_s, on Γ and Γ_s. Moreover, its derivative is given by:

$$\frac{d}{ds}(n_s \circ T_s) = \langle DV \cdot n_s, n_s \rangle \circ T_s n_s \circ T_s - DV^* \circ T_s n_s \circ T_s.$$

3.5. The volume evolution

Now we shall consider the volume evolution of the domain $\Omega_s = (I + u_s) \circ (I + u)^{-1}(\Omega) = T_s(\Omega)$, which we can also write as $\Omega_s = \varphi_s(\mathcal{O}) = (I + u_s)(\mathcal{O})$. We have the following general result (proved in Appendix C):

Lemma 3.2. *For any integer N, let $D \subset \mathbb{R}^N$ and $u \in W^{r+1,p}(D, \mathbb{R}^N)$ with $r > N/p$ then* $\det(I + Du) \in W^{r+1,p}(D)$ *and let $\Theta \subset D$ be an open domain with Lipschitz continuous boundary $\Sigma = \partial\Theta$, then we get*

$$|(I+u)(\Theta)| \;=\; |\Theta| \,+\, \int_\Sigma \langle u,\, M_u.n_\Sigma\rangle d\Sigma,$$

where $|\Theta| = \int_\Theta dx$, while the Matrix M_u is given by:

$$M_u = \int_0^1 \det(I + t\,Du)\,(I +\, t\,Du)^{-*}\,dt \;=\int_0^1 \mathrm{cof}[\;(I +\, t\,Du)\;]\,dt.$$

3.6. Shape derivatives

Assume that the transformation $T_s(V)$ is the flow mapping of a Lipschitz-continuous vector field $V(s,x)$. Then we get

$$\forall x \in \Gamma, \omega(s,x) = \det(DT_s(V))\,||(DT_s(V))^{-*}.n||$$

and

$$\forall x \in \Gamma,\; \frac{\partial}{\partial s}\omega(s,x)|_{s=0} \,=\, H(x)\,\langle V(0,x), n(x)\rangle,$$

where H is the mean curvature of Γ, $H = \mathrm{Tr}(D^2 b_\Omega)|_\Gamma = (\Delta b_\Omega)|_\Gamma$ and $v = \langle V(0,x), n(x)\rangle$ is the so-called normal speed of the moving boundary Γ_s.

3.7. Existence results for the material derivatives

Recall that $\mathcal{O}$ is the reference domain whose boundary is S and let $\mathcal{O}^c$ be its complement. The mapping $I + u_s$ is invertible from $\mathcal{O}$ onto Ω_s as soon as $\det(I + Du_s) > 0$ over $\mathcal{O}$. We extend the functions defined on Ω_s to the whole domain D as follows. Since the domain $\mathcal{O}$ is assumed smooth enough, it is known that there exists a continuous prolongation (or extension) mapping

$$P \in L(H^m(\mathcal{O}), H^m(D)), \quad s.t.\ \forall\phi \in H^m(\mathcal{O}),\; P.\phi|_{\mathcal{O}} = \phi.$$

Then any element $\Phi_s \in H^m(\Omega_s)$ can be extended to D by considering the continuous extension operator

$$P_s.\Phi_s = (\,P.(\Phi_s \circ (I+u_s)\;)\circ(I+u_s)^{-1}.$$

We transport the harmonic problem whose solution is $\phi_s \in H^1(\Omega_s)$. Let

$$\hat{\phi}^s :=\; \phi_s \circ (I + u_s) \in H^1(\mathcal{O}).$$

From Proposition 3.2 we know that

$$\mathrm{div}(\,A(s).\nabla\hat{\phi}^s\,) \,= 0 \;\text{ in }\; \mathcal{O}^c \tag{3.7}$$

where

$$A(s) = \det(I + Du_s)\,(I + Du_s)^{-1}.(I + Du_s)^{-*}.$$

Moreover, we have the following boundary condition

$$\langle n_S,\, A(s).\nabla\hat{\phi}^s\rangle \,=\, 0 \;\text{ on }\; S. \tag{3.8}$$

Concerning the elastic boundary condition on Γ_s, we have:

$$\mathcal{T}_s.n_s = \frac{1}{2}|\nabla_{\Gamma_s}\phi_s|^2 + f \text{ on } \Gamma_s, \tag{3.9}$$

where the forcing term f may be due to the gravity acceleration and can take the form $f(x) = \rho g x_3$ in $\mathbb{R}^3$. A change of variables gives

$$\mathcal{T}_s \circ (I+u_s).n_s \circ (I+u_s) = \frac{1}{2\det(I+Du_s)}\langle A(s).\nabla\hat{\phi}^s, \nabla\hat{\phi}^s\rangle + f \circ (I+u_s) \text{ on } S. \tag{3.10}$$

The stress tensor is the matrix

$$\mathcal{T}_s \circ (I+u_s) = \left(\frac{1}{\det(I+Du_s)}(I+Du_s).\Sigma(\sigma(u_s)).(I+D^*u_s)\right), \tag{3.11}$$

where

$$\Sigma(\sigma(u_s)) = C_{\lambda,\mu}.. \left[\frac{1}{2}\Big(Du_s + D^*u_s + Du_s.D^*u_s\Big)\right]. \tag{3.12}$$

Using Proposition 3.1, we have the following:

$$n_s \circ (I+u_s) = (I+Du_s)^{-*}.\nabla(b_{\Omega_s} \circ (I+u_s)).$$

Equations (3.7), (3.8), (3.9), (3.10), and (3.11) above form a system that we can rewrite as

$$\mathcal{F}(s,\ (u_s, \hat{\phi}^s)) = 0,$$

when the mapping

$$\begin{aligned} \mathcal{F} : [0,\, s_1[\times (H^2(\mathcal{O},\mathbb{R}^N) \times H^2(\mathcal{O}) \\ \to H^{-1}(\mathcal{O},\mathbb{R}^N) \times H^{-1}(\mathcal{O}) \times H^{-1/2}(S,\mathbb{R}^N) \times H^{-1/2}(S) \end{aligned}$$

verifies the Implicit Function theorem assumptions so that the derivative $u' := \frac{\partial}{\partial s}u_s|_{s=0}$ exists in $H^1(\mathcal{O}, R^N)$ and also $\frac{\partial}{\partial s}\hat{\phi}^s|_{s=0}$ exists in $H^1(\mathcal{O}, R)$.

3.8. Material derivatives

3.8.1. Displacement derivative. We consider the mapping $T_s = (I+u_s) \circ (I+u)^{-1}$ which maps Ω onto Ω_s and Γ onto Γ_s. Classically, we introduced the material derivatives of any element $\phi_s \in H^1(\Omega_s)$ as being the derivative (in $H^1(\Omega)$-norm)

$$\dot{\phi} = \frac{d}{ds}(\phi_s \circ T_s)|_{s=0}.$$

Concerning the elastic displacement, u_s is defined on the reference set $\mathcal{O}$ so we consider the element

$$\tilde{u}_s = u_s \circ (I+u_s)^{-1},$$

defined on Ω_s. Then the material derivative for this element is

$$\dot{\tilde{u}} = \frac{d}{ds}(\tilde{u}_s \circ T_s)|_{s=0}.$$

and we have

$$\tilde{u}_s \circ T_s = u_s \circ (I+u)^{-1},$$

which gives us

$$\dot{\tilde{u}} = \left(\frac{d}{ds}(\tilde{u}_s|_{s=0})\right) \circ (I+u)^{-1} = \tilde{u}' \circ (I+u)^{-1}.$$

3.8.2. Stress derivative. The transported stress tensor is $\mathcal{T}_s \circ T_s$. Recall that $T_s = \varphi_s \circ \varphi^{-1}$, where $\varphi_s = I + u_s$, and that $V(s) = \frac{\partial}{\partial s} T_s \circ T_s^{-1} = \frac{\partial}{\partial s}\varphi_s \circ \varphi_s^{-1}$.

The stress tensor $\mathcal{T}_s \circ T_s$ is the matrix

$$\mathcal{T}_s \circ T_s = \left(\frac{1}{\det(I+Du_s)}(I+Du_s).\Sigma(\sigma(u_s)).(I+D^*u_s)\right) \circ (I+u)^{-1} \tag{3.13}$$

where

$$\Sigma(\sigma(u_s)) = C_{\lambda,\mu}.. \left[\frac{1}{2}\Big(Du_s + D^*u_s + Du_s.D^*u_s\Big)\right]. \tag{3.14}$$

In (3.14) we assumed the four entries elasticity tensor to be governed by the *Lamé* coefficients λ and μ.

Taking derivative w.r.t s in (3.13), we obtain:

$$\begin{aligned}
\left(\left[\frac{\partial}{\partial s}\mathcal{T}_s \circ T_s\right]_{s=0}\right) \circ (I+u) = & -\frac{\operatorname{div}(u')}{\det(I+Du)}\ (I+Du).C_{\lambda,\mu}..(\sigma(u)).(I+D^*u) \\
& + \frac{1}{\det(I+Du)} D(u').C_{\lambda,\mu}..(\sigma(u)).(I+D^*u) \\
& + \frac{1}{\det(I+Du)}(I+Du).C_{\lambda,\mu}..(\sigma').(I+D^*u) \\
& + \frac{1}{\det(I+Du)}(I+Du).C_{\lambda,\mu}..(\sigma(u)).D^*(u')
\end{aligned}$$

where

$$\sigma' = \frac{1}{2}\Big(D(u') + D^*(u') + D(u').D^*u + Du.D^*(u')\Big).$$

Now we let

$$\begin{aligned}
\dot{\mathcal{T}} &= \left[\frac{\partial}{\partial s}\mathcal{T}_s \circ T_s\right]_{s=0} \\
&= \mathcal{T}' + \mathbf{D}\mathcal{T}.(u' \circ (I+u)^{-1})
\end{aligned}$$

where $\mathbf{D}$ is a three entries tensor, representing the gradient of the matrix $\mathcal{T}$. Its contraction with the vector $(u' \circ (I+u)^{-1})$ gives the matrix $\mathbf{D}\mathcal{T}.(u' \circ (I+u)^{-1})$. Then we have

$$\mathcal{T}' = \Big(-\frac{\operatorname{div}(u')}{\det(I+Du)}\ (I+Du).C_{\lambda,\mu}..(\sigma(u)).(I+D^*u)\Big) \circ (I+u)^{-1}$$

$$
\begin{aligned}
&+ \Big(\frac{1}{\det(I+Du)} D(u').C_{\lambda,\mu}..(\sigma(u)).(I+D^*u)\Big) \circ (I+u)^{-1} \\
&+ \Big(\frac{1}{\det(I+Du)} (I+Du).C_{\lambda,\mu}..(\sigma').(I+D^*u)\Big) \circ (I+u)^{-1} \\
&+ \Big(\frac{1}{\det(I+Du)} (I+Du).C_{\lambda,\mu}..(\sigma(u)).D^*(u')\Big) \circ (I+u)^{-1} \\
&- \mathbf{D}\mathcal{T}.(u' \circ (I+u)^{-1}).
\end{aligned} \tag{3.15}
$$

This expression is difficult to handle. Nevertheless, in the most "popular" framework (which consists in considering $u = 0$), it simplifies to the following expression:

$$\mathcal{T}' = C_{\lambda,\mu}..(\sigma') - \mathbf{D}\mathcal{T}.(u')$$

Since $u = 0$, we have $\mathcal{T} = 0$ and thus $\mathbf{D}\mathcal{T} = 0$. Moreover,

$$\sigma' = 1/2\,(Du' + D^*u').$$

Therefore

$$
\begin{aligned}
\mathcal{T}' &= C_{\lambda,\mu}..(Du' + D^*u') \\
\mathcal{T}' &= \lambda \operatorname{Tr} Du'\, I + \mu\,(Du' + D^*u') \\
&= \lambda \operatorname{Tr} Du'\, I + 2\mu\,\sigma', \quad 2\sigma' = Du' + D^*u'.
\end{aligned} \tag{3.16}
$$

4. Proofs

The expression of $\mathcal{T}'$ in terms of u' that we obtained above (3.15) will be needed in the proofs of all the theorems. Unfortunately, we can not make use of the nicer expression (3.16), since, even though we linearize the system around rest (i.e., the velocity of the fluid is $w = 0$), the elastic displacement u can not be zero (due to the fluid's pressure effect).

4.1. Proof of Theorem 2.1

The first step in the proof of the theorem is to write the variational form associated with system (1.14):

$$
\begin{cases}
\Delta\phi_s = 0 & \Omega_s^c \\
-\operatorname{Div}\mathcal{T}_s = 0 & \Omega_s \\
\nabla\phi_s \cdot n_s = 0 & \Gamma_s \\
\mathcal{T}_s.n_s = (p_0 + \frac{1}{2}\|\nabla\phi_s\|^2 - \rho g x_3) n_s & \Gamma_s \\
u = 0 & \Gamma' \\
\int_{\Gamma_{\text{out}}} \alpha_s \, d\Gamma_{\text{out}} = -(a+s) \int_{\Gamma_{\text{in}}} c(x)\, d\Gamma_{\text{in}}, \quad \text{for all} \quad s \geq 0
\end{cases} \tag{4.1}
$$

$\forall \Psi \in H^1(\mathcal{D})$, $\forall R \in H^1(\mathcal{D}, \mathbb{R}^3)$, we have that

$$\int_{\Omega_s} \mathcal{T}_s..DR\,dx + \int_{\Omega_s^c} \langle\nabla\phi_s, \nabla\Psi\rangle dx = \int_{\Gamma_s} \left\{ p_0 + \frac{1}{2}|\nabla\phi_s|^2 + \rho g\, x_3 \right\} \langle n_s, R\rangle d\Gamma_s. \tag{4.2}$$

Our goal is to compute the s derivatives (at $s = 0$). Let $\phi' = \dfrac{\partial}{\partial s}\phi_s|_{s=0}$, $\mathcal{T}' = \dfrac{\partial}{\partial s}\mathcal{T}_s|_{s=0}$, and $v = \langle V(0), n\rangle$ on Γ. Recall that $n_s = \nabla b_{\Omega_s}$ is the unit outer normal to Ω_s, and $H = \Delta b_\Omega$ is the mean curvature of Γ. Let $n' = \dfrac{d}{ds}(\nabla b_{\Omega_s})|_{s=0}$. Taking derivative with respect to s at $s = 0$ in (4.2), we obtain:

$$\begin{aligned}
&\int_\Omega \mathcal{T}'..DR\,dx + \int_\Gamma \mathcal{T}..DR\,v\,d\Gamma + \int_{\Omega^c}\langle\nabla\phi', \nabla\Psi\rangle dx - \int_\Gamma \langle\nabla\phi, \nabla\Psi\rangle\,v\,d\Gamma \\
&= \int_\Gamma \{\langle\nabla\phi', \nabla\phi\rangle\}\,\langle n, R\rangle d\Gamma + \int_\Gamma \left\{p_0 + \frac{1}{2}|\nabla\phi|^2 + \rho g\,x_3\right\}\langle n', R\rangle d\Gamma \\
&\quad + \int_\Gamma \frac{\partial}{\partial n}\left\{p_0 + \left(\frac{1}{2}|\nabla\phi|^2 + \rho g\,x_3\right)\langle\nabla b_\Omega, R\rangle\right\}\,v\,d\Gamma \\
&\quad + \int_\Gamma H\left(p_0 + \frac{1}{2}|\nabla\phi|^2 + \rho g\,x_3\right)\langle\nabla b_\Omega, R\rangle v\,d\Gamma. \qquad (4.3)
\end{aligned}$$

Choosing Ψ (respectively R) with compact support in Ω^c (respectively in Ω) and using the following integration by parts formula

$$\int_\Omega \mathcal{T}'..DR\,dx = -\int_\Omega \langle\vec{\mathrm{Div}}(\mathcal{T}'), R\rangle dx + \int_\Gamma \langle\mathcal{T}'.n, R\rangle d\Gamma$$

we recover the following equations for $(\phi', \mathcal{T}')$ on Ω^c and Ω.

$$-\Delta\phi' = 0 \ \text{ in } \ \Omega^c$$

$$-\vec{\mathrm{Div}}(\mathcal{T}') = 0 \ \text{ in } \ \Omega.$$

Now we are concerned with the boundary conditions. We first note that

$$\begin{aligned}
&\frac{\partial}{\partial n}\Big\{\Big(p_0 + \frac{1}{2}|\nabla\phi|^2 + \rho g\,x_3\Big)\langle\nabla b_\Omega, R\rangle\Big\} \qquad (4.4)\\
&= \Big\langle n,\, \nabla\Big\{\Big(p_0 + \frac{1}{2}|\nabla\phi|^2 + \rho g\,x_3\Big)\langle\nabla b_\Omega, R\rangle\Big\}\Big\rangle \\
&= (\langle n,\, D^2\phi.\nabla\phi\rangle + \rho g\,n_3)\langle n, R\rangle + \Big(p_0 + \frac{1}{2}|\nabla\phi|^2 + \rho g\,x_3\Big)\langle n\,, D^2 b_\Omega.R + D^*R.n\rangle.
\end{aligned}$$

Due to the fact that $D^2 b_\Omega$ is symmetric and that $D^2 b_\Omega.n = \nabla(\frac{1}{2}|\nabla b_\Omega|^2) = 0$, we have that the term

$$\langle n, D^2 b_\Omega.R\rangle = \langle D^2 b_\Omega.n, R\rangle = 0. \qquad (4.5)$$

Therefore, concerning the boundary conditions, and taking $\Psi \in H^1(D)$ and $R = 0$, we obtain:

$$\frac{\partial}{\partial n}\phi' = \mathrm{div}_\Gamma(\,v\,\nabla_\Gamma\phi\,). \qquad (4.6)$$

In addition, choosing $\Psi = 0$ and $R \in H^1(D, R^N)$ with $DR.n = 0$ on Γ, we obtain:

$$\langle n\,,\, D^*R.n\rangle = \langle DR.n,\, n\rangle = 0. \qquad (4.7)$$

Combining (4.4) with (4.5) and (4.7), we obtain the following identity:

$$\frac{\partial}{\partial n}\left\{\left(p_0+\frac{1}{2}|\nabla\phi|^2+\rho g\,x_3\right)\langle\nabla b_\Omega,R\rangle\right\}$$
$$=(\langle n,\,D^2\phi.\nabla\phi\rangle\;+\;\rho g\,n_3\,)\langle n,R\rangle. \tag{4.8}$$

Finally, using (4.8) in (4.3), we obtain the following variational problem at the boundary:

$$\forall R\in H^1(\Gamma,R^N)$$

$$\int_\Gamma\langle\mathcal{T}'.n,R\rangle\,d\Gamma\;+\;\int_\Gamma(\mathcal{T}..DR)\,v\,d\Gamma$$
$$=\int_\Gamma\langle\nabla\phi',\nabla\phi\rangle\,\langle n,R\rangle\,d\Gamma+\int_\Gamma\left\{p_0+\frac{1}{2}|\nabla\phi|^2+\rho g\,x_3\right\}\langle n',R\rangle d\Gamma$$
$$+\int_\Gamma\langle(n,\,D^2\phi.\nabla\phi\rangle\;+\;\rho g\,n_3\,)\langle n,R\rangle v\,d\Gamma$$
$$+\int_\Gamma H\left(p_0+\frac{1}{2}|\nabla\phi|^2+\rho g\,x_3\right)\langle\nabla b_\Omega,R\rangle\,v\,d\Gamma. \tag{4.9}$$

Now we have to perform a tangential by part integration in the term:

$$\int_\Gamma(\mathcal{T}..DR)\,v\,d\Gamma=\int_\Gamma\mathcal{T}_{i,j}\frac{\partial}{\partial x_j}R_i\,v\,d\Gamma=\int_\Gamma\langle\mathcal{T}_{i,.},\nabla R_i\rangle v\,d\Gamma.$$

But as for all i we have $\frac{\partial}{\partial n}R_i=0$, then

$$\nabla R_i=\nabla_\Gamma R_i\;+\;\frac{\partial}{\partial n}R_i\;\;\vec{n}=\nabla_\Gamma R_i,$$

and

$$\int_\Gamma(\mathcal{T}..DR)\,v\,d\Gamma=\int_\Gamma\langle\mathcal{T}_{i,.},\nabla_\Gamma R_i\rangle v\,d\Gamma$$
$$=-\int_\Gamma\mathrm{div}_\Gamma(v\,\mathcal{T}_{i,.})\;R_i\;\,d\Gamma+\int_\Gamma vH\langle\mathcal{T}_i.n,R_i\rangle\,d\Gamma$$
$$=-\int_\Gamma\langle\vec{\mathrm{Div}}_\Gamma(v\,\mathcal{T}),\;\,R\rangle\;\,d\Gamma,+\int_\Gamma vH\langle\mathcal{T}.n,R\rangle\,d\Gamma. \tag{4.10}$$

Combining (4.9) with (4.10), we obtain the following boundary condition for the stress function $\mathcal{T}'$: $\forall R\in H^1(\Gamma,R^N)$:

$$\int_\Gamma\langle\mathcal{T}'.n,R\rangle d\Gamma=\int_\Gamma\langle\vec{\mathrm{Div}}_\Gamma(v\,\mathcal{T}),\;\,R\rangle\,d\Gamma+\int_\Gamma\langle\nabla\phi',\nabla\phi\rangle\,\langle n,R\rangle\,d\Gamma$$
$$+\int_\Gamma\left\{p_0+\frac{1}{2}|\nabla\phi|^2+\rho g\,x_3\right\}\langle n',R\rangle\,d\Gamma$$
$$+\int_\Gamma(\langle n,\,D^2\phi.\nabla\phi\rangle\;+\;\rho g\,n_3\,)\langle n,R\rangle\,v\,d\Gamma.$$

Therefore, on Γ we obtain:

$$\mathcal{T}'.n = \vec{\mathrm{Div}}_\Gamma(v\,\mathcal{T}) \;\; + \langle \nabla\phi', \nabla\phi\rangle\,\vec{n} + \left\{ p_0 + \frac{1}{2}|\nabla\phi|^2 + \rho g\, x_3 \right\} \vec{n}' \\ + \left(\langle n,\, D^2\phi.\nabla\phi\rangle \,+\, \rho g\, n_3 \right) v\,\vec{n}. \tag{4.11}$$

We know that

$$\mathrm{Div}_\Gamma(v\mathcal{T}) = v\,\mathrm{Div}_\Gamma + \mathcal{T}.\nabla_\Gamma v. \tag{4.12}$$

Moreover, we have the following calculus of the tangent vector n': From [19], [15], we know that in some neighborhood $\mathcal{U}$ of $\Sigma = \cup_{0<s<s_1}\{s\}\times\partial\Omega_s$ the oriented distance function solves the convection equation

$$\frac{\partial}{\partial s} b_{\Omega_s} + \nabla b_{\Omega_s}.V(s)\circ p_{\Gamma_s} = 0$$

where $p_{\Gamma_s} = Id \,-\, b_{\Omega_s}\nabla b_{\Omega_s}$ is the projection mapping onto Γ_s.

Then we obtain that

$$\begin{aligned} n' := \frac{\partial}{\partial s}(\nabla b_{\Omega_s})_{s=0} &= \nabla\left(\frac{\partial}{\partial s} b_{\Omega_s}\right)_{s=0} = (\; \nabla(-\nabla b_{\Omega_s}.V(s)\circ p_{\Gamma_s})\;)_{s=0} \\ &= -\,(\,D^2 b_{\Omega_s}.V(s)\circ p_{\Gamma_s}\;)_{s=0} - (\;\; D^*(V(s)\circ p_{\Gamma_s}).\nabla b_{\Omega_s}\;\;)_{s=0} \\ &= -\;D^2 b_\Omega(x).V(0,x) \;-\; D^*_\Gamma V(0,x).n(x) \;\; = -\nabla_\Gamma v(x), \end{aligned} \tag{4.13}$$

where we recall that $v(x) = \langle V(0,x), n(x)\rangle$ on Γ is the normal speed of the boundary.

Combining (4.11) with (4.12) and (4.13), we obtain the following new expression on the boundary Γ:

$$\mathcal{T}'.n = \mathcal{T}.\nabla_\Gamma v \;\; + \langle \nabla_\Gamma\phi', \nabla_\Gamma\phi\rangle\,\vec{n} - \left\{ p_0 + \frac{1}{2}|\nabla_\Gamma\phi|^2 + \rho g\, x_3 \right\} \nabla_\Gamma v \\ + \left(\langle n,\, D^2\phi.\nabla\phi\rangle \,+\, \rho g\, n_3 \right) v\,\vec{n} + v\,\mathrm{Div}_\Gamma\,\mathcal{T} \tag{4.14}$$

$$= \left[\mathcal{T} - \left\{ p_0 + \frac{1}{2}|\nabla_\Gamma\phi|^2 + \rho g\, x_3 \right\} I \right].\nabla_\Gamma v \;\; + \langle \nabla_\Gamma\phi', \nabla_\Gamma\phi\rangle\,\vec{n} \\ + \left(\langle n,\, D^2\phi.\nabla\phi\rangle \,+\, \rho g\, n_3 \right) v\,\vec{n} + v\,\mathrm{Div}_\Gamma\,\mathcal{T} \tag{4.15}$$

Here we want to point out that the term $\nabla_\Gamma v$ contains the mean curvature of the boundary Γ. More specifically, recall that $V(0) = u'\circ(I+u)^{-1}$, and thus

$$v = \langle u'\circ(I+u)^{-1}, n\rangle. \tag{4.16}$$

Therefore, we have the following expression for $\nabla_\Gamma v$:

$$\nabla_\Gamma v = \nabla_\Gamma(\langle(u'\circ(I+u)^{-1}), n\rangle) = D^*_\Gamma(u'\circ(I+u)^{-1}).n \,+\, D^2 b_\Omega.(u'\circ(I+u)^{-1})_\Gamma \tag{4.17}$$

where $D^2 b_\Omega$ is the symmetrical matrix whose eigenvalues are the main curvatures λ_1 and λ_2, and whose trace is $\mathrm{Tr}(D^2 b_\Omega) = H$, the mean curvature of the boundary.

Now combining (4.15) with (4.16) and (4.17), we obtain:

$$\begin{aligned}\mathcal{T}'.n = &\left[\mathcal{T} - \left\{p_0 + \frac{1}{2}|\nabla_\Gamma \phi|^2 + \rho g x_3\right\} I\right].(D_\Gamma^*(u' \circ (I+u)^{-1}).n \\ &+ D^2 b_\Omega.(u' \circ (I+u)^{-1})_\Gamma) \\ &+ \langle \nabla_\Gamma \phi', \nabla_\Gamma \phi\rangle \vec{n} + (\langle n, D^2\phi.\nabla\phi\rangle + \rho g\, n_3)\langle u' \circ (I+u)^{-1}, n\rangle \vec{n} \\ &+ \langle u' \circ (I+u)^{-1}, n\rangle \operatorname{Div}_\Gamma \mathcal{T}. \qquad (4.18)\end{aligned}$$

We assume that the fluid speed v is steady, but not zero. We are in the case where the flow is irrotational, so that v derives from a harmonic potential in Ω^c, that is $v_s = \nabla\phi_s$. From (4.18), with the following notation

$$\mathbf{\Phi} = \phi', \quad U = u' \circ (I+u)^{-1}, \quad \text{and} \quad p = p_0 + \frac{1}{2}|\nabla_\Gamma\phi|^2 + \rho g x_3,$$

we obtain the following linearized system for the fluid-structure coupling $(\mathbf{\Phi}, u')$:

$$\begin{aligned}\Delta\mathbf{\Phi} &= 0 && \text{in } \Omega^c \\ \frac{\partial}{\partial n}\mathbf{\Phi} &= -\operatorname{div}_\Gamma(\langle U, n\rangle \nabla_\Gamma \phi) && \text{on } \Gamma \\ \operatorname{Div}(\mathcal{T}') &= 0 && \text{in } \Omega\end{aligned}$$

$$\begin{aligned}\mathcal{T}'.n = &[\mathcal{T} - pI].(D_\Gamma^* U.n + D^2 b_\Omega.U_\Gamma) + \langle \nabla_\Gamma \mathbf{\Phi}, \nabla_\Gamma\phi\rangle \vec{n} \\ &+ (\langle n, D^2\phi.\nabla\phi\rangle + \rho g\, n_3)\langle U, n\rangle \vec{n} + \langle U, n\rangle \operatorname{Div}_\Gamma \mathcal{T} \quad \text{on } \Gamma \qquad (4.19)\end{aligned}$$

with the boundary conditions

$$\begin{aligned}\frac{\partial}{\partial n}\mathbf{\Phi} &= c(x) && \text{on } \Gamma_{\text{in}} \\ \frac{\partial}{\partial n}\mathbf{\Phi} &= -\frac{\int_{\Gamma_{\text{in}}} c(x)d\Gamma}{\int_{\Gamma_{\text{out}}} d\Gamma} && \text{on } \Gamma_{\text{out}},\end{aligned}$$

where $\mathcal{T}'$ and $\mathcal{T}$ are given by (3.15) and (1.2), respectively.

Our linearized system (4.19) does recover the linear equations for $(\Phi, \mathcal{T}')$ on Ω and Ω^c. Nevertheless, we can see that the boundary conditions are quite complicated. First off, there is a double coupling on the boundary Γ. Then, we clearly see the presence of the curvatures on the common interface Γ.

4.2. Proof of Theorem 2.2

The proof follows immediately after the proof of Theorem 2.1. Now we are in the situation when, considering $a = 0$, then at $s = 0$ the forcing condition on Γ_{in} is zero and thus $\phi = 0$. Nevertheless, some pressure term remains ($p = p_0 + \rho g\, x_3$), and thus the linearized fluid-structure problem is

$$\begin{aligned}\Delta\mathbf{\Phi} &= 0 \quad \text{in } \Omega^c \\ \frac{\partial}{\partial n}\mathbf{\Phi} &= 0 \quad \text{on } \Gamma\end{aligned}$$

$$\mathrm{Div}(\mathcal{T}') = 0 \quad \text{in} \quad \Omega$$

$$\begin{aligned}\mathcal{T}'.n =& [\mathcal{T} - pI].(D^*_\Gamma U.n \, + \, D^2 b_\Omega.U_\Gamma) \\ &+ \, \rho g \, n_3 \langle U, n\rangle \vec{n} + \langle U, n\rangle \, \mathrm{Div}_\Gamma \, \mathcal{T} \quad \text{on} \quad \Gamma \end{aligned} \tag{4.20}$$

where notation and the boundary conditions are the same as in the proof of Theorem 2.1.

4.3. Proof of Theorem 2.3

Now we are concerned with the Navier-Stokes flow-elastic structure interaction (1.11):

$$\begin{cases} -\nu \, \Delta \vec{w}_s \, + \, D w_s.w_s \, + \, \nabla p_s \, = \, 0 & \Omega_s^c \\ \mathrm{div} \, w_s = 0 & \Omega_s^c \\ - \, \mathrm{Div} \, \mathcal{T}_s = 0 & \Omega_s \\ w_s = 0 & \Gamma_s \\ \mathcal{T}_s.n_s = p_s \vec{n}_s - \epsilon(w_s).\vec{n}_s & \Gamma_s \\ u = 0 & \Gamma' \\ \int_{\Gamma_{\mathrm{out}}} \alpha_s \, d\Gamma_{\mathrm{out}} = -(a+s) \int_{\Gamma_{\mathrm{in}}} c(x) \, d\Gamma_{\mathrm{in}}, \quad \text{for all} \quad s \geq 0 & \end{cases}$$

where n_s is the unit outer normal vector along Γ_s, $2\epsilon(w_s) = Dw_s + Dw_s^*$, and $\mathcal{T}_s : \bar{\Omega}_s \to \mathbb{S}^3$ is the Cauchy stress tensor (associated to s), given by

$$\mathcal{T}_s = \Big(\frac{1}{\det(\nabla \varphi_s)} \nabla \varphi_s \cdot \Sigma(\sigma(u_s)) \cdot (\nabla \varphi_s)^* \Big) \circ \varphi_s^{-1}. \tag{4.21}$$

We start by writing the variational form associated with system (1.11). Since $w_s \in H^1_0(\mathcal{D}; \mathbb{R}^3)$, with $\mathrm{div}\, w_s = 0$, $\forall \theta_s \in H^1_0(\Omega_s^c; \mathbb{R}^3)$, with $\mathrm{div}\, \theta_s = 0$, and $\forall R \in H^1(\mathcal{D}, \mathbb{R}^3)$, we have

$$\begin{aligned}\int_{\Omega_s} (\mathcal{T}_s..DR) dx + \int_{\Omega_s^c} (\nu \, D w_s..D\theta_s + \, \langle D w_s.w_s, \theta_s \rangle) \, dx + \int_{\Omega_s^c} \mathrm{div}(D^* \theta_s.w_s) dx \\ = \int_{\Gamma_s} \langle \{ \, p_s \vec{n}_s - \epsilon(w_s).\vec{n}_s \, \}, \, R \rangle \, d\Gamma_s. \end{aligned} \tag{4.22}$$

Regarding θ_s, we have $\theta_s = (DT_s.\theta) \circ T_s^{-1}$, $\forall \theta \in H^1_0(\Omega^c) \cap H^2(\Omega^c)$, with $\mathrm{div}\, \theta = 0$ on Ω^c and $\theta = 0$ on Γ.

The fluid pressure p_s is given by the Neumann problem obtained from system (1.11):

Lemma 4.1. *p_s is solution to the following Neumann problem:*

$$-\Delta p_s = \mathrm{div}(\, D w_s.w_s \,) \quad in \quad \Omega_s^c$$

$$\frac{\partial}{\partial n_s} p_s = -H_s \, \mathrm{div}_{\Gamma_s} w_s + \Big\langle \frac{\partial^2}{\partial n_s^2} w_s, n_s \Big\rangle = \Big\langle \frac{\partial^2}{\partial n_s^2} w_s, n_s \Big\rangle \; on \quad \Gamma_s$$

where $H_s = \Delta b_{\Omega_S}$ is the mean curvature of Γ_s.

In the next step, we have to compute the s derivatives (at $s = 0$). Let $w' = \frac{\partial}{\partial s} w_s\Big|_{s=0}$, $p' = \frac{\partial}{\partial s} p_s\Big|_{s=0}$, and $u' = \frac{\partial}{\partial s} u_s\Big|_{s=0}$ represent the shape derivatives.

First, note that the boundary integral turns into a volume integral:

$$\int_{\Gamma_s} \langle \{ p_s \vec{n}_s - \epsilon(w_s).\vec{n}_s \}, R \rangle \, d\Gamma_s = - \int_{\Omega_s^c} \operatorname{div}(p_s R - \epsilon(w_s).R) \, dx$$

and thus taking derivative w.r.t. s in at $s = 0$, we obtain:

$$\begin{aligned} &\left[\frac{\partial}{\partial s} \int_{\Gamma_s} \langle \{ p_s \vec{n}_s - \epsilon(w_s).\vec{n}_s \}, R \rangle \, d\Gamma_s \right]_{s=0} \\ &\quad = - \int_{\Omega^c} \operatorname{div}(p' R - \epsilon(w').R) \, dx + \int_\Gamma \operatorname{div}(pR - \epsilon(w).R) \, v \, d\Gamma \\ &\quad = \int_\Gamma \{ \langle p' R - \epsilon(w').R , n \rangle + \operatorname{div}(pR - \epsilon(w).R) \, v \} \, d\Gamma. \end{aligned} \tag{4.23}$$

Now we take the s derivative of the weak formulation (4.22) and using (4.23) and the fact that $\theta_0 = \theta$ at $s = 0$, we obtain:

$$\begin{aligned} &\int_\Omega (\mathcal{T}'..DR) dx + \int_\Gamma (\mathcal{T}..DR) \, v \, d\Gamma \\ &\quad + \int_{\Omega^c} (\nu \, Dw'..D\theta + \langle Dw'.w + Dw.w', \theta \rangle) \, dx - \int_\Gamma (\nu \, Dw..D\theta + \langle Dw.w, \theta \rangle) \, v \, d\Gamma \\ &\quad + \int_{\Omega^c} (\nu \, Dw..D\theta' + \langle Dw.w, \theta' \rangle + \operatorname{div}(D^*\theta'.w)) \, dx - \int_\Gamma \operatorname{div}(D^*\theta.w) v \, d\Gamma \\ &= \int_\Gamma \{ \langle p' R - \epsilon(w').R , n \rangle + \operatorname{div}(pR - \epsilon(w).R) \, v \} \, d\Gamma + \int_\Gamma D\theta.n w' \, d\Gamma \end{aligned} \tag{4.24}$$

where $\theta' = \frac{\partial}{\partial s} \theta_s \Big|_{s=0} = \frac{\partial}{\partial s} [(DT_s.\theta) \circ T_s^{-1}] \Big|_{s=0} = DV(0) - D\theta.V(0)$.

Recalling the by part integration formula:

$$\int_\Omega (\mathcal{T}'..DR) dx = - \int_\Omega \langle \vec{\operatorname{Div}}(\mathcal{T}'), R \rangle dx + \int_\Gamma \langle \mathcal{T}'.n, R \rangle d\Gamma,$$

and taking θ (respectively R) with compact support in Ω^c (respectively in Ω), we obtain the following linearized equations:

$$\begin{cases} -\nu \, \Delta w' + Dw'.w + Dw.w' + \nabla p' = 0 & \text{in } \Omega^c \\ -\vec{\operatorname{Div}}(\mathcal{T}') = 0 & \text{in } \Omega. \end{cases}$$

Regarding the terms involving θ' that appear in the Ω^c-integrals in (4.24), they all show up in combination with w. Since we will linearize the system near "rest", we will consider the fluid velocity $w = 0$. Hence, all these terms will disappear. The same will happen with the boundary integral $\int_\Gamma \operatorname{div}(D^*\theta.w) v \, d\Gamma$. Nevertheless, we want to point out that when linearizing around $w \neq 0$, all the terms mentioned above can not be neglected, since they will bring extra terms in the linearized equations.

Now we look at the boundary integrals. Since $w = 0$ on Γ, then we have

$$\int_\Gamma \langle \mathcal{T}'.n, R\rangle d\Gamma + \int_\Gamma (\mathcal{T}..DR)\, v\, d\Gamma - \int_\Gamma \nu\langle Dw'.n, \theta\rangle - \int_\Gamma \nu\, Dw..D\theta\ v\, d\Gamma$$
$$= \int_\Gamma \{\langle p'\, n - \epsilon(w').n\,,\, R\rangle + \operatorname{div}(\, pR - \epsilon(w).R\,)\, v\,\}\, d\Gamma.$$

Now

$$\begin{aligned}\int_\Gamma \operatorname{div}(\, pR - \epsilon(w).R\,)\, v\ d\Gamma = &- \int_\Gamma \langle (pR - \epsilon(w).R)\,,\, \nabla_\Gamma v\rangle\, d\Gamma\\ &+ \int_\Gamma H\langle pR - \epsilon(w).R, n\rangle\, v\ d\Gamma\\ &+ \int_\Gamma \langle D(\, pR - \epsilon(w).R\,).n, n\rangle\, v\ d\Gamma.\end{aligned}$$

Choosing R such that $DR.n = 0$, we obtain the following identity:

$$\langle D(\, pR).n, n\rangle = p\langle DR.n, n\rangle + \langle (\,\nabla p.R^*).n, n\rangle = \frac{\partial}{\partial n} p\, \langle R, n\rangle.$$

Concerning the last term we have

$$\begin{aligned}\langle D(\,\epsilon(w).R\,).n, n\rangle &= \partial_i\, (\,\epsilon(w)_{j,k} R_k\ )\ n_i n_j\\ &= \partial_i \epsilon(w)_{j,k} R_k\ n_i n_j\ +\ \epsilon(w)_{j,k}\ n_i n_j \partial_i R_k\\ &= \partial_i \epsilon(w)_{j,k} R_k\ n_i n_j\ +\ \epsilon(w)_{j,k} n_j\ (DR.n)_k\end{aligned}$$

[which by the previous choice of R simplifies to:]

$$\begin{aligned}&= \partial_i \epsilon(w)_{j,k} R_k\ n_i n_j = \frac{\partial}{\partial n}\epsilon(w)_{j,k}\ (R.n^*)_{j,k}\\ &= \left\langle \frac{\partial}{\partial n}\epsilon(w).n,\ R\right\rangle = \langle (D\epsilon(w).n).n, R\rangle.\end{aligned}$$

Finally, the boundary integrals give:

$$\begin{aligned}&\int_\Gamma \langle \mathcal{T}'.n, R\rangle d\Gamma + \int_\Gamma (\mathcal{T}..DR)\, v\, d\Gamma\\ &\quad - \int_\Gamma \nu\langle Dw'.n, \theta\rangle - \int_\Gamma \nu\, Dw..D\theta\ v\, d\Gamma + \int_\Gamma D\theta.nw'\, d\Gamma\\ &= \int_\Gamma \{\langle (p'\, I - \epsilon(w')).n\,,\, R\rangle + \int_\Gamma H\langle pR - \epsilon(w).R, n\rangle v\ d\Gamma\\ &\quad - \int_\Gamma \langle (\,(pI\ - \epsilon(w)\,).\nabla_\Gamma v,\ R\rangle\, d\Gamma + \int_\Gamma \left\langle \left(\frac{\partial}{\partial n} p\ \ I - \frac{\partial}{\partial n}\epsilon(w)\right).n,\ R\right\rangle v\ \ d\Gamma.\end{aligned}$$

Moreover, using the fact that

$$\int_\Gamma (\mathcal{T}..DR)\, v\, d\Gamma\ = - \int_\Gamma (\langle \vec{\operatorname{Div}}_\Gamma(v\,\mathcal{T}),\ R\rangle\ +\ vH\langle \mathcal{T}.n,\ R\rangle\,) d\Gamma,$$

we obtain the following equations on the boundary:

$$\mathcal{T}'.n = (p'\,I - \epsilon(w')).n + (pI - \epsilon(w)).\nabla_\Gamma v + \left(\frac{\partial}{\partial n}p\;\;I - \frac{\partial}{\partial n}\epsilon(w)\right).n\,v - \vec{\mathrm{Div}}_\Gamma(v\,\mathcal{T}) \tag{4.25}$$

and

$$w' = -Dw.nv. \tag{4.26}$$

We assume now that $a = 0$ so that at $s = 0$, and we get the "rest" system with $\vec{w} = 0$. Nevertheless, there is still pressure p in the fluid (since the fluid has density $\rho > 0$). Since $v = u' \circ (I+u)^{-1}.n$, using the previously introduced $U = u' \circ (I+u)^{-1}$, we obtain the following boundary condition:

$$\begin{aligned}\mathcal{T}'.n &= (p'\,I - \epsilon(w')).n + \langle \nabla p, n\rangle \vec{n} v + p\nabla_\Gamma v - \vec{\mathrm{Div}}_\Gamma(v\,\mathcal{T}) \\ &= (p'\,I - \epsilon(w')).n + \langle \nabla p, n\rangle \vec{n} v + p\nabla_\Gamma v - v\vec{\mathrm{Div}}_\Gamma(\mathcal{T}) - \mathcal{T}\nabla_\Gamma v \\ &= (p'\,I - \epsilon(w')).n + \langle \nabla p, n\rangle \vec{n} v + (pI - \mathcal{T}).(D^*_\Gamma U.n + D^2 b_\Omega.U_\Gamma) - v\vec{\mathrm{Div}}_\Gamma(\mathcal{T})\end{aligned} \tag{4.27}$$

where we recall that $\mathrm{Tr}(D^2 b_\Omega) = H$, the mean curvature of Γ. Thus at this point we note clearly the presence of the mean curvature on the interface.

The linearized Navier-Stokes equation whose w', p' is solution, when $w = 0$ becomes the linear Stokes system. Therefore, we obtain the following linearization around "rest":

$$\begin{cases} -\Delta w' + \nabla p' = 0 & \Omega^c \\ \mathrm{div}\; w' = 0 & \Omega^c \\ w' = 0 & \Gamma \\ -\vec{\mathrm{Div}}(\mathcal{T}') = 0 & \Omega \\ \mathcal{T}'.n = (p'I - \epsilon(w')).n + \langle \nabla p, n\rangle\langle U, n\rangle \vec{n} + (pI - \mathcal{T})(D^*_\Gamma U.n + D^2 b_\Omega.U_\Gamma) & \\ \qquad -\langle U, n\rangle \vec{\mathrm{Div}}_\Gamma(\mathcal{T}) & \Gamma \\ w'.n_{\mathrm{in}} = c(x) & \Gamma_{\mathrm{in}} \\ w'.n_{\mathrm{out}} = -\int_{\Gamma_{\mathrm{in}}} c(x)\; d\Gamma / \int_{\Gamma_{\mathrm{out}}} d\Gamma & \Gamma_{\mathrm{out}} \end{cases} \tag{4.28}$$

where, as before, $\mathcal{T}'$ and $\mathcal{T}$ are given by (3.15) and (1.2), respectively.

Again, just as in the case of potential fluid-structure coupling, we note that the linearization of the system turns out to be quite different from the usual coupling of classical linear modelings and it shows that the common boundary Γ (and implicitly the mean curvature of the boundary) plays a key role in the analysis of the coupling.

Appendix A. Nonlinear, 3D elasticity

At rest, the elastic body occupies a reference configuration $\overline{\mathcal{O}} \in \mathbb{R}^3$, where $\mathcal{O}$ is a bounded, open, connected set in $\mathbb{R}^3$ with sufficiently smooth boundary $\mathcal{S} \cup \Gamma'$. When subjected to applied forces, the elastic body occupies a deformed configuration $\Omega = \varphi(\overline{\mathcal{O}})$, with smooth boundary $\Gamma \cup \Gamma'$ (where Γ' is fixed). The deformation of the reference configurations is given by the map $\varphi : \overline{\mathcal{O}} \to \mathbb{R}^3$, that is smooth enough, injective (except possibly on the boundary of the set $\mathcal{O}$), and orientation-preserving (i.e., $\det D\varphi(x) > 0$, for all $x \in \overline{\mathcal{O}}$).

Together with the deformation φ, we introduce the displacement $u : \overline{\mathcal{O}} \to \mathbb{R}^3$, defined as usual as $\varphi = I + u$, where I denotes the identity map $I : \overline{\mathcal{O}} \to \mathbb{R}^3$.

It is well known that a body occupying a deformed configuration $\overline{\Omega}$, and subjected to zero applied body forces in its interior Ω and to applied surface forces on the boundary Γ, is in static equilibrium if the fundamental stress principle of Euler and Cauchy is satisfied:

$$\begin{cases} -\operatorname{Div} \mathcal{T} = 0 & \text{in } \Omega \\ \mathcal{T}.n^{\varphi} = g^{\varphi} & \text{on } \Gamma \end{cases} \tag{A.1}$$

where g^{φ} represents the density of the applied surface force, n^{φ} is the unit outer normal vector along Γ, and the tensor $\mathcal{T}$ is the Cauchy stress tensor. The above equilibrium equations over $\overline{\Omega}$ are equivalent to the equilibrium equations over the reference configuration $\overline{\mathcal{O}}$:

$$\begin{cases} -\operatorname{Div} \mathcal{P} = 0 & \text{in } \mathcal{O} \\ \mathcal{P}.n = g & \text{on } \mathcal{S} \end{cases} \tag{A.2}$$

where n denotes the unit outer normal vector along $\mathcal{S}$, $g da = g^{\varphi} da^{\varphi}$, and $\mathcal{P} : \overline{\mathcal{O}} \to \mathbb{M}^3$ is the Piola transform of the Cauchy stress tensor field, defined by

$$\mathcal{P}(x) = \mathcal{T}(x^{\varphi}) \mathrm{Cof} \nabla T(x) = \det(D\varphi(x)) \mathcal{T}(x^{\varphi}) (D\varphi)^{-*}. \tag{A.3}$$

From the constitutive equations, we have that $\mathcal{P}(x) = D\varphi(x)\Sigma(\sigma(u(x)))$, where Σ defines the second Piola-Kirchhoff stress tensor. In terms of the displacement u, Σ is given by

$$\Sigma(\sigma(u)) = \lambda(\mathrm{tr}\sigma(u))I + 2\mu\sigma(u) \tag{A.4}$$

where λ and μ are the Lamé constants of the material, and the Green-St. Venant strain tensor $\sigma(u)$ is given by

$$\sigma(u) = \frac{1}{2}(Du^* + Du + Du^* Du). \tag{A.5}$$

Therefore equations (A.2) can be rewritten as

$$\begin{cases} -\operatorname{div}[(I + Du)\Sigma(\sigma(u))] = 0 & \text{in } \mathcal{O} \\ (I + Du)\Sigma(\sigma(u))n = g & \text{on } \mathcal{S}. \end{cases} \tag{A.6}$$

The advantage of the equilibrium equations over the reference configuration (A.2) or (A.6) over (A.1) is the fact that they are written in terms of the Lagrange variable x that is attached to the reference configuration, instead of the Euler variable $x^{\varphi} = \varphi(x)$, which is precisely one of the unknowns.

Nevertheless, we want to stress the fact that equations (A.1) play a critical role when dealing with elastic body-fluid systems, where the coupling is taking place on the boundary interface between the two media. This interface is precisely the boundary Γ of the deformed configuration of the elastic body Ω and thus the coupling requires the continuity of the velocities and the normal stress tensors across Γ. Therefore, we need a relationship between the Cauchy stress tensor $\mathcal{T}$ and the strain tensor $\sigma(u)$, that will provide us with the correct matching of the two dynamics on the common interface.

Recalling the relations between $\mathcal{P}$, $\mathcal{T}$, and $\Sigma(u)$ we obtain that

$$\mathcal{T} = \Big(\frac{1}{\det(D\varphi)} D\varphi \cdot \Sigma(\sigma(u)) \cdot (D\varphi)^*\Big) \circ \varphi^{-1}. \tag{A.7}$$

Appendix B. Proof of Proposition 3.1

Proof. Let $\phi \in C^\infty_C(D)$. Using the change of variable $y = T(x)$ (or $x = S(y)$), we obtain:

$$\int_D (\operatorname{div} E) \circ T(x)\, \phi(x)\, dx = \int_D \operatorname{div} E(y)\, \phi \circ S(y)\, \det(DS)(y)\, dy$$

$$= -\int_D \langle E(y),\, \nabla(\, \phi \circ S(y)\, \det(DS)(y)\,)\rangle\, dy$$

$$= -\int_D \langle E(y),\, \nabla(\, \phi \circ S(y)\,)\, \det(DS)(y)\,)\, +\, \phi \circ S(y)\, \nabla(\, \det(DS)(y)\,)\,\rangle\, dy$$

[Using the identity $\nabla(\phi \circ S) = (DS)^*.(\nabla\phi) \circ S$, we obtain:]

$$= -\int_D \langle E(y),\, (DS)^*.(\nabla\phi) \circ S\, \det(DS)(y)\,)\, +\, \phi \circ S(y)\, \nabla(\, \det(DS)(y)\,)\,\rangle\, dy$$

[Transposing of the matrix DS^*, we can rewrite as follows:]

$$= -\int_D \{\langle \det(DS)(y)\,)\, DS.E(y),\, (\nabla\phi) \circ S\,\rangle$$
$$+ \langle \nabla(\, \det(DS)(y)\,)\, E(y),\, \phi \circ S(y)\rangle\,\}dy$$

[Performing the change of variable $y = T(x)$, we obtain:]

$$= -\int_D \{\langle \det(DT)\det(DS) \circ T DS \circ T.E \circ T, \nabla\phi\rangle$$
$$+ \langle \det(DT)(\nabla \det(DS)) \circ T\, E \circ T, \phi\rangle\}dx.$$

[As $(DS) \circ T = (DT)^{-1} \;\Rightarrow \det(DS) \circ T = \det(\,(DS) \circ T\,) = \det(\,(DT)^{-1}\,) = (\,\det DT\,)^{-1}$. Then we have:]

$$= -\int_D \{\langle (DT)^{-1}.E \circ T, \nabla\phi\rangle + \langle \det(DT)(\nabla \det(DS)) \circ T\, E \circ T, \phi\rangle\}dx.$$

[Using the fact that $(\nabla \det(DS)) \circ T = (DT)^{-*}.\nabla(\det(DS) \circ T\,) =$

$$(DT)^{-*}.\nabla\left(\frac{1}{\det DT}\right) = -\frac{1}{(\det DT)^2}\,(DT)^{-*}.\nabla(\det DT),\text{ we obtain:]}$$

$$= -\int_D \{\langle (DT)^{-1}.E \circ T, \nabla\phi\rangle - \langle \det(DT)^{-1}\ (DT)^{-*}.\nabla(\det DT).E \circ T, \phi\rangle\}dx$$

$$= \int_D \{\mathrm{div}((DT)^{-1}.E \circ T)\ + \det(DT)^{-1}\langle\, (DT)^{-*}.\nabla(\det DT), E \circ T\rangle\}\phi dx$$

$$= \int_D \{\mathrm{div}((DT)^{-1}.E \circ T)\ + \det(DT)^{-1}\langle\, \nabla(\det DT), (DT)^{-1}.E \circ T\rangle\}\phi dx$$

$$= \int_D \det(DT)^{-1}\{\det(DT)\ \mathrm{div}((DT)^{-1}.E \circ T)$$
$$+ \langle\, \nabla(\det DT), (DT)^{-1}.E \circ T\rangle\}\phi dx.$$

[For any scalar function a and any vector function $\vec{A}$ we have $\mathrm{div}(a\,\vec{A}) = a\ \mathrm{div}\,\vec{A} + \langle \nabla a,\, \vec{A}\rangle_{\mathbb{R}^3}$. Therefore we have that

$$\det(DT)\ \mathrm{div}((DT)^{-1}.E \circ T)\ + \langle\, \nabla(\det DT), (DT)^{-1}.E \circ T\rangle$$
$$= \mathrm{div}(\det(DT)\,(DT)^{-1}.E \circ T),$$

which gives us the desired conclusion:]

$$= \int_D \det(DT)^{-1}\ \ \mathrm{div}(\det(DT)\,(DT)^{-1}.E \circ T)\ \phi dx. \qquad \square$$

Appendix C. Proof of Lemma 3.2

Proof. Let $F_t = I + t\,u$. Then its speed flow is $W(t, .) = (\frac{\partial}{\partial t}F_t) \circ F_t^{-1} = u \circ F_t^{-1}$. Moreover we have ([15, 29]):

$$\frac{\partial}{\partial t}\det D(F_t) = (\mathrm{div}\,W(t)) \circ F_t\ \ \det D(F_t),$$

and then

$$\det(I + Du) = 1 + \int_0^1 (\mathrm{div}\,W(t)) \circ F_t\ \ \det D(F_t)\,dt. \tag{C.1}$$

Using (C.1) and (3.4), we obtain:

$$|(I + u)(\Theta)| = \int_\Theta \det(I + Du)\,dx$$
$$= |\Theta| + \int_0^1 \left(\int_\Theta (\mathrm{div}\,W(t)) \circ F_t\ \ \det D(F_t)\,dx\right)\,dt$$
$$= |\Theta| + \int_0^1 \left(\int_\Theta \mathrm{div}(\ \det D(F_t)\,D(F_t)^{-1}.W(t) \circ F_t\ )\,dx\ \right)\,dt$$
$$= |\Theta| + \int_0^1 \left(\int_\Sigma \langle\ \det D(F_t)\,D(F_t)^{-1}.W(t) \circ F_t,\ n_\Sigma\rangle\ d\Sigma\right)\,dt$$

$$= |\Theta| + \int_0^1 \left(\int_\Sigma \langle\, u,\ \det D(F_t)\, D(F_t)^{-*}.n_\Sigma \rangle\ d\Sigma \right) dt$$

$$= |\Theta| + \int_\Sigma \left\langle u, \left(\int_0^1 \det D(F_t)\, D(F_t)^{-*}\, dt \right) .n_\Sigma \right\rangle d\Sigma. \qquad \square$$

Acknowledgment

Many thanks to two anonymous referees for their valuable comments and suggestions.

References

[1] R.A. Adams, *Sobolev Spaces.* Academic Press, New York-London, 1975.

[2] G. Avalos and R. Triggiani, *Semigroup Well-posedness in the Energy Space of a Parabolic-Hyperbolic Coupled Stokes-Lamé PDE System of Fluid-Structure Interaction.* Discrete Contin. Dyn. Syst. Ser. S **2** (2009), no. 3, 417–447.

[3] V. Barbu, Z. Grujic, I. Lasiecka and A. Tuffaha, *Smoothness of Weak Solutions to a Nonlinear Fluid-structure Interaction Model.* Indiana Univ. Math. J. **57** (2008), no. 3, 1173–1207.

[4] L. Bociu, *Local and global wellposedness of weak solutions for the wave equation with nonlinear boundary and interior sources of supercritical exponents and damping.* Nonlinear Analysis: Theory, Methods & Applications **71** (2009), no. 12, e560–e575.

[5] L. Bociu and I. Lasiecka, *Blow-up of weak solutions for the semilinear wave equations with nonlinear boundary and interior sources and damping.* Appl. Math. **35** (2008), 281–304.

[6] L. Bociu and I. Lasiecka, *Uniqueness of Weak Solutions for the Semilinear Wave Equations with Supercritical Boundary/Interior Sources and Damping.* Discrete Contin. Dyn. Syst. **22** (2008), no. 4, 835–860.

[7] L. Bociu and I. Lasiecka, *Hadamard wellposedness for nonlinear wave equations with supercritical sources and damping.* J. Differential Equations (to appear).

[8] L. Bociu, M. Rammaha and D. Toundykov, *On a wave equation with supercritical interior and boundary sources and damping terms.* Math. Nachr. (to appear).

[9] S. Boisgérault and J.-P. Zolésio, *Shape derivative of sharp functionals governed by Navier-Stokes flow.* Partial differential equations (Praha, 1998), Vol. 406 of Chapman & Hall/CRC Res. Notes Math., Chapman & Hall/CRC, Boca Raton, FL, 2000, 49–63.

[10] S. Boisgérault and J.-P. Zolésio, *Boundary variations in the Navier-Stokes equations and Lagrangian functionals.* Shape optimization and optimal design (Cambridge, 1999), Vol. 216 of Lecture Notes in Pure and Appl. Math., Dekker, New York, 2001, 7–26.

[11] P.G. Ciarlet, *Mathematical Elasticity Volume I: Three-dimensional Elasticity.* Noth Holland, Amsterdam-New York-Oxford-Tokyo, 1988.

[12] D. Coutand and S. Shkoller, *Motion of an Elastic Solid inside and Incompressible Viscous Fluid.* Arch. Ration. Mech. Anal. **176** (2005), no. 1, 25–102.

[13] G. Da Prato and J.-P. Zolésio, *Existence and optimal control for wave equation in moving domain.* Stabilization of flexible structures (Montpellier, 1989), Vol. 147 of Lecture Notes in Control and Inform. Sci., Springer, Berlin, 1990, 167–190.

[14] G. Da Prato and J.-P. Zolésio, *Boundary control for inverse free boundary problems.* Boundary control and boundary variation (Sophia-Antipolis, 1990), Vol. 178 of Lecture Notes in Control and Inform. Sci., Springer, Berlin, 1992, 163–173.

[15] M.C. Delfour and J.-P. Zolésio, *Shapes and Geometries: Analysis, Differential Calculus and Optimization.* Advances in Design and Control, SIAM 2001.

[16] M.C. Delfour and J.-P. Zolésio, *Shape analysis via oriented distance functions.* J. Funct. Anal. **123** (1994), no. 1, 129–201.

[17] M.C. Delfour and J.-P. Zolésio, *Dynamical free boundary problem for an incompressible potential fluid flow in a time-varying domain.* J. Inverse Ill-Posed Probl. **12** (2004), no. 1, 1–25.

[18] F.R. Desaint and J.-P. Zolésio, *Manifold derivative in the Laplace-Beltrami equation.* J. Funct. Anal. **151** (1997), no. 1, 234–269.

[19] F. Desaint and J.-P. Zolésio, *Shape boundary derivative for an elastic membrane.* Advances in mathematical sciences: CRM's 25 years (Montreal, PQ, 1994), Vol. 11 of CRM Proc. Lecture Notes, Amer. Math. Soc., Providence, RI, 1997, 481–491.

[20] F.R. Desaint and J.-P. Zolésio, *Shape derivative for the Laplace-Beltrami equation.* Partial differential equation methods in control and shape analysis (Pisa), vol. 188 of Lecture Notes in Pure and Appl. Math., Dekker, New York, 1997, 111–132.

[21] F. Desaint and J.-P. Zolésio, *Sensitivity analysis of all eigenvalues of a symmetrical elliptic operator.* Boundary control and variation (Sophia Antipolis, 1992), Vol. 163 of Lecture Notes in Pure and Appl. Math., Dekker, New York, 1994, 141–160.

[22] C. Grandmont, *Existence et unicité de solutions d'un problème de couplage fluide-structure bidimensionnel stationnaire.* C. R. Acad. Sci. Paris Sér. I Math. **326** (1998), no. 5, 651–656.

[23] C. Grandmont, *Existence for a Three-Dimensional Steady State Fluid-Structure Interaction Problem.* J. Math. Fluid. Mech. **4** (2002), 76–94.

[24] J. Sokołowski and J.-P. Zolésio, *Shape design sensitivity analysis of plates and plane elastic solids under unilateral constraints.* J. Optim. Theory Appl. **54** (1987), no. 2, 361–382.

[25] J. Sokołowski and J.-P. Zolésio, *Shape sensitivity analysis of contact problem with prescribed friction.* Nonlinear Anal. **12** (1988), no. 12, 1399–1411.

[26] J. Sokołowski and J.-P. Zolésio, *Shape sensitivity analysis of unilateral problems.* SIAM J. Math. Anal. **18** (1987), no. 5, 1416–1437.

[27] M. Souli and J.-P. Zolésio, *Shape derivative of discretized problems.* Comput. Methods Appl. Mech. Engrg. **108** (1993), no. 3-4, 187–199.

[28] M. Souli and J.-P. Zolésio, *Finite element method for free surface flow problems.* Comput. Methods Appl. Mech. Engrg. **129** (1996), no. 1-2, 43–51.

[29] J.-P. Zolésio, *Weak shape formulation of free boundary problems.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **21** (1994), no. 1, 11–44.

Lorena Bociu
CNRS-INLN
1361 Routes des Lucioles
Sophia Antipolis
F-06560 Valbonne, France

and

University of Nebraska-Lincoln
Department of Mathematics
203 Avery Hall
880130 Lincoln, NE 68588-0130, USA
e-mail: lbociu2@math.unl.edu

Jean-Paul Zolésio
CNRS-INLN
1361 Routes des Lucioles
Sophia Antipolis
F-06560 Valbonne, France

and

INRIA
Route des Lucioles – BP 93
F-06902 Sophia Antipolis Cedex, France
e-mail: Jean-Paul.Zolesio@sophia.inria.fr

Operator Theory:
Advances and Applications, Vol. 216, 121–138

Some Results on the Identification of Memory Kernels

F. Colombo and D. Guidetti

Communicated by Y. Kurylev

Abstract. The present paper is base on a talk given by one of the authors to the 7th International ISAAC Congress held in London, UK, in 2009. A few years ago the authors have introduced a strategy to prove global in time existence and uniqueness results for semilinear integrodifferential inverse problems. Here we discuss the strategies used to treat some problems related to the identification of convolution memory kernels in semilinear integrodifferential models. Moreover, we explain the novelty with respect to some of the existing methods with respect to our strategy whose main ideas are contained in the paper [F. Colombo, D. Guidetti, A global in time existence and uniqueness result for a semilinear integrodifferential parabolic inverse problem in Sobolev Spaces, Math. Models Methods Appl. Sci., 17 (2007), 1–29]. Convolution kernels are important to take into account memory effects, but in the case of the heat equation they are also used to make the speed of propagation of the heat finite. Among the models we discuss in this paper we mention: Phase-field models with memory, the heat equation with memory, a model in the theory of combustion, the beam equation with memory and a model arising in the theory of nuclear reactors.

Mathematics Subject Classification (2000). Primary 35R30; Secondary 45K05.

Keywords. Global in time existence and uniqueness results, phase-field model, nuclear reactor model, beam equation, strongly damped wave equation with memory.

1. Introduction and notation

The literature related to inverse problems is very wide. Recent contributions can be found in the books [2, 3, 19, 20, 27, 28, 32, 33, 34] which we mention without claim of completeness. Such books treats different aspects of inverse problems.

In recent years the theory of phase-field models with memory has had interesting developments also from the inverse problems point of view, we refer for example to [6, 7, 14] and the literature therein for more details. Inverse problems are in general ill-posed problems and to obtain just uniqueness of a solution to a given inverse problem can be considered a good result. To obtain global in time existence and uniqueness of a solution is in general the most difficult part of the problem. The aim of this paper is to discuss a strategy, introduced for the first time by the authors in 2007, which allows us to prove global in time existence and uniqueness results for a class of semilinear integrodifferential models. Such strategy was applied to the heat equation with memory in the paper [16], then we have applied such method to several models in [8, 9, 10, 11, 13, 15]. Let us begin by considering the classification of the nonlinearities we are able to study with our method. Let us consider the inverse problem related to the heat equation with memory in its semilinear version.

Problem 1. Determine $u : [0,T] \times \Omega \longrightarrow \mathbb{R}$, and $h : [0,T] \longrightarrow \mathbb{R}$, satisfying

- $D_t u(t,x) = \Delta u(t,x) + \int_0^t h(t-s)\Delta u(s,x)\,ds + F(u(t,x))$,
- $u(0,x) = u_0(x), \quad x \in \overline{\Omega}$,
- $D_\nu u(t,x) = 0, \quad (t,x) \in [0,T] \times \partial\Omega$,
- $\int_\Omega \phi(x) u(t,x)\,dx = g(t), \qquad \forall t \in [0,T]$,

where F is a given nonlinear function of the unknown u and u_0, ϕ, g are given data.

The additional restriction on u given by $\int_\Omega \phi(x)u(t,x)\,dx = g(t)$ is necessary to determine both u and h, otherwise there would be no possibilities to make to problem well-posed. Since both h and u are unknown, in the evolution equation, we have two types of nonlinearities: the first one is of convolution type, the integral $\int_0^t h(t-s)\Delta u(s,x)\,ds$ contains such nonlinearity, while the second one is in the term $F(u)$. In the case F is independent of u or when $F(u)$ is a linear function of u, we have only a nonlinearity of convolution type. In this case the problem becomes easier to treat because there is a well-known strategy to face such problem. The difficulties are only technical.

In this paper we will discuss the following case:

- Strategy I, inverse problems with a non linearity of convolution type: global in time results.
- The strategy II, inverse problems with two non linearities: local in time results.
- The strategy III, inverse problems with two non linearities: global in time results.

Notations. If X and Y are Banach spaces, we indicate by $\mathcal{L}(X,Y)$ the class of linear continuous mappings from X to Y. We simply write $\mathcal{L}(X)$ in case $X = Y$. We shall usually indicate by $\|\cdot\|_Y$ the norm in the Banach space Y. In general, if $y \in Y$, we shall write $\|y\|$ instead of $\|y\|_Y$ when no confusion arises.
If A is a densely defined linear operator on X and X' is the dual space, we shall indicate by A' the dual operator of A.

We denote by $\mathbb{R}^+$ the set of positive real numbers, Ω is an open subset of $\mathbb{R}^n$, lying on one side of its boundary $\partial\Omega$, which is a submanifold of $\mathbb{R}^n$ of class at least C^1. We denote by $\nu(x)$ the unit vector normal to $\partial\Omega$ in x, pointing outside Ω, and by D_ν the normal derivative. If $s \in \mathbb{N}_0$, $p \in [1,+\infty]$, $W^{s,p}(\Omega)$ is the usual Sobolev space.
If $s \in \mathbb{Z}$, $s \geq 2$ and Ω is an open subset of $\mathbb{R}^n$, with smooth boundary, we set

$$W_B^{s,p}(\Omega) := \{f \in W^{s,p}(\Omega) : D_\nu f \equiv 0\},$$
$$W_{BB}^{s,p}(\Omega) := \{f \in W^{s,p}(\Omega) : D_\nu f \equiv D_\nu \Delta f \equiv 0\}.$$

The Besov spaces are denoted by $B^s_{p,q}(\Omega)$ for $s > 0$, $1 \leq p < +\infty$ and $1 \leq q < +\infty$ (see [36]).
The symbol $(\cdot,\cdot)_{\theta,p}$ denotes the real interpolation functor $(0<\theta<1,\ 1\leq p\leq +\infty)$.
Let $p \in [1,+\infty)$, $T \in \mathbb{R}^+$, $m \in \mathbb{N}_0$ and X be a Banach space.
If $f \in W^{m,p}(0,T;X)$, (see [1]) we set

$$\|f\|_{W^{m,p}(0,T;X)} := \sum_{j=0}^{m-1} \|f^{(j)}(0)\| + \|f^{(m)}\|_{L^p(0,T;X)}.$$

Let X be a Banach space and let A be a linear operator whose domain $D(A)$ is contained in X. For the sake of brevity we define the Banach space

$$X(T,p) = W^{1,p}(0,T;X) \cap L^p(0,T;D(A)),$$

where $T \in \mathbb{R}^+$, $p \in [1,+\infty]$. If $u \in X(T,p)$ we set

$$\|u\|_{X(T,p)} = \|u\|_{W^{1,p}(0,T;X)} + \|u\|_{L^p(0,T;D(A))}.$$

In the sequel we will denote by $a \wedge b$ the number $\min\{a,b\}$, where $a,b \in \mathbb{R}$.
Let $h \in L^1(0,T)$ and $f : (0,T) \to X$, where X is a Banach space. We define the convolution

$$(h * f)(t) := \int_0^t h(t-s) f(s) ds,$$

whenever the integral has a meaning. We conclude this section by recalling some interesting papers on inverse problems [5, 12, 18, 23, 24, 25, 26, 30].

2. Some strategies to treat integrodifferential inverse problems

2.1. Strategy I, inverse problems with a non linearity of convolution type: global in time results

Let us consider the abstract formulation of the heat problem relating it to a Banach space X and let us suppose that A is the infinitesimal generator of an analytic semigroup in X.

Problem 2. Determine $u : [0,T] \longrightarrow X$ and $h : [0,T] \longrightarrow \mathbb{R}$, satisfying

- $u'(t) = Au(t) + \int_0^t h(t-s)Au(s)\,ds + F(t)$,
- $u(0) = u_0$,
- $\Phi(u(t)) = g(t), \quad \forall t \in [0,T]$,

where Φ denotes a bounded linear functional on X and u_0, F, g are given data.

The method to solve the problem is well known and can be formulated in the following steps.

(1) We consider an abstract formulation of the inverse problem.
(2) We choose a functional setting and we select the related optimal regularity theorem for the linearized version of the problem.
(3) We prove that the abstract version of the problem is equivalent to a suitable fixed point system.
(4) The fixed point system contains integral operators, we have to estimate them in the weighted spaces we are considering. The exponential weight $e^{\sigma t}$, $\sigma \in \mathbb{R}^+$, $t \in [0, T]$ is usually used.
(5) By the Contraction Principle we get existence and uniqueness of a solution to our inverse problem.
(6) We apply the abstract results to the concrete problem.

2.2. The strategy II, inverse problems with two non linearities: local in time results

In this case we follow the same strategy but here we cannot use weighted spaces. The problem we consider is as follows.

Problem 3. Determine $u : [0, T] \longrightarrow X$ and $h : [0, T] \longrightarrow \mathbb{R}$, satisfying

- $u'(t) = Au(t) + \int_0^t h(t-s)Au(s)\,ds + F(u(t))$,
- $u(0) = u_0$,
- $\Phi(u(t)) = g(t), \quad \forall t \in [0, T]$,

where Φ denotes a bounded linear functional on X and u_0, g are given data and F is a non linear function of u.

The reason for which we cannot use weighted spaces is due to the fact that the nonlinear term $F(u)$ cannot be suitably estimated when we introduce some weights in the function spaces we are considering. So condition (4), in strategy I, has to be replaced by

(4′) The fixed point system contains integral operators, we have to estimate them in the spaces we are considering.

With strategy II we can prove local in time existence and uniqueness results and also global in time uniqueness results. The unsolved problem remains the global existence of a solution. To get global in time existence and uniqueness it is necessary to modify the strategy and make assumptions on the nonlinear term $F(u)$ as follows. For the strategy in the case of local results see [31] and for the use of weighted spaces see [17].

2.3. The strategy III, inverse problems with two non linearities: global in time results

The main ideas to solve the Problem 3 in this case is to prove that there exists a local in time solution of the inverse problem in Sobolev spaces without weights.

We linearize the convolution term and we find a priori estimates for u and for the convolution kernel h. We can split the strategy in three main steps.

Step 1: We prove local in time existence and uniqueness.

(a) We use the Sobolev spaces $W^{2,p}(0,T;X)$.
(b) Find a suitable equivalent fixed point system.
(c) The fixed point system contains integral operators, we have to estimate them in the Sobolev spaces we have chosen.
(d) We apply the Contraction Principle to prove that there exists a unique *local in time* solution. Thanks to the equivalence theorem previously obtained we get existence and uniqueness of the solution to our inverse problem which is local in time.
(e) We prove a global in time uniqueness result without any condition on $F(u)$.

Step 2: We linearize the convolution term: $\int_0^t h(t-s)Au(s)\,ds$.

(f) We linearize the convolution term thanks to the *local in time* existence and uniqueness theorem.
We observe that a unique solution $(\hat{u}, \hat{h})$ exists in $[0, \tau]$ for some $\tau > 0$.
Set $v(t) := u'(t)$, $v_\tau(t) = v(\tau + t)$ and $h_\tau(t) = h(\tau + t)$ and consider, for $0 < t < \tau$ the splitting

$$\int_0^{\tau+t} h(\tau + t - s)Av(s)ds = h_\tau * A\hat{v}(t) + \hat{h} * Av_\tau(t) + \tilde{F}(t),$$

where the symbol $*$ stands for the convolution. $\tilde{F}(t)$ is a given data and depends on the known functions $(\hat{u}, \hat{h})$.

Step 3: A priori estimates with the condition $F_u(u)$ bounded.
The above way of rewriting the convolution term allows us to avoid the weighted spaces that have a bad behavior when we deal with the non linearity $F(u)$.

(g) We deduce the a priori estimates for $v_\tau(t)$ and $h_\tau(t)$ for $0 < t < \tau$ assuming that F_u is a bounded function.
(h) In a finite number of steps we extend the solution to the interval $[0, T]$.

Remark 2.1 (An open problem associated to strategy III). The nonlinearity $F(u) = u(x,t) - u^3(x,t)$ appears in several models we are interested in finding global in time existence and uniqueness results for

$$u_t(x,t) = \Delta u(x,t) + \int_0^t h(t-s)\Delta u(x,s)ds + u(x,t) - u^3(x,t) \tag{2.1}$$

with the additional restriction on u in integral form with the associated suitable initial-boundary conditions. For the inverse problem we make the following considerations.

- The term $u(x,t) - u^3(x,t)$ it is monotone and it is suitable to find a priori estimates for the direct problem.

- If we fix u, and we try to solve the evolution equation (2.1) with respect to h we have to observe that the term $\int_0^t h(t-s)\Delta u(x,s)ds$ gives a Volterra equation of the first kind. That is we have to face a Ill-posed problem.
- If we differentiate with respect to time the evolution equation we obtain a Volterra equation of the second kind for the unknown h, since the derivative of the convolution term gives:

$$h(t)\Delta u_0 + \int_0^t h(t-s)\Delta u'(x,s)ds.$$

So we get a well-posed problem for h, but the nonlinear terms becomes

$$u'(x,t) - 3u^2(x,t)u'(x,t).$$

The problem in that the differentiation with respect to time spoils the monotonicity of the nonlinear term $u(x,t) - u^3(x,t)$ and so we are not able to find the a priori estimates for the unknowns u and h.

2.4. The results of Strategy II applied to a phase-field model with memory

Here we present a phase-field model with memory studied in [14] which has been solved in Sobolev spaces. For a different phase-field model with memory studied in Hölder spaces see [6, 7] and the literature therein.

Let Ω be an open bounded set in $\mathbb{R}^3$ with sufficiently regular boundary $\partial\Omega$ occupied by an isotropic, rigid and homogeneous heat conductor. We consider only small variations of the absolute temperature and its gradient. The material which exhibit phase transitions, due to the temperature variations, are described by two state variables: the absolute temperature Θ and the phase-field χ at each point $x \in \Omega$ and $t \in [0,T]$ for $T > 0$, where χ takes approximately value -1 in the liquid and $+1$ in the solid. In our model we prefer to work with the *temperature variational field* θ defined by:

$$\theta = \frac{\Theta - \Theta_c}{\Theta_c}, \tag{2.2}$$

where Θ_c is the reference temperature at which the transition occurs. The energy balance equation is

$$\partial_t e + \nabla \cdot \mathbf{J} = \mathcal{F}, \tag{2.3}$$

where e is the internal energy, $\mathbf{J}$ is the heat flux and $\mathcal{F}$ is the external heat supply. Taking into account a linearized version of the Coleman–Gurtin theory, we assume the constitutive equations:

$$e(t,x) = e_c + c_v\Theta_c\theta(t,x) + \int_0^t a(s)\theta(t-s)\,ds + \Theta_c\lambda(\chi(t,x)), \tag{2.4}$$

$$\mathbf{J} = -k\nabla\theta(t,x) - \int_0^t b(s)\nabla\theta(t-s)\,ds, \tag{2.5}$$

where a and b account for the memory effects, e_c, c_v and k are the internal energy at equilibrium, the specific heat and the conductivity, respectively. Moreover, λ is a suitable regular given function. By (2.3), (2.4) and (2.5) we get

$$\begin{aligned}&\partial_t\Big(e_c + c_v\Theta_c\theta(t,x) + \textstyle\int_0^t a(s)\theta(t-s)\,ds + \Theta_c\lambda(\chi(t,x))\Big)\\&-\nabla\cdot\Big(k\nabla\theta(t,x) + \textstyle\int_0^t b(s)\nabla\theta(t-s)\,ds\Big) = \mathcal{F}(t,x), \quad t\in[0,T],\ x\in\Omega.\end{aligned} \tag{2.6}$$

We couple the evolution equation (2.6) with the Cahn–Hilliard type equation which rules the phase evolution

$$\varepsilon\partial_t\chi(t,x) = \Delta[-\Delta\chi(t,x) + \chi^3(t,x) + \gamma'(\chi(t,x)) - \lambda'(\chi)\theta(t,x)], \quad t\in[0,T],\ x\in\Omega, \tag{2.7}$$

where γ is a smooth given function and $\varepsilon > 0$ is a parameter. We will assume that λ and γ are linear functions of their arguments. Associated to the evolution equations we will consider also the initial and the Neumann boundary conditions to be introduced just below. As we have already observed, the kernels a and b cannot be measured directly and the physical observable that can be easily measured is the temperature, so a and b have to be indirectly determined by additional measurements on θ made on suitable parts of the material.

We consider additional measurements on the temperature which can be represented as

$$\Phi_j(\theta)(t) := \int_\Omega \phi_j(x)\theta(t,x)\,dx = g_j(t), \quad \forall t\in[0,T], \quad j=1,2, \tag{2.8}$$

where $\phi_j : \Omega \to \mathbb{R}$ are given compact support functions depending on the type thermometer used for the additional measurements on θ and $g_j : [0,T] \to \mathbb{R}$, $j = 1,2$, represent the result of the additional measurements on θ. We have given two conditions because we have to identify the two unknown kernels a and b.

With the above notations we can give the definition of the inverse problem we are investigating in the sequel.

Problem 4. Let $p\in(1,+\infty)$. Determine θ, χ, a, b, and k with

$$\theta \in W^{2,p}(0,\tau;L^p(\Omega)) \cap W^{1,p}(0,\tau;W^{2,p}(\Omega)), \tag{2.9}$$

$$\chi \in W^{2,p}(0,\tau;L^p(\Omega)) \cap W^{1,p}(0,\tau;W^{4,p}(\Omega)), \tag{2.10}$$

$$a \in W^{1,p}(0,\tau), \qquad b\in L^p(0,\tau), \qquad k\in\mathbb{R}^+, \tag{2.11}$$

satisfying the system

$$\begin{cases} D_t(\theta + \lambda\chi + a*\theta)(t,x) - \Delta[k\theta(t,x) + b*\theta](t,x)\\ = f(t,x), \quad (t,x)\in[0,\tau]\times\Omega,\\ \varepsilon D_t\chi(t,x) = \Delta[-\Delta\chi + \phi(\chi) - \lambda\theta](t,x), \quad (t,x)\in[0,\tau]\times\Omega,\\ D_\nu\theta(t,x') = D_\nu\chi(t,x') = D_\nu\Delta\chi(t,x') = 0, \quad (t,x')\in[0,\tau]\times\partial\Omega,\\ \theta(0,x) = \theta_0(x), \qquad \chi(0,x) = \chi_0(x), \quad x\in\Omega,\\ \Phi_j[\theta(t)] = g_j(t), \quad j\in\{1,2\}, \quad t\in[0,\tau], \end{cases} \tag{2.12}$$

under suitable regularity and compatibility conditions on the data.

We have set, for simplicity:

$$\phi(\chi) := \chi^3 - \gamma'(\chi).$$

The main result related to the inverse Problem 4 states that under suitable regularity and compatibility conditions on the data there exists a unique local in time solution in the Sobolev setting. More precisely, let us introduce the set of conditions:

(C1) Ω is an open bounded subset of $\mathbb{R}^n$, lying on one side of its boundary $\partial\Omega$, which is a submanifold of $\mathbb{R}^n$ of class C^4;
(C2) $\lambda \in \mathbb{R}$, $\varepsilon \in \mathbb{R}^+$;
(C3) $p \in (1, +\infty)$, $n \in \mathbb{N}$, $n < 4p$;
(C4) $\phi \in C^\infty(\mathbb{R})$;
(C5) $\chi_0 \in W^{4,p}_{BB}(\Omega)$;
(C6) $\theta_0 \in W^{2,p}_B(\Omega)$;
(C7) for some $T \in \mathbb{R}^+$, $f \in W^{1,p}(0, T; L^p(\Omega))$;
(C8) for $j \in \{1, 2\}$, $u \in L^p(\Omega)$, $\Phi_j[u] = \int_\Omega \phi_j(x)u(x)dx$, with $\phi_j \in L^{p'}(\Omega)$;
(C9) for $j \in \{1, 2\}$, $g_j \in W^{2,p}(0, T)$;
(C10) $v_0 := \varepsilon^{-1}\Delta[-\Delta\chi_0 + \phi(\chi_0) - \lambda\theta_0] \in (L^p(\Omega), W^{4,p}_{BB}(\Omega))_{1-1/p,p}$;
(C11) $\chi^{-1} := \Phi_2[\theta_0]\Phi_1[\Delta\theta_0] - \Phi_1[\theta_0]\Phi_2[\Delta\theta_0] \neq 0$;
(C12) $k_0 := \chi[\Phi_1[\theta_0]\{\Phi_2[f(0) - \lambda v_0] - g_2'(0)\} - \Phi_2[\theta_0]\{\Phi_1[f(0) - \lambda v_0] - g_1'(0)\}] \in \mathbb{R}^+$;
(C13) $\Phi_j(\theta_0) = g_j(0)$, $j \in \{1, 2\}$;
(C14) $u_0 := f(0) - \lambda v_0 - a_0\theta_0 + k_0\Delta\theta_0 \in (L^p(\Omega), W^{2,p}_B(\Omega))_{1-1/p,p}$;
(C15) $a_0 := \chi[\{\Phi_2[f(0) - \lambda v_0] - g_2'(0)\}\Phi_1[\theta_0] - \{\Phi_1[f(0) - \lambda v_0] - g_1'(0)\}\Phi_2[\theta_0]] \in \mathbb{R}^+$.

The main result that is obtained by strategy II is the following.

Theorem 2.2. *Assume that the assumptions* (C1)–(C15) *are satisfied. Then there exists* $\tau \in (0, T]$ *such that the system* (2.12) *has a unique solution* (θ, χ, a, b, k) *satisfying the conditions* (2.9)–(2.11).

3. The results originally obtained by Strategy III

Here we state the main results for the heat equation with memory in its abstract version.

Problem 5 (The Inverse Abstract Problem (IAP)). Determine $\tau \in (0, T]$ and

- $u \in W^{2,p}(0, \tau; X) \cap W^{1,p}(0, \tau; D(A))$,
- $h \in L^p(0, \tau)$,

satisfying the system

- $u'(t) = Au(t) + h * Bu(t) + f(u(t)) + G(t), \quad t \in (0, \tau)$
- $u(0) = u_0$,
- $\Phi(u) = g(t), \quad t \in (0, \tau)$.

The set of regularity and compatibility conditions on the data in order to get a well-posed problem are as follows. Let $p \in (1, +\infty)$.

(H1) $D(A) \hookrightarrow Y \hookrightarrow X$, $D(A)$ is dense in X and there exist $C > 0$ and $\theta \in [0, 1)$, such that, $\forall u \in D(A)$,

$$\|u\|_Y \leq C\|u\|_X^{1-\theta}\|u\|_{D(A)}^{\theta}.$$

(H2) A is the infinitesimal generator of an analytic semigroup in X

(H3) $B \in \mathcal{L}(D(A), X)$.

(H4) $u_0 \in D(A)$.

(H5) $\Phi \in X'$.

(H6) $f \in C^1(Y, X)$ and $f' : Y \to \mathcal{L}(Y, X)$ is Lipschitz continuous in bounded subsets of Y.

(H7) $G \in W^{1,p}(0, T; X)$.

(H8) $v_0 := Au_0 + f(u_0) + G(0) \in (X, D(A))_{1-1/p,p}$.

(H9) $\Phi(Bu_0) \neq 0$.

(H10) $g \in W^{2,p}(0, T)$ with $\Phi(u_0) = g(0)$ and $\Phi(v_0) = g'(0)$.

(H11) $f' : Y \to \mathcal{L}(Y, X)$ is bounded, with f' Fréchet derivative of f.

Theorem 3.1. (Local in time existence) *Let the assumptions* (H1)–(H10) *hold. Then there exists* $\tau \in (0, T]$*, depending on the data, such that Problem* 5 *has a solution* $(u, h) \in [W^{2,p}(0, \tau; X) \cap W^{1,p}(0, \tau; D(A))] \times L^p(0, \tau)$.

Theorem 3.2. (Global in time uniqueness) *Let the assumptions* (H1)–(H10) *hold. Then, if* $\tau \in (0, T]$*, and Problem* 5 *has two solutions* $(u_j, h_j) \in [W^{2,p}(0, \tau; X) \cap W^{1,p}(0, \tau; D(A))] \times L^p(0, \tau)$ $(j \in \{1, 2\})$*, then* $u_1 = u_2$ *and* $h_1 = h_2$.

Theorem 3.3. (Global in time existence and uniqueness) *Let the assumptions* (H1)–(H11) *hold. Let* $T > 0$*. Then Problem* 5 *has a unique solution* $(u, h) \in [W^{2,p}(0, T; X) \cap W^{1,p}(0, T; D(A))] \times L^p(0, T)$.

We point out that the global in time existence and uniqueness result has been obtained in the abstract setting and it is not based on a maximum principle that in most of the concrete cases does not hold.

3.1. An application of the abstract results to the heat equation with memory

Problem 6. Determine $\tau \in (0, T]$ and $u \in W^{2,p}(0, \tau; L^p(\Omega)) \cap W^{1,p}(0, \tau; W^{2,p}(\Omega))$ and $h \in L^p(0, \tau)$, satisfying the system

$$\begin{cases} \partial_t u(t, x) = \Delta u(t, x) + h * \Delta u(t, x) + f(u(t, x)) + G(t, x), \\ u(0, x) = u_0(x), \;\; x \in \Omega, \\ D_\nu u(t, x) = 0, \quad t \in (0, \tau), \;\; x \in \partial\Omega, \\ \Phi(u) := \int_\Omega \phi(x)u(t, x)dx = g(t). \end{cases} \tag{3.1}$$

Under the following conditions on the data:

(h1) Ω is an open bounded subset of $\mathbb{R}^n$, lying on one side of its boundary $\partial\Omega$, which is a submanifold of $\mathbb{R}^n$ of class C^2;

(h2) $p \in (1, +\infty)$, $n \in \mathbb{N}$, with $n < 2p$;

(h3) $u_0 \in W^{2,p}_B(\Omega)$;
(h4) $\phi \in W^{2,p'}_B(\Omega)$;
(h5) $f \in C^\infty(\mathbb{R})$;
(h6) $v_0 := \Delta u_0 + f(u_0) + G(0) \in (L^p(\Omega), W^{2,p}_B(\Omega))_{1-1/p,p}$;
(h7) $g \in W^{2,p}(0,T)$ with $\Phi(u_0) = g(0)$, $\Phi(v_0) = g'(0)$;
(h8) $\Phi(\Delta u_0) := \int_\Omega \phi(x)\Delta u_0(x)dx \neq 0$;
(h9) $G \in W^{1,p}(0,T; L^p(\Omega))$;
(h10) f_u is bounded.

Theorem 3.4. (Local in time existence) *Let* (h1)–(h9) *hold. Then there exists* $\tau \in (0,T]$, *depending on the data, s.t. the Inverse Problem* 6 *has a solution* $u \in W^{2,p}(0,\tau; L^p(\Omega)) \cap W^{1,p}(0,\tau; W^{2,p}(\Omega))$, $h \in L^p(0,\tau)$.

Theorem 3.5. (Global in time uniqueness) *Let the assumptions* (h1)–(h9) *hold. Then, if* $\tau \in (0,T]$, *and the Inverse Problem* 6 *has two solutions*

$$u_j \in W^{2,p}(0,\tau; L^p(\Omega)) \cap W^{1,p}(0,\tau; W^{2,p}(\Omega)), \qquad (j \in \{1,2\})$$

$$h_j \in L^p(0,\tau), \qquad (j \in \{1,2\})$$

then $u_1 = u_2$ *and* $h_1 = h_2$.

Theorem 3.6. (Global in time existence and uniqueness) *Let the assumptions* (h1)–(h10) *hold. Assume that* $p > 1$. *Let* $T > 0$. *Then the Inverse Problem* 6 *has a unique solution* $u \in W^{2,p}(0,T; L^p(\Omega)) \cap W^{1,p}(0,T; W^{2,p}(\Omega))$, $h \in L^p(0,T)$.

3.2. An application of the abstract results to a parabolic problem of order $2m$

Our main results are proved in an abstract setting so that they can be applied to the more general case of operators of order $2m$ that contains as a particular case the heat conduction problem for $m = 1$.

Problem 7. (The inverse problem for operators of order $2m$) Let $T > 0$. Determine $\tau \in (0,T]$ and

$$u \in W^{2,p}(0,\tau; L^p(\Omega)) \cap W^{1,p}(0,\tau; W^{2m,p}(\Omega)), \qquad h \in L^p(0,\tau), \tag{3.2}$$

satisfying the system

$$\begin{cases} \partial_t u(t,x) = A(x,\partial_x)u(t,x) + h * B(x,\partial_x)u(t,x) \\ +\mathcal{F}((\partial_x^\alpha u(t,x))_{|\alpha|\le 2m-1}) + G(t,x), \\ u(0,x) = u_0(x), \ x \in \Omega, \\ B_j(x,\partial_x)u(t,x) = 0, \ \ 1 \le j \le m, \ t \in (0,\tau), \ x \in \partial\Omega, \end{cases} \tag{3.3}$$

with the additional information on u:

$$\Phi(u) := \int_\Omega \phi(x)u(t,x)dx = g(t). \tag{3.4}$$

We solve the inverse problem under the following conditions on the data:

(K1) $m, n \in \mathbb{N}$, Ω is an open bounded subset of $\mathbb{R}^n$, lying on one side of its boundary $\partial\Omega$, which is a submanifold of $\mathbb{R}^n$ of class C^{2m}.

(K2) $A(x, \partial_x)$ is a strongly elliptic operator of order $2m$, with coefficients in $C(\overline{\Omega})$, for $j = 1, \ldots, m$, $B_j(x, \partial_x)$ is a linear differential operator of order $m_j \leq 2m - 1$, with coefficients in $C^{2m-m_j}(\partial\Omega)$, $\{B_j(x, \partial_x) : 1 \leq j \leq m\}$ is a normal system of boundary operators in the sense of [35], Definition 3.7.1, the operator $A(x, \partial_x)$ with vanishing boundary conditions $B_j(x, \partial_x)$ $(1 \leq j \leq m)$ has $Arg(\lambda) = \theta$ as a ray of minimal growth of the resolvent in the sense of [35], Definition 3.8.1 for all $\theta \in [-\pi/2, \pi/2]$.

(K3) $p \in (1, +\infty)$, with $n < p$, $2m(1 - 1/p) \neq m_j + 1/p \quad \forall j = 1, \ldots, m$.

(K4) $u_0 \in W_B^{2m,p}(\Omega) := \{u \in W^{2m,p}(\Omega) : B_j(x, \partial_x)u \equiv 0 \quad \forall j = 1, \ldots, m\}$.

(K5) $\phi \in L^{p'}(\Omega)$.

(K6) $\mathcal{F} \in C^1(\mathbb{R}^{N(m)})$, with $N(m)$ indicating the cardinality of $\{\alpha \in \mathbb{N}_0^n : |\alpha| \leq 2m-1\}$, and we denote by $(y_\alpha)_{|\alpha| \leq 2m-1}$ a general element of $\mathbb{R}^{N(m)}$; moreover, its first-order derivatives are Lipschitz continuous on the bounded subsets of $\mathbb{R}^{N(m)}$.

(K7) $G \in W^{1,p}(0, T; L^p(\Omega))$.

(K8) $v_0 := A(x, \partial_x)u_0 + \mathcal{F}((\partial_x^\alpha u_0)_{|\alpha| \leq 2m-1}) + G(0) \in B_{p,p,B}^{2m(1-1/p)}(\Omega)$, where $B_{p,p,B}^{2m(1-1/p)}(\Omega) := \{v \in B_{p,p}^{2m(1-1/p)}(\Omega) : B_j(x, \partial_x)u \equiv 0, \ \forall j = 1, \ldots, m, m_j + 1/p < 2m(1 - 1/p)\}$.

(K9) $g \in W^{2,p}(0, T)$ with $\Phi(u_0) = g(0)$ and $\Phi(v_0) = g'(0)$.

(K10) $B(x, \partial_x)$ is a linear differential operator of order not exceeding $2m$, with coefficients in $C(\overline{\Omega})$.

(K11) $\Phi(B(x, \partial_x)u_0) = \int_\Omega \phi(x)B(x, \partial_x)u_0(x)dx \neq 0$.

(K12) $\nabla\mathcal{F}$ is bounded in $\mathbb{R}^{N(m)}$.

Theorem 3.7. (Local in time existence) *Let the assumptions* (K1)–(K11) *hold. Then there exists* $\tau \in (0, T]$, *depending on the data, such that Problem* 7 *has a solution* $(u, h) \in [W^{2,p}(0, \tau; L^p(\Omega)) \cap W^{1,p}(0, \tau; W^{2m,p}(\Omega))] \times L^p(0, \tau)$.

Theorem 3.8. (Global in time uniqueness) *Let the assumptions* (K1)–(K11) *hold. Then, if* $\tau \in (0, T]$, *and Problem* 7 *has two solutions* $(u_j, h_j) \in [W^{2,p}(0, \tau; L^p(\Omega)) \cap W^{1,p}(0, \tau; W^{2m,p}(\Omega))] \times L^p(0, \tau)$ $(j \in \{1, 2\})$, *then* $u_1 = u_2$ *and* $h_1 = h_2$.

Theorem 3.9. (Global in time existence and uniqueness) *Let the assumptions* (K1)–(K12) *hold. Let* $T > 0$. *Then Problem* 7 *has a unique solution* $(u, h) \in [W^{2,p}(0, T; L^p(\Omega)) \cap W^{1,p}(0, T; W^{2m,p}(\Omega))] \times L^p(0, T)$.

4. Some models to which Strategy III applies

4.1. The strongly damped wave equation with memory

For the proofs of the results, presented here and related to strongly damped wave equation with memory see [8]. For the identification of the memory kernel in the strongly damped wave equation where the additional restriction on the state variable u is given by a flux condition see the paper [13].

The model and the physical problem. Let Ω be an open bounded set in $\mathbb{R}^n$, $n = 1, 2, 3$ and $T > 0$. For $(t, x) \in [0, T] \times \Omega$ we consider the initial and boundary value problem for a semilinear strongly damped wave equation

$$\begin{cases} u_{tt}(t,x) = \Delta u_t(t,x) + \Delta u(t,x) + \int_0^t h(t-s)\Delta u(s,x)ds \\ +f(u(t,x), \nabla u(t,x)) + G(t,x), \\ u(0,x) = u_0(x), \quad x \in \Omega, \\ u_t(0,x) = u_1(x), \quad x \in \Omega, \\ D_\nu u(t,x) = 0, \quad (t,x) \in [0,T] \times \partial\Omega, \end{cases} \tag{4.1}$$

where D_ν denotes the outward unit normal derivative in $\partial\Omega$ and we suppose that the boundary $\partial\Omega$ is sufficiently smooth in the sense that will be clarified in the sequel. The functions f and G are given. In the case $\Omega \subset \mathbb{R}^n$ with $n = 1, 2$ system (4.1) rules the transversal vibration of a homogeneous string and the longitudinal vibrations of a homogeneous bar, respectively. The term $-\Delta u_t(t, x)$ takes into account the so-called strong damping due to viscous effects and indicates that the stress is proportional ont only to the strain, but also to the strain rates as in the linearized Kelvin-Voigt material.

The convolution kernel h accounts for memory effects as usual. The fundamental point, when dealing with memory effects, is that the kernel h cannot be considered a known function, since there are no ways to measure it directly. What we do is to reconstruct h by additional measurements u, taken on a suitable subset of the body Ω. We suppose that such additional information on u can be represented in integral form as

$$\int_\Omega \phi(x)u(t,x)\,dx = g(t), \qquad \forall t \in [0,T], \tag{4.2}$$

where ϕ and g are given functions representing the type of device used to measure u and the results of the measurements, respectively. The inverse problem we consider in its more general form is the following.

Problem 8. Determine $u : [0, T] \times \Omega \longrightarrow \mathbb{R}$ and the convolution kernel $h : [0, T] \longrightarrow \mathbb{R}$ satisfying (4.1) and (4.2), given the initial values $u_0(x)$ and $u_1(x)$.

Definition 4.1. (The inverse problem for the damped wave equation with memory in Sobolev spaces) Let $T > 0$. Determine $\tau \in (0, T]$ and

$$u : W^{3,p}(0,\tau; L^p(\Omega)) \cap W^{2,p}(0,\tau; W^{2,p}(\Omega)), \qquad h \in L^p(0,\tau),$$

satisfying the system (4.1)–(4.2).

We solve the inverse problem under the following conditions on the data:

- (K1) Ω is an open bounded subset of $\mathbb{R}^n$, lying on one side of its boundary $\partial\Omega$, which is a submanifold of $\mathbb{R}^n$ of class C^2 (in the physical case $n = 1, 2, 3$).
- (K2) $p \in (1, +\infty)$, $n \in \mathbb{N}$, with $n < p$, $p \neq 3$.
- (K3) u_0, $u_1 \in W^{2,p}_B(\Omega) := \{u \in W^{2,p}(\Omega) : D_\nu u \equiv 0\}$.
- (K4) $\phi \in L^{p'}(\Omega)$.
- (K5) $f \in C^1(\mathbb{R})$ and f' is Lipschitz continuous in bounded subsets of $\mathbb{R}$.

(K6) $G \in W^{1,p}(0,T;L^p(\Omega))$

(K7) $v_1 := \Delta u_1 + \Delta u_0 + f(u_0, \nabla u_0) + G(0,\cdot) \in B^{2(1-1/p)}_{p,p,B}(\Omega)$, where

$$B^{2(1-1/p)}_{p,p,B}(\Omega) = \begin{cases} B^{2(1-1/p)}_{p,p}(\Omega) & \text{if } p < 3, \\ \{v \in B^{2(1-1/p)}_{p,p}(\Omega) : D_\nu v \equiv 0\} & \text{if } p > 3. \end{cases}$$

(K8) $\int_\Omega \phi \Delta u_0 dx \neq 0$.

(K9) $g \in W^{3,p}(0,T)$ with $\int_\Omega \phi u_0 dx = g(0)$ and $\int_\Omega \phi v_1 dx = g'(0)$.

(K10) f' is globally bounded.

Theorem 4.2. (Local in time existence) *Let the assumptions* (K1)–(K9) *hold. Then there exists* $\tau \in (0,T]$*, depending on the data, such that the inverse problem given by Definition* 4.1 *has a solution* $(u,h) \in [W^{3,p}(0,\tau;L^p(\Omega)) \cap W^{2,p}(0,\tau;W^{2,p}(\Omega))] \times L^p(0,\tau)$.

Theorem 4.3. (Global in time uniqueness) *Let the assumptions* (K1)–(K9) *hold. Then, if* $\tau \in (0,T]$*, and the inverse problem given by Definition* 4.1 *has two solutions*

$$(u_j,h_j) \in [W^{3,p}(0,\tau;L^p(\Omega)) \cap W^{2,p}(0,\tau;W^{2,p}(\Omega))] \times L^p(0,\tau) \quad (j \in \{1,2\}),$$

then $u_1 = u_2$ *and* $h_1 = h_2$.

Theorem 4.4. (Global in time existence and uniqueness) *Let the assumptions* (K1)–(H10) *hold. Let* $T > 0$*. Then the inverse problem given by Definition* 4.1 *has a unique solution* $(u,h) \in [W^{3,p}(0,T;L^p(\Omega)) \cap W^{2,p}(0,T;W^{2,p}(\Omega))] \times L^p(0,T)$.

4.2. A nuclear reactor model

For the proofs of the results presented here see [10]. For the nuclear reactor models see for example [4] and [29]. Let $t \in [0,T]$, for $T > 0$, $x \in \Omega$ where Ω is an open bounded set representing the nuclear reactor. In the sequel we indicate the boundary of Ω by $\partial\Omega$. We denote by u the deviation of the temperature from the equilibrium, $\mathcal{W}$ is the logarithm of the total reactor power.

Definition 4.5. (The inverse problem in Sobolev spaces) Let $T > 0$. Determine $\tau \in (0,T]$ and

$$u \in W^{2,p}(0,\tau;L^p(\Omega)) \cap W^{1,p}(0,\tau;W^{2,p}(\Omega)), \quad \mathcal{W} \in W^{3,p}(0,\tau), \quad h \in L^p(0,\tau), \tag{4.3}$$

satisfying system

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x) + h * \Delta u(t,x) + \eta(x)(e^{\mathcal{W}(t)} - 1), \\ u(0,x) = u_0(x), \ x \in \Omega, \\ D_\nu u(t,x) = 0, \quad t \in (0,\tau), \ x \in \partial\Omega, \\ \mathcal{W}'(t) = -\int_\Omega \alpha(x) u(t,x) dx, \\ \mathcal{W}(0) = \mathcal{W}_0, \\ \int_\Omega \phi(x) u(t,x) dx = g(t). \end{cases} \tag{4.4}$$

Definition 4.6. (The linearized version of the inverse problem in Sobolev spaces) Let $T > 0$. Determine

$$u \in W^{2,p}(0,T;L^p(\Omega)) \cap W^{1,p}(0,T;W^{2,p}(\Omega)), \quad \mathcal{W} \in W^{3,p}(0,T), \quad h \in L^p(0,T), \tag{4.5}$$

satisfying system (4.4) where the term $\eta(x)(e^{\mathcal{W}(t)} - 1)$ is replaced by $\eta(x)\mathcal{W}(t)$.

Let us assume the following conditions (for the physical case take $n = 3$).

(h1) Ω is an open bounded subset of $\mathbb{R}^n$, lying on one side of its boundary $\partial\Omega$, which is a submanifold of $\mathbb{R}^n$ of class C^2.
(h2) $p \in (1, +\infty)$, $n \in \mathbb{N}$, with $n < p$, $p \neq 3$.
(h3) $\mathcal{W}_0 \in \mathbb{R}$, $\quad u_0 \in W_B^{2,p}(\Omega)$.
(h4) $\phi \in L^{p'}(\Omega)$, $\quad \alpha \in L^{p'}(\Omega)$, $\quad \eta \in L^p(\Omega)$.
(h5) $f \in C^1(\mathbb{R})$ and f' is Lipschitz continuous in bounded subsets of $\mathbb{R}$.
(h6) $v_0 := \Delta u_0 + \eta f(\mathcal{W}_0) \in B_{p,p,B}^{2(1-1/p)}(\Omega)$.
(h7) $g \in W^{2,p}(0,T)$ with $\int_\Omega \phi(x)u_0(x)dx = g(0)$ and $\int_\Omega \phi(x)v_0(x)dx = g'(0)$.
(h8) $\int_\Omega \phi(x)\Delta u_0(x)dx \neq 0$.
(h9) f' globally bounded.

We recall that for $\theta \in (0,1)$ we have the interpolation result (see [36])

$$\begin{aligned}(L^p(\Omega), W_B^{2,p}(\Omega))_{\theta,p} &= B_{p,p,B}^{2(1-1/p)}(\Omega) \\ &= \begin{cases} B_{p,p}^{2(1-1/p)}(\Omega) & \text{if } p < 3, \\ \{v \in B_{p,p}^{2(1-1/p)}(\Omega) : D_\nu v \equiv 0\} & \text{if } p > 3. \end{cases}\end{aligned}$$

Theorem 4.7. (Local in time existence in the case of Definition 4.5) *Let the assumptions* (h1)–(h8) *hold. Then there exists $\tau \in (0,T]$, depending on the data, such that the inverse problem given by Definition* 4.5 *has a solution* $(u, \mathcal{W}, h) \in [W^{2,p}(0,\tau;L^p(\Omega)) \cap W^{1,p}(0,\tau;W^{2,p}(\Omega))] \times W^{3,p}(0,\tau) \times L^p(0,\tau)$.

Theorem 4.8. (Global in time uniqueness in the case of Definition 4.5) *Let the assumptions* (h1)–(h8) *hold. Then, if $\tau \in (0,T]$, and the inverse problem given by Definition* 4.5 *has two solutions*

$$(u_j, \mathcal{W}_j, h_j) \in [W^{2,p}(0,\tau;L^p(\Omega)) \cap W^{1,p}(0,\tau;W^{2,p}(\Omega))] \times W^{3,p}(0,\tau) \times L^p(0,\tau)$$

for $j \in \{1,2\}$, then $u_1 = u_2$, $\mathcal{W}_1 = \mathcal{W}_2$, and $h_1 = h_2$.

In the case we consider the linearized version of the inverse problem in Definition 4.6, instead of position (P6) we have to assume

$$\eta f(\mathcal{W}) := \eta(x)\mathcal{W}(t).$$

Theorem 4.9. (Global in time existence and uniqueness in the case of Definition 4.6) *Let the assumptions* (h1)–(h9) *hold. Let $T > 0$. Then the inverse problem given by Definition* 4.6 *has a unique solution* $(u, \mathcal{W}, h) \in [W^{2,p}(0,T;L^p(\Omega)) \cap W^{1,p}(0,T;W^{2,p}(\Omega))] \times W^{3,p}(0,T) \times L^p(0,T)$.

5. The beam equation with memory

We conclude this overview of models with the beam equation with memory, the proofs of the following results are in the paper [15]. For non parabolic models see also [21] and [22].

Definition 5.1. (The inverse problem for the beam equation) Let $T > 0$. Determine $\tau \in (0, T]$ and

$$(U, h) \in [C^2([0,T]; L^2(\Omega)) \cap C^1([0,T]; H^2(\Omega)) \cap C([0,T]; H^4(\Omega))] \times L^1(0,T)$$

satisfying the system

$$\left\{\begin{array}{l} U_{tt}(t,x) + \Delta^2 U(t,x) - \Delta U(t,x) = \int_0^t h(t-s)\Delta U(s,x)ds \\ +F(t,x,U(t,x), D_x U(t,x), D_x^2 U(t,x)), \qquad (t,x) \in (0,\tau) \times \Omega, \\ U(t,x) = g_0(t,x), \qquad (t,x) \in [0,T] \times \partial\Omega, \\ D_\nu U(t,x) = g_1(t,x), \qquad (t,x) \in [0,T] \times \partial\Omega, \\ U(0,x) = U_0(x), \qquad x \in \Omega, \\ U_t(0,x) = U_1(x), \qquad x \in \Omega, \\ \int_\Omega \phi(x) U(t,x)\, dx = G(t), \qquad t \in [0,T]. \end{array}\right. \tag{5.1}$$

We will indicate with γ the trace operator in $\partial\Omega$. We study the problem in Definition 5.1 under the following assumptions.

(K1) Ω is an open bounded subset of $\mathbb{R}^n$, with $n \le 3$, lying on one side of its boundary $\partial\Omega$, which is a submanifold of $\mathbb{R}^n$ of class C^5.

(K2) We indicate with (t, x, u, p, q) the generic element of $[0,T] \times \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2}$. We assume that $F \in C^1([0,T] \times \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2})$; moreover, the first-order derivatives are Lipschitz continuous with respect to u, p and q, uniformly in bounded subsets of $[0,T] \times \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2}$.

(K3) $U_0 \in H^{4+\varepsilon}(\Omega)$, $U_1 \in H^{2+\varepsilon}(\Omega)$, for some $\varepsilon \in \mathbb{R}^+$.

(K4) For some $\varepsilon \in \mathbb{R}^+$:

$$g_0 \in W^{1+\varepsilon,1}(0,T; H^{\frac{7}{2}}(\partial\Omega)) \cap W^{\frac{11}{4}+\varepsilon,1}(0,T; L^2(\partial\Omega)),$$

$$g_1 \in W^{1+\varepsilon,1}(0,T; H^{\frac{5}{2}}(\partial\Omega)) \cap W^{\frac{9}{4}+\varepsilon,1}(0,T; L^2(\partial\Omega)).$$

(K5) Compatibility conditions on g_0, g_1, U_0 and U_1:

$$\gamma U_0 = g_0(0,.), \quad D_\nu U_0 = g_1(0,.), \quad \text{in } \partial\Omega,$$

$$\gamma U_1 = D_t g_0(0,.), \quad D_\nu U_1 = D_t g_1(0,.) \quad \text{in } \partial\Omega.$$

(K6) $\phi \in H_0^2(\Omega)$.

(K7) $\int_\Omega \phi(x)\Delta U_0(x)dx \neq 0$.

(K8) $G \in W^{3,1}(0,T)$.

(K9) $\int_\Omega \phi(x)U_0(x)dx = G(0)$, $\int_\Omega \phi(x)U_1(x)dx = G'(0)$, $\int_\Omega \phi(x)V_0(x)dx = G''(0)$, with

$$V_0 := -\Delta^2 U_0 + \Delta U_0 + F(0, U_0, D_x U_0, D_x^2 U_0).$$

(K10) The first-order derivatives of F are uniformly bounded in $[0,T] \times \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2}$.

We shall prove the following results:

Theorem 5.2. (Local in time existence) *Let assumptions* (K1)–(K9) *hold. Then there exists* $\tau \in (0, T]$*, depending on the data, such that the problem in Definition* 5.1 *has a solution*

$$(U, h) \in [C^2([0, \tau]; L^2(\Omega)) \cap C^1([0, \tau]; H^2(\Omega)) \cap C([0, \tau]; H^4(\Omega))] \times L^1(0, \tau).$$

Theorem 5.3. (Global in time uniqueness) *Let assumptions* (K1)–(K9) *hold. Then, if* $\tau \in (0, T]$*, and the problem in Definition* 5.1 *has two solutions*

$$(U_j, h_j) \in [C^2([0, \tau]; L^2(\Omega)) \cap C^1([0, \tau]; H^2(\Omega)) \cap C([0, \tau]; H^4(\Omega))] \times L^1(0, \tau),$$

$(j \in \{1, 2\})$*, then* $U_1 = U_2$ *and* $h_1 = h_2$*.*

Theorem 5.4. (Global in time existence and uniqueness) *Let assumptions* (K1)–(K10) *hold. Then the problem in Definition* 5.1 *has a unique solution*

$$(U, h) \in [C^2([0, T]; L^2(\Omega)) \cap C^1([0, T]; H^2(\Omega)) \cap C([0, T]; H^4(\Omega))] \times L^1(0, T).$$

References

[1] R. Adams, *Sobolev spaces*, Pure and Applied Mathematics, vol. 65, Plenum Press (1975).

[2] A.B. Bakushinsky, M.Yu. Kokurin, *Iterative methods for approximate solution of inverse problems*, Mathematics and Its Applications (New York), 577, Springer, Dordrecht, 2004.

[3] M. Bertero, P. Boccacci, *Introduction to inverse problems in imaging*, Institute of Physics Publishing, Bristol, 1998.

[4] T.A. Bronikowski, *An integrodifferential system which occurs in reactor dynamics*, Arch. Rational Mech. Anal., **37** (1970), 363–380.

[5] A. Charalambopoulos, A. Kirsch, K.A. Anagnostopoulos, D. Gintides, K. Kiriaki, *The factorization method in inverse elastic scattering from penetrable bodies*, Inverse Problems, **23** (2007), 27–51.

[6] F. Colombo, *Direct and inverse problems for a phase-field model with memory*, J. Math. Anal. Appl., **260** (2001), 517–545.

[7] F. Colombo, *Quasilinear Systems with Memory Kernels*, J. Math. Anal. Appl., **311** (2005), 40–68.

[8] F. Colombo, *An inverse problem for the strongly damped wave equation with memory*, Nonlinearity, **20** (2007), 659–683.

[9] F. Colombo, *An inverse problem for a parabolic integrodifferential model in the theory of combustion*, Physica D, **236** (2007), 81–89.

[10] F. Colombo, *Stability and global in time results for an inverse problem related to a nuclear reactor model*, Inverse Problems, **25** (2009), 105007 (20pp).

[11] F. Colombo, *An inverse problem for a linearized model in the theory of combustion*, Inverse and Ill posed problems, **18** (2010) 167–187.

[12] F. Colombo, D. Guidetti, *A unified approach to nonlinear integrodifferential inverse problems of parabolic type*, Z. Anal. Anwendungen, **21** (2002), 431–464.

[13] F. Colombo, D. Guidetti, *Identification of the memory kernel in the strongly damped wave equation by a flux condition*, Communication on Pure and Applied Analysis, **8** (2009), 601–620. Co2

[14] F. Colombo, D. Guidetti, *An inverse problem for a phase-field model in Sobolev spaces*, Progress in Nonlinear Differential equations and Their Applications, **64** (2005), 189–210.

[15] F. Colombo, D. Guidetti, *An inverse problem for the beam equation with memory with non homogeneous boundary conditions*, Inverse Problems, **24** (2008), no. 6, 065015, 28 pp.

[16] F. Colombo, D. Guidetti, *A global in time existence and uniqueness result for a semilinear integrodifferential parabolic inverse problem in Sobolev Spaces*, Math. Models Methods Appl. Sci., **17** (2007), 1–29.

[17] F. Colombo, D. Guidetti, A. Lorenzi, *Integrodifferential identification problems for thermal materials with memory in non-smooth plain domains*, Dynam. Systems Appl., **12** (2003), 533–559.

[18] H. Eckel, R. Kress, *Nonlinear integral equations for the inverse electrical impedance problem*, Inverse Problems, **23** (2007), 475–491.

[19] G.M.L. Gladwell, *Inverse problems in vibration*, second edition. Solid Mechanics and its Applications, 119. Kluwer Academic Publishers, Dordrecht, 2004.

[20] C.W. Groetsch, *Inverse problems. Activities for undergraduates.* Mathematical Association of America, Washington, DC, 1999.

[21] D. Guidetti, *Reconstruction of a bounded variation kernel in an abstract wave equation*, to appear in "Forum Mathematicum".

[22] D. Guidetti, A. Lorenzi, *A mixed type identification problem related to a phase-field model with memory*, Osaka J. Math., **44** (2007), 961–990.

[23] J.I. Frankel, M. Keyhani *A global in time treatment for inverse heat conduction problems*, Journal of Heat Transfer, **119** (1997), 673–683.

[24] J.I. Frankel, G.E. Osborne, *A new time treatment for solving partial integrodifferential equations of radiative transport*, IMA J. Numer. Anal., **19** (1999), 91–103.

[25] T. Hohage, M.L. Rapun, F.J. Sayas, *Detecting corrosion using thermal measurements*, Inverse Problems, **23** (2007), 53–72.

[26] M. Ikehata, *An inverse source problem for the heat equation and the enclosure method*, Inverse Problems, **23** (2007), 183–202.

[27] V. Isakov, *Inverse problems for partial differential equations*, Applied Mathematical Sciences, 127. Springer-Verlag, New York, 1998.

[28] A. Kirsch, *An introduction to the mathematical theory of inverse problems*, Applied Mathematical Sciences, 120. Springer-Verlag, New York, 1996.

[29] J.J. Levin, J.A. Nohel, *The integrodifferential equations of a class of nuclear reactors with delated neutrons*, Arch. Rational Mech. Anal., **31** (1968), 151–172.

[30] H. Liu, M. Yamamoto, J. Zou, *Reflection principle for the Maxwell equations and its application to inverse electromagnetic scattering*, Inverse Problems, **23** (2007), 2357–2366.

[31] A. Lorenzi, E. Sinestrari, *An inverse problem in the theory of materials with memory*, Nonlinear Anal. **12** (1988), 1317–1335.

[32] J. Kaipio, E. Somersalo, *Statistical and computational inverse problems*, Applied Mathematical Sciences, 160. Springer-Verlag, New York, 2005.

[33] A.I. Prilepko, D.G. Orlovsky, I.A. Vasin, *Methods for solving inverse problems in mathematical physics,* Monographs and Textbooks in Pure and Applied Mathematics, 231. Marcel Dekker, Inc., New York, 2000.

[34] V.G. Romanov, *Investigation methods for inverse problems*, Inverse and Ill-posed Problems Series. VSP, Utrecht, 2002.

[35] H. Tanabe, Equations of evolution, *Mon. and Studies in Math. vol.* 6, Pitman, 1979.

[36] H. Triebel, *Interpolation theory, function spaces, differential operators,* North-Holland, Amsterdam, New York, Oxford, 1978.

F. Colombo
Dipartimento di Matematica
Politecnico di Milano
Via Bonardi 9
I-20133 Milano, Italy
e-mail: fabrizio.colombo@polimi.it

D. Guidetti
Dipartimento di Matematica
Università di Bologna
Piazza di Porta S. Donato 5
I-40126 Bologna, Italy
e-mail: davide.guidetti@unibo.it

Operator Theory:
Advances and Applications, Vol. 216, 139–151

A $\boldsymbol{k}$-uniform Maximum Principle When 0 is an Eigenvalue

Genni Fragnelli and Dimitri Mugnai

Communicated by F. Bucci and I. Lasiecka

Abstract. In this paper we consider linear operators of the form $L + \lambda I$ between suitable functions spaces, when 0 is an eigenvalue of L with constant associated eigenfunctions. We introduce a new notion of "quasi"-uniform maximum principle, named $\boldsymbol{k}$-uniform maximum principle, which holds for λ belonging to certain neighborhoods of 0 depending on $k \in \mathbb{R}^+$. Our approach actually also covers the case of a "quasi"-uniform antimaximum principle, and is based on an $L^\infty - L^2$ estimate. As an application, we prove some generalization of known results for elliptic Neumann problems and new results for parabolic problems with time-periodic boundary conditions.

Mathematics Subject Classification (2000). Primary 35B50; Secondary 35J25, 35J40, 35K10, 34C25.

Keywords. Maximum principle, antimaximum principle, $\boldsymbol{k}$-uniform maximum principle, polyharmonic operators, periodic solutions.

1. Introduction and abstract setting

In the recent paper [6], Campos, Mawhin and Ortega showed a very general maximum and antimaximum principle for linear differential equations whose prototypes were given by linear ODE's with periodic boundary conditions and the linear damped wave equation (or telegraph equation) in one spatial dimension with double periodic boundary conditions. In that paper the abstract setting relies on a $L^\infty - L^1$ estimate for solution-datum of the form

$$\|u\|_{L^\infty(\Omega)} \leq M\|f\|_{L^1(\Omega)},$$

which is common and natural for ODE's and for the wave equation in 1D. On the other hand, for the classical theory of elliptic problems like

$$\begin{cases} \Delta u + \lambda u = f(x) & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 1$, and B denotes Dirichlet or Neumann boundary conditions, a more natural setting would be an $L^2 - L^2$ estimate of the form

$$\|u\|_{L^2(\Omega)} \leq M\|f\|_{L^2(\Omega)},$$

since data belonging to L^1 are not the good ones to perform a standard variational approach (we refer to [3] for a well-established theory for this case). In this classical framework many results have been established for problem (1.1), the typical *maximum principle* sounding as:

(MP): if $\lambda < \lambda_1$, the first eigenvalue of $-\Delta$ under the corresponding boundary condition, then for any $f \geq 0$ the associated solution u is nonpositive in Ω.

On the other hand, a related stronger version, namely the *strong maximum principle*, holds:

(SMP): if in addition $f \neq 0$, then u is strictly negative in Ω.

However, it is now well known that jumping after λ_1 changes the situation a lot: indeed, Clément and Peletier in [7] were the first to show the following *antimaximum principle*:

(AMP): for any $f \geq 0$ in $L^p(\Omega)$, $p > N$, there exists $\delta = \delta(f) > 0$ such that if u solves (1.1) with $\lambda \in (\lambda_1, \lambda_1 + \delta)$ under Dirichlet or Neumann boundary conditions, then $u \geq 0$ in Ω.

They also showed that under Neumann boundary conditions it is possible to take δ independent of f, thus showing a *uniform antimaximum principle* **(UAMP)**, only when $N = 1$. Refinements of the **(UAMP)** are established for higher-order ODE's with periodic boundary conditions in [5], for general second-order PDE's with Neumann or Robin boundary conditions in [16] (where it is proved that **(UAMP)** holds only if $N = 1$), in [8], [9], [18] for polyharmonic operators in low dimensions (essentially for all those dimensions for which the natural Sobolev space containing weak solutions are embedded in $C^0(\Omega)$), while in [25] it is showed that the condition $p > N$ in [7] is sharp for the validity of an antimaximum principle when $L = \Delta$ under Dirichlet boundary conditions, in the sense that requiring a right-hand side $f \in L^2(\Omega)$ forces to assume $N = 1$.

On the other hand, the result of [6] seems to be much more general, since the authors show that it is possible to state maximum and antimaximum principles in a unitary way. Roughly speaking, having in mind Neumann boundary conditions, so that $\lambda_1 = 0$, they start with the following definition of maximum principle, which is actually formulated therein for data f belonging to $L^1(\Omega)$, but which we rephrase here for functions in $L^2(\Omega)$.

Definition 1.1. Given $\lambda \in \mathbb{R} \setminus \{0\}$, we say that the operator $L + \lambda I$ satisfies a *maximum principle* if for every $f \in L^2(\Omega)$ the equation

$$Lu + \lambda u = f, \quad u \in \mathrm{Dom}(L) \subset C^0 \tag{1.2}$$

has a unique solution with $\lambda u \geq 0$ for any $f \geq 0$. Moreover, the maximum principle is said to be *strong* if $\lambda u(x) > 0$ for any $x \in \Omega$ whenever $f \geq 0$ and $f(x) > 0$ in a subset of Ω with positive measure.

Thus it is clear that the authors are actually dealing with a "classical" maximum principle when $\lambda < \lambda_1 = 0$ and with a "classical" antimaximum principle when $\lambda > 0$; more precisely, we remark that for $\lambda > 0$ their definition includes a **(UAMP)** *tout court.*

Without going into the detailed description of L, but thinking for instance of Lu as u'' with Neumann boundary conditions, the main result in [6] is the following

Theorem 1.2 ([6]). *There exist λ_- and λ_+ such that*

$$-\infty \leq \lambda_- < 0 < \lambda_+ \leq +\infty$$

and $L + \lambda I$ has a maximum principle if and only *if $\lambda \in [\lambda_-, 0) \cup (0, \lambda_+]$. Moreover the maximum principle is strong if $\lambda \in (\lambda_-, 0) \cup (0, \lambda_+)$.*

As already said, an $L^\infty - L^1$ estimate is the main ingredient of their proofs; for this reason, having in mind weak solutions to (1.1), this setting is natural for all those problems in which L^1 is contained in the dual of the Sobolev space where weak solutions are sought. In this context, the easy problem

$$\begin{cases} \Delta u + \lambda u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is a bounded domain of $\mathbb{R}^2$ or $\mathbb{R}^3$, *cannot* be handled by Theorem 1.2 if $f \in L^2(\Omega)$, which is the most reasonable assumption since $L^2(\Omega) \subset (H^1(\Omega))'$, while $L^1(\Omega) \not\subset (H^1(\Omega))'$; indeed, we remark that the inclusion $L^1(\Omega) \subset (H^1(\Omega))'$ actually holds only in dimension 1, and this fact was used in [6] to consider (1.3) for $N = 1$ as a special case of polyharmonic problems.

On the other hand, classical regularity results for elliptic PDE's guarantee that if $f \in L^2(\Omega)$ and Ω is a bit regular, say of class C^2 just for simplicity, then the corresponding solution u of (1.3) with $\lambda \neq 0$ belongs to $H^2(\Omega)$, and there exists $C = C(\Omega) > 0$ such that

$$\|u\|_{H^2(\Omega)} \leq C \|f\|_{L^2(\Omega)}, \tag{1.4}$$

for example see [4, Theorem IX.26]. By Morrey's Theorem, if $N = 1, 2, 3$, then $H^2(\Omega) \hookrightarrow C^{0,\alpha}(\bar{\Omega})$, so that (1.4) implies

$$\|u\|_{C^0(\Omega)} \leq M \|f\|_{L^2(\Omega)},$$

where, of course, $\|u\|_{C^0(\Omega)} = \max_{\bar{\Omega}} |u| = \|u\|_{L^\infty(\Omega)}$.

Our purpose is to combine the spirit of all the results cited so far showing that, although a **(UAMP)** cannot hold, in higher dimensions a "quasi-**(UAMP)**" does, in the sense of Definition 1.3 below.

In order to make our setting precise, we start describing the abstract framework we are working within. By Ω we denote a bounded domain of $\mathbb{R}^n$ endowed with a positive and finite measure μ, and we write $L^p(\Omega) := L^p(\Omega, \mu)$, $p \in [1, \infty]$. Given $f \in L^2(\Omega)$ and $k > 0$, we define

$$\overline{f} := \frac{1}{\mu(\Omega)} \int_\Omega f d\mu, \qquad \tilde{f} := f - \overline{f},$$
$$\mathcal{L} := \{f \in L^2(\Omega) : \overline{f} = 0\}, \qquad \tilde{\mathcal{C}} := C^0(\bar{\Omega}) \cap \mathcal{L},$$

and

$$\mathcal{F}_k := \{f \in L^2(\Omega) : \|\tilde{f}\|_{L^2(\Omega)} \leq k\|f\|_{L^1(\Omega)}\}.$$

It is clear that any $f \in L^2(\Omega)$ belongs to a suitable $\mathcal{F}_k$ and to $\mathcal{F}_\ell$ for any $\ell \geq k$, and that $\cup_k \mathcal{F}_k \subset L^1(\Omega)$ with strict inclusion.

We now consider a linear operator $L : \mathrm{Dom}(L) \subset C^0(\bar{\Omega}) \to L^2(\Omega)$ satisfying the following properties:

$$\mathrm{Ker}(L) = \{\text{constant functions}\}, \qquad \mathrm{Im}(L) = \mathcal{L}, \tag{1.5}$$

$$\begin{cases} \text{the problem } Lu = \tilde{f} \text{ has a unique solution } \tilde{u} \in \tilde{\mathcal{C}} \\ \text{and } \exists\, M = M(L) > 0 \text{ such that } \|\tilde{u}\|_{C^0(\Omega)} \leq M\|\tilde{f}\|_{L^2(\Omega)}. \end{cases} \tag{1.6}$$

We remark that these requirements are the natural extensions to our setting of the assumptions made in [6]. Therefore, having in mind Definition 1.1, we give the following

Definition 1.3. Given $\lambda \in \mathbb{R} \setminus \{0\}$, we say that the operator $L + \lambda I$ satisfies a *k-uniform maximum principle* (***k*-(UMP)** for short) if for every $f \in \mathcal{F}_k$ equation (1.2) has a unique solution with $\lambda u \geq 0$ for any $f \geq 0$. We say that a *strong* ***k*-(UMP)** holds if $\lambda u(x) > 0$ for any $x \in \Omega$ whenever $f \geq 0$ and $f(x) > 0$ in a subset of Ω with positive measure.

Remark 1.4. As in the case of Definition 1.1, the case $\lambda < 0$ corresponds to a classical maximum principle, while the case $\lambda > 0$ states the validity of an antimaximum principle, which is "almost" uniform due to the fact that $f \in \mathcal{F}_k$ and not to the whole of $L^2(\Omega)$.

In view of the cited results stating that for the Laplace operator with Neumann conditions a **(UAMP)** can hold only in dimension 1, we want to prove that a ***k*-(UMP)** *does* hold also in some higher dimensions. In this context, we believe that our result, stated in Theorem 1.5, can shed new light in the general understanding of the matter.

In particular, our result concerns the existence of a neighborhood $\mathcal{U}$ of 0 such that

$$\mathcal{U} \setminus \{0\} \subset \{\lambda \in \mathbb{R} : L + \lambda I \text{ satisfies a } \boldsymbol{k}\textbf{-(UMP)}\}.$$

Thus, having in mind a "quasi"-uniform **(AMP)** for L, we can conclude that there exists $\delta = \delta(k) > 0$ such that for any $\lambda \in (0, \delta)$ and for any $f \in \mathcal{F}_k$, $f \geq 0$, the solution of (1.2) is nonnegative, at least if $N = 1, 2, 3$, in contrast to the validity of the **(UAMP)** for problem (1.1), which can hold only if $N = 1$, as already remarked in [7].

More precisely, our abstract result is the following.

Theorem 1.5. *Assume that* (1.5) *and* (1.6) *hold and fix* $k > 0$*. Then there exists* $\Lambda = \Lambda(k) > 0$ *such that* $L + \lambda I$ *has a* $\boldsymbol{k}$**-(UMP)** *if* $\lambda \in [-\Lambda, \Lambda]$*. Moreover a strong* $\boldsymbol{k}$**-(UMP)** *holds if* $\lambda \in (-\Lambda, \Lambda) \setminus \{0\}$*.*

We remark that in this way we can extend the result about an antimaximum principle for the Laplace operator to higher dimensions, say $N = 2, 3$, also in a quasi-uniform way, being impossible to extend it in a uniform way by the cited results. Indeed, although in [25] it is shown that the condition $p > n$ is sharp (being $f \in L^p(\Omega)$) and that one cannot have $\delta(f)$ to be bounded away from 0 uniformly for all positive f, we can prove that in $\mathcal{F}_k$ there is the desired uniformity. In some sense, it seems that the validity of **(UAMP)** is strongly related to the fact that $L^1 \neq L^2$!

On the other hand, we must also underline the fact that if solutions exist in the right Sobolev space, independently of the Lebesgue spaces containing f, standard maximum principle can be proved also for inhomogeneous inequalities, possibly set on Riemannian manifolds (see the recent [2], [20], [21]), and also when everything is settled in anisotropic Sobolev and Lebesgue spaces with variable exponent ([11]).

Final Remark. In [6] the authors could prove a complete characterization of the set of λ's for which the **(UAMP)** holds (see the "if and only if" part in Theorem 1.2); thus our Theorem 1.5 is not a complete generalization of their Theorem 1.2. However, we believe that this is a first step for further improvements in our setting, which seems more natural to face the maximum (or antimaximum) principle for PDE's.

2. Proof of Theorem 1.5

In this section we want to extend the technique and the spirit of [6] to our functional setting.

We start recalling that the resolvent of L is the operator $R_\lambda : L^2(\Omega) \to C^0(\bar{\Omega})$ which is the inverse of $L + \lambda I$, whenever it exists. Moreover, we introduce the operator $\tilde{R}_0 : \mathcal{L} \to \tilde{\mathcal{C}}$ defined by

$$\tilde{u} = \tilde{R}_0 \tilde{f} \Longleftrightarrow L\tilde{u} = \tilde{f},$$

which is well defined by assumption (1.6).

The first lemma we prove gives a condition that ensures the existence of the resolvent of L.

Lemma 2.1. *There exists* $\Lambda_1 > 0$ *such that for all* $\lambda \in [-\Lambda_1, \Lambda_1] \setminus \{0\}$ *the resolvent* $R_\lambda : L^2(\Omega) \to C^0(\Omega)$ *of* L *is well defined. Moreover, there exists* $C > 0$ *such that if* $\tilde{f} \in \mathcal{L}$ *and* $\lambda \in [-\Lambda_1, \Lambda_1] \setminus \{0\}$ *then*

$$\|R_\lambda \tilde{f}\|_{C^0(\Omega)} \le C \|\tilde{f}\|_{L^2(\Omega)},$$

where $C := \frac{M}{1 - \Lambda_1 \|\tilde{R}_0\|_{\tilde{\mathcal{C}} \to \tilde{\mathcal{C}}}}$ *and* M *is the constant appearing in* (1.6).

Here $\|\tilde{R}_0\|_{\tilde{\mathcal{C}} \to \tilde{\mathcal{C}}}$ denotes the norm of the restriction of the operator $\tilde{R}_0$ from $\tilde{\mathcal{C}}$ to $\tilde{\mathcal{C}}$, which is well defined, since $\tilde{\mathcal{C}} \subset \mathcal{L}$.

Proof. Rewrite (1.2) as the system

$$\begin{cases} L\tilde{u} + \lambda \tilde{u} = \tilde{f}, \\ \lambda \overline{u} = \overline{f}. \end{cases} \tag{2.1}$$

Applying $\tilde{R}_0$, the first equation in (2.1) can be rewritten as

$$(I + \lambda \tilde{R}_0)\tilde{u} = \tilde{R}_0 \tilde{f}. \tag{2.2}$$

Now, if $|\lambda| \|\tilde{R}_0\|_{\tilde{\mathcal{C}} \to \tilde{\mathcal{C}}} < 1$, then $I + \lambda \tilde{R}_0$ is invertible from $\tilde{\mathcal{C}}$ to $\tilde{\mathcal{C}}$ (note that $\tilde{R}_0 \tilde{f} \in \tilde{\mathcal{C}}$) and (2.2) is solved by

$$\tilde{u} = (I + \lambda \tilde{R}_0)^{-1} \tilde{R}_0 \tilde{f}.$$

In conclusion, $R_\lambda f = (I + \lambda \tilde{R}_0)^{-1} \tilde{R}_0 \tilde{f} + \frac{\overline{f}}{\lambda}$.

Now, take $\Lambda_1 \in \left(0, \dfrac{1}{\|\tilde{R}_0\|_{\tilde{\mathcal{C}} \to \tilde{\mathcal{C}}}}\right)$. Thus for all λ such that $|\lambda| \le \Lambda_1$, one has, from the triangle inequality, (2.2) and (1.6),

$$\begin{aligned} \|\tilde{u}\|_{L^\infty(\Omega)} - \Lambda_1 \|\tilde{R}_0\|_{\tilde{\mathcal{C}} \to \tilde{\mathcal{C}}} \|\tilde{u}\|_{L^\infty(\Omega)} &\le \|\tilde{u}\|_{L^\infty(\Omega)} - |\lambda| \|\tilde{R}_0\|_{\tilde{\mathcal{C}} \to \tilde{\mathcal{C}}} \|\tilde{u}\|_{L^\infty(\Omega)} \\ &\le \|(I + \lambda \tilde{R}_0)\tilde{u}\|_{L^\infty(\Omega)} = \|\tilde{R}_0 \tilde{f}\|_{L^\infty(\Omega)} \\ &= \|\tilde{u}\|_{L^\infty(\Omega)} \le M \|\tilde{f}\|_{L^2(\Omega)}. \end{aligned}$$

The thesis follows. □

Note that the proof above provides the estimate

$$\Lambda_1 < \frac{1}{\|\tilde{R}_0\|_{\tilde{\mathcal{C}} \to \tilde{\mathcal{C}}}}.$$

Once proved that R_λ exists, we can prove the following essential result about maximum and antimaximum principles when the data belong to $\mathcal{F}_k$.

Lemma 2.2. *Take* $k > 0$*; then there exists* $\Lambda_2 := \Lambda_2(k) \in (0, \Lambda_1]$ *such that for all* $\lambda \in [-\Lambda_2, \Lambda_2] \setminus \{0\}$ *the operator* $L + \lambda I$ *has a* $\boldsymbol{k}$-**(UMP)**. *Moreover, a strong* $\boldsymbol{k}$-**(UMP)** *holds if* $\lambda \in (-\Lambda_2, \Lambda_2) \setminus \{0\}$.

Proof. If $f \in \mathcal{F}_k$, $f \geq 0$, then $\overline{f} = \frac{1}{\mu(\Omega)}\|f\|_{L^1(\Omega)}$. Thus, using the second equation in (2.1), one has

$$\begin{aligned}\lambda u = \lambda R_\lambda(\tilde{f} + \overline{f}) = \lambda R_\lambda(\tilde{f}) + \overline{f} = \lambda R_\lambda(\tilde{f}) + \frac{1}{\mu(\Omega)}\|f\|_{L^1(\Omega)} \\ \geq \frac{1}{\mu(\Omega)}\|f\|_{L^1(\Omega)} - |\lambda|\|R_\lambda \tilde{f}\|_{L^\infty(\Omega)}.\end{aligned}$$

By the previous lemma it results that if $\lambda \in [-\Lambda_1, \Lambda_1] \setminus \{0\}$, then

$$\begin{aligned}\lambda u &\geq \frac{1}{\mu(\Omega)}\|f\|_{L^1(\Omega)} - |\lambda|\frac{M}{1 - \Lambda_1\|\tilde{R}_0\|_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}}}\|\tilde{f}\|_{L^2(\Omega)} \\ &\geq \left(\frac{1}{\mu(\Omega)} - k|\lambda|\frac{M}{1 - \Lambda_1\|\tilde{R}_0\|_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}}}\right)\|f\|_{L^1(\Omega)}.\end{aligned}$$

The thesis follows taking

$$\Lambda_2 = \min\left\{\Lambda_1, \frac{1 - \Lambda_1\|\tilde{R}_0\|_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}}}{kM\mu(\Omega)}\right\}.$$

□

Proof of Theorem 1.5. Fixed $k > 0$, by Lemma 2.2 the theorem is proved simply taking $\Lambda = \Lambda_2(k)$. □

3. Applications

In this section we present three differential problems where Theorem 1.5 can be applied. The first two examples are almost straightforward, after the considerations made in Section 1, and consist in extending to higher dimensions the uniform maximum principle proved in [6] for elliptic operators, of course under our version of ***k*-(UMP)**.

The third application requires some additional calculations, but we think it is an interesting one: in fact, in the last example we consider some classes of time-periodic parabolic problems, which have raised a growing interest in the last years, especially in their nonlinear versions, mainly for the large number of biological applications they describe (see [1], [12], [13], [15], [19], [23], and also [14] and [22] for other cases), and for which a general approach for the validity of a uniform maximum (or antimaximum) principle seemed to miss so far.

On the other hand, we prove the validity of a ***k*-(UMP)** only in dimension 1, which seems to follow coherently the previous results (see Remark 3.4).

3.1. Laplace operator

Let us consider the classical Neumann problem

$$\begin{cases}\Delta u + \lambda u = f & \text{in } \Omega,\\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega,\end{cases} \tag{3.1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, $N \in \{1,2,3\}$, and $f \in L^2(\Omega)$. Then it is well known that problem (3.1) with $\lambda = 0$ has a solution if and only if $\int_\Omega f = 0$. On the other hand, setting $L = \Delta$, it is clear that $\mathrm{Ker}(L) = \{\text{constant functions}\}$ and that $\lambda_1 = 0$. In addition, it is evident that the problem

$$\begin{cases} Lu = \tilde{f} & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

has a unique solution

$$u \in H^2(\Omega) \cap H_0^1(\Omega) \hookrightarrow C^0(\Omega) \tag{3.2}$$

satisfying the additional condition $\int_\Omega u = 0$, that is $u \in \tilde{\mathcal{C}}$, according to the notations introduced in Section 1. Moreover, as already remarked at the beginning, by classical regularity theory, there exists $\tilde{M} > 0$ such that $\|u\|_{H^2(\Omega)} \leq \tilde{M}\|f\|_{L^2(\Omega)}$. Hence, by (3.2), all the abstract requirements (1.5) and (1.6) for L are fulfilled, where the underlying measure μ is simply Lebesgue's measure in Ω.

Applying Theorem 1.5 we immediately get the following

Proposition 3.1. *Let $N \in \{1,2,3\}$; for any $k > 0$ there exists $\Lambda = \Lambda(k) > 0$ such that if $\lambda \in [-\Lambda, 0) \cup (0, \Lambda]$, then $\Delta + \lambda I$ under homogeneous Neumann boundary conditions has a **k-(UMP)**. Moreover, a strong **k-(UMP)** holds if $\lambda \in (-\Lambda, 0) \cup (0, \Lambda)$.*

In [6] it was already proved that this result was valid for $N = 1$, also giving a complete characterization of the values of λ's for which the result holds true. However, the authors underlined the fact that they could not prove it for $N > 1$, so that they were naturally turned to consider polyharmonic operators in low dimensions. We consider the same operators in the following section.

3.2. Polyharmonic operator

Let us now consider a classical elliptic Neumann problem in presence of an m-polyharmonic operator, $m \in \mathbb{N}$, in a smooth bounded domain Ω of $\mathbb{R}^N$, $N \in \{1, \dots, 4m-1\}$,

$$\begin{cases} \Delta^m u + \lambda u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} \cdots = \frac{\partial \Delta^{m-1} u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.3}$$

so that $\mathrm{Ker}(L) = \{\text{constant functions}\}$, with $L = \Delta^m$, and as already remarked in [6], the assumption (1.5) for L is satisfied. Moreover, by elliptic regularity the weak solution $u \in H^m(\Omega)$ of (3.3) actually belongs to $H^{2m}(\Omega)$ and is such that the estimate $\|\tilde{u}\|_{H^{2m}(\Omega)} \leq \tilde{M}\|f\|_{L^2(\Omega)}$ holds for a suitable constant $\tilde{M}$. Since $N \leq 4m-1$, $H^{2m}(\Omega) \hookrightarrow C^0(\bar{\Omega})$, so that also assumption (1.6) holds. Theorem 1.5 can be immediately applied, thus extending the result showed in [6] when $N \leq 2m-1$ to higher dimensions. As in the previous case, in [6] there was a complete characterization of the λ's, while here we give only a sufficient condition for the validity of the **k-(UMP)**:

Proposition 3.2. *Let $m \in \mathbb{N}$ and $N \in \{1, 2, \ldots, 4m-1\}$; for any $k > 0$ there exists $\Lambda = \Lambda(k) > 0$ such that if $\lambda \in [-\Lambda, 0) \cup (0, \Lambda]$, then $\Delta^m + \lambda I$ under homogeneous Neumann boundary conditions has a* ***k*-(UMP)**. *Moreover, a strong* ***k*-(UMP)** *holds if $\lambda \in (-\Lambda, 0) \cup (0, \Lambda)$.*

3.3. Periodic parabolic problems

In this last part we consider the following parabolic problem:

$$\begin{cases} u_t - \alpha u_{xx} + \lambda u = f & \text{in } \Omega \times (0, \infty), \\ u_x = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(0) = u(T). \end{cases} \tag{3.4}$$

Here Ω is a bounded interval of $\mathbb{R}$, $\alpha > 0$, $T > 0$, $\lambda \in \mathbb{R}$ and $f \in L^2(Q_T)$, where we have put $Q_T = \Omega \times (0, T)$ for shortness. Using the notation of Section 1, we set $Lu := u_t - \alpha u_{xx}$ with

$$D(L) = \left\{ u \in H^1(0, T; H^2(\Omega)) \,:\, u_x = 0 \text{ on } \partial\Omega \text{ for a.e. } t \in (0, T) \right\}.$$

As usual, we define weak solutions of (3.4) as functions $u \in L^2(0, T; H^1(\Omega))$ such that

$$\frac{d}{dt} \int_\Omega uv\,dx + \alpha \int_\Omega u_x v_x\,dx + \lambda \int_\Omega uv\,dx = \int_\Omega fv\,dx \tag{3.5}$$

for a.e. t in $(0, T)$ and for all $v \in H^1(\Omega)$. Moreover u has to satisfy $u(0) = u(T)$.

First, let us note that by parabolic regularity, any solution of (3.4) actually belongs to $C([0, T]; H^1(\Omega))$ (this is an obvious consequence of [4, Theorem X.11] applied to time-periodic solutions of Neumann problems). This fact lets us prove that $\mathrm{Ker}(L) = \{\text{constant functions}\}$: indeed, if u is a solution of

$$\begin{cases} u_t - \alpha u_{xx} = 0 & \text{in } \Omega \times (0, \infty), \\ u_x = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(0) = u(T), \end{cases} \tag{3.6}$$

integrating over $(0, T)$ the related equality given by (3.5) with $\lambda = f = 0$ and $v = u$, gives

$$0 = \int_0^T \frac{d}{dt} \int_\Omega u^2\,dx\,dt + \alpha \int_{Q_T} |u_x|^2\,dx\,dt = \int_\Omega [u^2(T) - u^2(0)]\,dx + \int_{Q_T} |u_x|^2\,dx\,dt,$$

from which we get that u is a constant by the periodicity condition.

Moreover, we now prove that $f \in \mathrm{Im}(L)$ if and only if

$$f \in \mathcal{L} = \left\{ f \in L^2(Q_T) \,:\, \int_{Q_T} f\,dx\,dt = 0 \right\}.$$

Indeed, if $f \in \mathrm{Im}(L)$ and u is the related solution, integrating over $(0, T)$ the definition of weak solution with $v = 1$, gives

$$\int_0^T \frac{d}{dt} \int_\Omega u\,dx\,dt = \int_{Q_T} f\,dx\,dt,$$

and by periodicity this implies $\int_{Q_T} f = 0$, so that $\operatorname{Im}(L) \subseteq \mathcal{L}$. *Viceversa*, let $f \in \mathcal{L} \subset L^2(Q_T) \subset (L^2(0,T); H^1(\Omega))'$; then, by [27, Theorem 32.D] there exists a weak solution $u \in L^2(0,T; H^1(\Omega))$ of

$$\begin{cases} u_t - \alpha u_{xx} = f & \text{in } \Omega \times (0,\infty), \\ u_x = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(0) = u(T), \end{cases} \tag{3.7}$$

and thus $\operatorname{Im}(L) = \mathcal{L}$.

Any other solution of (3.7) is found adding a constant to u, since their difference solves (3.6); thus there exists a unique solution

$$\tilde{u} \in \tilde{\mathcal{C}} := \left\{ v \in C^0(\overline{Q_T}) \, : \, \int_{Q_T} v \, dx \, dt = 0 \right\}.$$

Moreover, by parabolic regularity, we get that $\tilde{u} \in C^0([0,T]; H^1(\Omega))$. In addition, the following estimate holds:

$$\|\tilde{u}\|_{L^\infty(0,T;H^1(\Omega))} \le C \big\{ \|\tilde{u}(0)\|_{H^1(\Omega)} + \|f\|_{L^2(Q_T)} \big\} \tag{3.8}$$

for a universal constant $C = C(\alpha, T, \Omega)$. We remark that this is again an adaptation of classical estimates to solutions of periodic problems, for example, see [24, Theorem 8.13].

We are not able to apply Theorem 1.5 to any problem of the form (3.4). Thus, at this point we assume to deal with data (α, T, Ω) such that the related constant $C(\alpha, T, \Omega)$ appearing in (3.8) is strictly less than 1, i.e.,

$$C = C(\alpha, T, \Omega) < 1. \tag{3.9}$$

We remark that condition (3.9) can be satisfied if, for example, α is sufficiently large.

Thus from (3.8) we easily get

$$\|\tilde{u}\|_{L^\infty(0,T;H^1(\Omega))} \le \frac{C}{1-C} \|f\|_{L^2(Q_T)}. \tag{3.10}$$

By Poincaré-Wirtinger inequality (see [4, Chapter VIII]) we know that

$$\|\tilde{u}(t)\|_{L^\infty(\Omega)} \le \sqrt{|\Omega|} \|\tilde{u}(t)\|_{H^1(\Omega)} \quad \forall \, t,$$

so that (3.10) implies

$$\|\tilde{u}\|_{L^\infty(Q_T)} \le \frac{C\sqrt{|\Omega|}}{1-C} \|f\|_{L^2(Q_T)};$$

thus (1.6) is satisfied with $M = \frac{C\sqrt{|\Omega|}}{1-C}$.

Without other assumptions we can now apply Theorem 1.5 to problem (3.4) to get:

Theorem 3.3. *Assume* (3.9), *fix* $k > 0$ *and set* $Lu := u_t - \alpha u_{xx}$. *Then there exists* $\Lambda = \Lambda(k) > 0$ *such that* $L + \lambda I$ *has a* $\boldsymbol{k}$**-(UMP)** *if* $\lambda \in [-\Lambda, 0) \cup (0, \Lambda]$. *Moreover a strong* $\boldsymbol{k}$**-(UMP)** *holds if* $\lambda \in (-\Lambda, 0) \cup (0, \Lambda)$.

Remark 3.4. To our best knowledge, there are not many results concerning **(UMP)** or **(UAMP)** for parabolic problems like (3.4). For example, we quote [10], where the authors prove a result which resembles an antimaximum principle but for certain Cauchy problems with homogeneous Dirichlet boundary conditions. Their result, however, is different in nature from ours, since they show what we could call a kind of *eventual* antimaximum principle, in the sense that they prove that solutions of Cauchy-Dirichlet problems are positive for large times, also when the datum is negative.

We are aware of the recent paper [17], where the authors consider a periodic parabolic problem under both homogeneous Dirichlet or Neumann conditions, and they show an **(AMP)** also in presence of a weight. On the other hand, if $N = 1$, they assume that the right-hand side of the parabolic equation belongs to L^p with $p > 3$, and in addition their result is not uniform. On the contrary, with our approach we can handle the case $f \in L^2$ and we can prove a ***k*-(UMP)**, so that certain uniformity for the validity of a maximum or antimaximum principle with data in L^2 is guaranteed, although with some restrictions on the coefficient α and on the interval Ω.

Of course, a result analogous to Theorem 3.3 can be proved if $-\Delta$ in dimension 1 is replaced in a higher dimension N by a polyharmonic operator with the natural Neumann boundary conditions, provided that N is so small that the associated spatial Sobolev space is embedded in the space of continuous functions; thus one can consider the following problem:

$$\begin{cases} u_t + \alpha(-\Delta^m)u + \lambda u = f & \text{in } \Omega \times (0, \infty), \\ \frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = \cdots = \frac{\partial \Delta^{m-1} u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(0) = u(T), \end{cases}$$

where all the assumptions made above for $m = 1$ are obviously generalized according to the new setting, and in particular $N \leq 2m-1$, so that $C^0([0,T]; H^m(\Omega)) \subset C^0(\overline{Q_T})$. The details are left to the reader.

Acknowledgment

The research of G.F. is supported by the M.I.U.R. National Projects *Metodi di viscosità, metrici e di controllo in equazioni alle derivate parziali nonlineari* and *Metodi geometrici per dinamiche controllate e applicazioni.*

The research of D.M. is supported by the M.I.U.R. National Project *Metodi variazionali ed equazioni differenziali nonlineari.*

References

[1] W. Allegretto, P. Nistri, *Existence and optimal control for periodic parabolic equations with nonlocal term*, IMA J. Math. Control Inform **16** (1999), 43–58.

[2] P. Antonini, D. Mugnai, P. Pucci, *Quasilinear elliptic inequalities on complete Riemannian manifolds*, J. Math. Pures Appl. **87** (2007), no. 6, 582–600.

[3] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, J.L. Vazquez, *An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations*, Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser. **22** (1995), no. 2, 241–273.

[4] H. Brezis, *Analyse fonctionnelle: Théorie et applications*, Masson, Paris, 1983.

[5] A. Cabada, S. Lois, *Maximum principles for fourth and sixth order periodic boundary value problems*, Nonlinear Anal. **29** (1997), no. 10, 1161–1171.

[6] J. Campos, J. Mawhin, R. Ortega, *Maximum principles around an eigenvalue with constant eigenfunctions*, Commun. Contemp. Math. **10** (2008), no. 6, 1243–1259.

[7] P. Clément, L.A. Peletier, *An anti-maximum principle for second-order elliptic operators*, J. Differential Equations **34** (1979), no. 2, 218–229.

[8] P. Clément, G. Sweers, *Uniform anti-maximum principle for polyharmonic boundary value problems*, Proc. Amer. Math. Soc. **129** (2000), no. 2, 467–474.

[9] P. Clément, G. Sweers, *Uniform anti-maximum principles*, J. Differential Equations **164** (2000), no. 1, 118–154.

[10] J.I. Díaz, J. Fleckinger–Pellé, *Positivity for large time of solutions of the heat equation: the parabolic antimaximum principle*, Discrete Contin. Dyn. Syst. **10** (2004), no. 1-2, 193–200.

[11] R. Fortini, D. Mugnai, P. Pucci, *Maximum Principles for anisotropic elliptic inequalities*, Nonlinear Analysis TMA **70** (2009), 2917–2929.

[12] G. Fragnelli, *Positive periodic solutions for a system of anisotropic parabolic equations*, J. Math. Anal. Appl. **367** (2010), 204-228.

[13] G. Fragnelli, A. Andreini, *Non-negative periodic solution of a system of $(p(x); q(x))$-Laplacian parabolic equations with delayed nonlocal terms*, Differential Equations and Dynamical Systems **15** (2007), 231–265.

[14] G. Fragnelli, D. Mugnai, *Nonlinear delay equations with nonautonomous past*, Discrete Contin. Dyn. Syst. **21** (2008), no. 4, 1159-1183.

[15] G. Fragnelli, P. Nistri, D. Papini, *Positive periodic solutions and optimal control for a distributed biological model of two interacting species*, to appear in Nonlinear Anal. Real World Appl., 19 pages.

[16] T. Godoy, J.P. Gossez, S. Paczka, *A minimax formula for principal eigenvalues and application to an antimaximum principle*, Calc. Var. Partial Differential Equations **21** (2004), no. 1, 85–111.

[17] T. Godoy, E. Lami Dozo, S. Paczka, *On the antimaximum principle for parabolic periodic problems with weight*, Rend. Semin. Mat. Univ. Politec. Torino **60** (2002), no. 1, 33–44.

[18] H.C. Grunau, G. Sweers, *Optimal conditions for anti-maximum principles*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **30** (2001), no. 3-4, 499–513.

[19] P. Hess, *Periodic-parabolic boundary value problems and positivity*, Pitman Research Notes 247, John Wiley, New York, 1991.

[20] D. Mugnai, P. Pucci, *Maximum Principles for inhomogeneous elliptic inequalities on complete Riemannian manifolds*, Adv. Nonlinear Stud. **9** (2009), 429–452.

[21] P. Pucci, J. Serrin, *The maximum principle*, Progress in Nonlinear Differential Equations and Their Applications 73. Basel: Birkhäuser, 2007.

[22] P. Quittner, P. Souplet, *Superlinear parabolic problems. Blow-up, global existence and steady states*, Birkhäuser Advanced Texts, Birkhäuser Verlag, Basel, 2007.

[23] T.I. Seidman, *Periodic solutions of a non-linear parabolic equation*, J. Differential Equations **19** (1975), no. 2, 242–257.

[24] S. Salsa, *Partial differential equations in action. From modelling to theory*, Berlin: Springer, 2008.

[25] G. Sweers, L^n *is sharp for the anti-maximum principle*, J. Differential Equations **134** (1997), no. 1, 148–153.

[26] Y. Wang, J. Yin, Z. Wu, *Periodic solutions of evolution p-Laplacian equations with nonlinear sources*, J. Math. Anal. Appl. **219** (1998), no. 1, 76–96.

[27] E. Zeidler, *Nonlinear Functional Analysis and its Applications. II/B. Nonlinear monotone operators*, Springer, 1990.

Genni Fragnelli
Dipartimento di Matematica
Università di Bari "Aldo Moro"
via E. Orabona 4
I-70125 Bari, Italy
e-mail: fragnelli@dm.uniba.it

Dimitri Mugnai
Dipartimento di Matematica e Informatica
Università degli Studi di Perugia
Via Vanvitelli 1
I-06123 Perugia, Italy
e-mail: mugnai@dmi.unipg.it

Operator Theory:
Advances and Applications, Vol. 216, 153–166

Steady-state Solutions for a General Brusselator System

Marius Ghergu

Abstract. We study the steady-state solutions associated with a general Brusselator system in a smooth and bounded domain. Various existence and nonexistence results are obtained in terms of parameters.

Mathematics Subject Classification (2000). Primary 35J25; Secondary 47H11; 58C15.

Keywords. Brusselator system, steady-states; Turing patterns.

1. Introduction

The Brusselator model was introduced in 1968 by Prigogine and Lefever [11] as a model for an autocatalytic oscillating chemical reaction. It consists of the following four intermediate reaction steps

$$A \to X, \qquad B + X \to Y + D, \qquad 2X + Y \to 3X, \qquad X \to E.$$

The global reaction is $A + B \to D + E$ and corresponds to the transformation of input products A and B into output products D and E. After some scaling and change of variables, the mathematical model corresponding to the Brusselator system is

$$\begin{cases} u_t - d_1\Delta u = a - (b+1)u + u^2 v & \text{in } \Omega \times (0,\infty), \\ v_t - d_2\Delta v = bu - u^2 v & \text{in } \Omega \times (0,\infty), \end{cases}$$

subject to homogeneous Neumann boundary conditions. Here $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a smooth and bounded domain, the unknowns u, u represent the concentration of the intermediate reactants X and Y having the diffusion rates $d_1, d_2 > 0$, and a, $b > 0$ are fixed concentrations.

Turing [12] suggested that under certain conditions, chemicals can react and diffuse in such a way to produce steady-state heterogeneous spatial patterns of chemical or morphogen concentrations. He showed that a system of two reacting and diffusing chemicals could give rise to spatial patterns from initial near-homogeneity. The idea behind Turing's model is the so-called *diffusion-driven instability* and consists of the existence of a low-range diffusing activator and a wide-range diffusing inhibitor. The activator production is inhibited by the presence of inhibitors and enhanced by the presence of the activator while the inhibitor is not self-enhancing, that is, its production is not linked to the presence of other inhibitors, but to the presence of activators.

Lately, many Turing-type models described by coupled systems of reaction-diffusion equations have been used for generating patterns in both organic and inorganic systems.

In this work we shall consider the following elliptic system

$$\begin{cases} -d_1\Delta u = a-(b+1)u+u^p v & \text{in } \Omega, \\ -d_2\Delta v = bu-u^p v & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

The case $p=2$ corresponds to the steady-state of the Brusselator system. This case was studied in [1] and [10] by using the scaling

$$U = u/a, \qquad V = av/b, \qquad \lambda = 1/d_2, \qquad \theta = d_1/d_2.$$

Here we consider a more general nonlinearity of power type u^p, $p>0$. First, it is easy to check that $(u,v)=(a,ba^{1-p})$ is a constant solution of (1.1). We shall see that if $0<p\le 1$ then this is the only solution of (1.1). In turn, when $p>1$ the existence of a non-constant solution to (1.1) is more delicate. It depends on all parameters a,b,d_1 and d_2 involved in (1.1).

In this work, unlike the approach in [4] (see also [5]), we shall keep the initial parameters a,b,d_1,d_2,p unaltered as this better emphasizes their influence in the qualitative study of (1.1).

One of the novelties in the present work is that we provide upper and lower bounds for positive solutions to (1.1) and thus we obtain various existence, non-existence, and regularity results without any restriction on the dimension $N\ge 1$ of the domain. This is a common difficulty when dealing with steady-state for reaction-diffusion systems (see for instance [2, 3, 10]). In case of Sel'kov model this restriction on N and on other parameters related to it has been removed by Lieberman [6]. Also the result in [6, Theorem 4.1] applies to Brusselator system but here we provide precise bounds by means of a simple argument. As a consequence, we derive uniform bounds for some range of parameters b,d_1 or d_2. Throughout this paper we denote by $0=\mu_0<\mu_1<\mu_2<\cdots<\mu_n<\cdots$ the eigenvalues of $-\Delta$ in Ω with homogeneous Neumann boundary condition. For any $k\ge 0$ we also denote by $m(\mu_k)$ the multiplicity of μ_k.

2. A priori estimates

Basic to our subsequent analysis is the following result which is due to Lou and Ni (see [7, Lemma 2.1]).

Lemma 2.1. *Let* $g \in C^1(\overline{\Omega} \times \mathbb{R})$.

1. *If* $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ *satisfies*

$$\Delta w + g(x, w) \geq 0 \ \ in \ \ \Omega, \qquad \frac{\partial w}{\partial n} \leq 0 \ \ on \ \ \partial\Omega,$$

and $w(x_0) = \max_{\overline{\Omega}} w$, *then* $g(x_0, w(x_0)) \geq 0$.

2. *If* $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ *satisfies*

$$\Delta w + g(x, w) \leq 0 \ \ in \ \ \Omega, \qquad \frac{\partial w}{\partial n} \geq 0 \ \ on \ \ \partial\Omega,$$

and $w(x_0) = \min_{\overline{\Omega}} w$, *then* $g(x_0, w(x_0)) \leq 0$.

We are now in a position to state our main result in this section.

Theorem 2.2 (pointwise estimates). *Assume* $1 < p < \infty$. *Then, any non-constant solution* (u, v) *of* (1.1) *satisfies*

$$\frac{a}{b+1} \leq u \leq a + \frac{d_1 b}{d_2}\left(\frac{b+1}{a}\right)^{p-1} \qquad in \ \ \Omega, \tag{2.1}$$

$$b\left[a + \frac{d_1 b}{d_2}\left(\frac{b+1}{a}\right)^{p-1}\right]^{1-p} \leq v \leq b\left(\frac{b+1}{a}\right)^{p-1} \qquad in \ \ \Omega. \tag{2.2}$$

Proof. Consider first a minimum point $x_0 \in \overline{\Omega}$ of u. By Lemma 2.1(ii) it follows

$$a - (b+1)u(x_0) + u(x_0)^p v(x_0) \leq 0$$

which implies $u(x_0) \geq a/(b+1)$. Hence

$$u \geq \frac{a}{b+1} \qquad \text{in } \ \Omega. \tag{2.3}$$

At maximum point of v we have $bu - u^p v \geq 0$, that is, $v \leq bu^{1-p}$. By virtue of (2.3) we deduce

$$v \leq b\left(\frac{b+1}{a}\right)^{p-1} \qquad \text{in } \ \Omega. \tag{2.4}$$

Let $w = d_1 u + d_2 v$. Adding the first two relations in (1.1) we have

$$-\Delta w = a - u \ \text{ in } \ \Omega, \qquad \frac{\partial w}{\partial \nu} = 0 \ \text{ on } \ \partial\Omega.$$

Let now $x_1 \in \overline{\Omega}$ be a maximum point of w. According to Lemma 2.1(i) we have $a - u(x_1) \geq 0$, that is, $u(x_1) \leq a$. By virtue of (2.4), for all $x \in \overline{\Omega}$ we have

$$d_1 u(x) \leq w(x) \leq w(x_1) \leq d_1 a + d_2 b\left(\frac{b+1}{a}\right)^{p-1} \qquad \text{in } \Omega.$$

This yields

$$u \le a + \frac{d_1 b}{d_2}\left(\frac{b+1}{a}\right)^{p-1} \qquad \text{in } \Omega. \tag{2.5}$$

We have proved that u satisfies (2.1). Again by Lemma 2.1(ii), at minimum points of v we have $bu - u^p v \le 0$, which yields $v \ge bu^{1-p}$. Combining this inequality with (2.5) we obtain the first estimate in (2.2). This concludes our proof. □

Theorem 2.3. *Let $a, b, D_1, D_2 > 0$ be fixed. There exist two positive constants $C_1, C_2 > 0$ depending on a, b, D_1, D_2 such that for all*

$$d_1 \ge D_1, \quad 0 < d_2 \le D_2,$$

any solution (u, v) of (1.1) satisfies

$$C_1 < u, v < C_2 \qquad \text{in } \overline{\Omega}.$$

From the estimates (2.1)–(2.2) in Theorem 2.3 we derive the following:

Theorem 2.4. *Assume that $p > 1$ and let $a, b, D_1, D_2 > 0$ be fixed. Then, there exist two positive constants $C_1, C_2 > 0$ depending on a, b, D_1, D_2 such that for all*

$$d_1 \ge D_1, \quad 0 < d_2 \le D_2,$$

any solution (u, v) of (1.1) satisfies

$$C_1 < u, v < C_2 \qquad \text{in } \overline{\Omega}.$$

Furthermore, by standard elliptic arguments and Theorem 2.4 we now obtain:

Theorem 2.5. *Assume $p > 1$ and let $a, b, D_1, D_2 > 0$ be fixed. Then, for any positive integer $k \ge 1$ there exists a constant*

$$C = C(a, b, D_1, D_2, k, N, \Omega) > 0$$

such that for all

$$d_1 \ge D_1, \quad 0 < d_2 \le D_2,$$

any solution (u, v) of (1.1) satisfies

$$\|u\|_{C^k(\overline{\Omega})} + \|v\|_{C^k(\overline{\Omega})} \le C.$$

In particular, any solution of (1.1) belongs to $C^\infty(\overline{\Omega}) \times C^\infty(\overline{\Omega})$.

We now consider energy estimates for non-constant solutions to (1.1) in the case $p > 1$. We have the following result.

Theorem 2.6 (energy estimates). *Assume $p > 1$. Then, any non-constant solution (u, v) of system (1.1) satisfies*

(i) $$\frac{(\mu_1 d_2)^2}{2(\mu_1 d_1)^2 + 2\mu_1 d_1 + 1} \le \frac{\|\nabla u\|^2_{L^2(\Omega)}}{\|\nabla v\|^2_{L^2(\Omega)}} \le \left(\frac{d_2}{d_1}\right)^2;$$

(ii) $$\|\nabla u\|^2_{L^2(\Omega)} \le \frac{ab^2 d_2}{(2d_1)^2}\left(\frac{b+1}{a}\right)^{p-1} |\Omega|;$$

(iii) $\|\nabla v\|^2_{L^2(\Omega)} \le \dfrac{ab^2}{4d_2^2}\left(\dfrac{b+1}{a}\right)^{p-1}|\Omega|$.

Proof. (i) Remark first that if (u,v) is a solution of (1.1), then, integrating the two equations in (1.1) over Ω and adding them up we have

$$\int_\Omega u(x)dx = a|\Omega|. \tag{2.6}$$

Adding the two equations in system (1.1) we obtain

$$-\Delta(d_1u + d_2v) = a - u \quad \text{ in } \Omega. \tag{2.7}$$

We next multiply with u in (2.7) and integrate over Ω. We find

$$\int_\Omega \nabla(d_1u + d_2v)\nabla v = \int_\Omega u(a-u) = -\int_\Omega (u-a)^2.$$

This yields

$$d_1\int_\Omega |\nabla u|^2 + d_2\int_\Omega \nabla u\nabla v = -\int_\Omega (u-a)^2,$$

so

$$d_2\int_\Omega \nabla u\nabla v = -\int_\Omega (u-a)^2 - d_1\int_\Omega |\nabla u|^2. \tag{2.8}$$

Using (2.8) we now compute

$$\begin{aligned}0 \le \int_\Omega |\nabla(d_1u+d_2v)|^2 &= d_1^2\int_\Omega |\nabla u|^2 + 2d_1d_2\int_\Omega \nabla u\nabla v + d_2^2\int_\Omega |\nabla v|^2\\ &= -d_1^2\int_\Omega |\nabla u|^2 - 2d_1\int_\Omega (u-a)^2 + d_2^2\int_\Omega |\nabla v|^2.\end{aligned}$$

In particular this implies

$$d_2^2\int_\Omega |\nabla v|^2 - d_1^2\int_\Omega |\nabla u|^2 \ge 0,$$

which proves the second part of the inequality in (i). For the first part, we multiply with $d_1u + d_2v$ in (2.7) and obtain

$$\begin{aligned}\int_\Omega |\nabla(d_1u+d_2v)|^2 &= \int_\Omega (a-u)(d_1u+d_2v)\\ &= -d_1\int_\Omega (u-a)^2 - d_2\int_\Omega (u-a)(v-\bar{v})\end{aligned}$$

Combining the last equality with (2.8) we obtain

$$d_2^2\int_\Omega |\nabla v|^2 = d_1\int_\Omega (u-a)^2 - d_2\int_\Omega (u-a)(v-\bar{v}) + d_1^2\int_\Omega |\nabla u|^2 \tag{2.9}$$

Note that

$$-d_2(u-a)(v-\bar{v}) \le \frac{1}{2\mu_1}(u-a)^2 + \frac{d_2^2\mu_1}{2}(v-\bar{v})^2$$

On the other hand, by Poincaré's inequality we have

$$\int_\Omega (u-a)^2 \le \frac{1}{\mu_1}\int_\Omega |\nabla u|^2, \quad \int_\Omega (v-\bar{v})^2 \le \frac{1}{\mu_1}\int_\Omega |\nabla v|^2.$$

Using these two inequalities in (2.9) we find

$$\begin{aligned} d_2^2\int_\Omega |\nabla v|^2 &\le d_1^2\int_\Omega |\nabla u|^2 + \left(d_1+\frac{1}{2\mu_1}\right)\int_\Omega (u-a)^2 + \frac{d_2^2\mu_1}{2}\int_\Omega (v-\bar{v})^2 \\ &\le d_1^2\int_\Omega |\nabla u|^2 + \frac{1}{\mu_1}\left(d_1+\frac{1}{2\mu_1}\right)\int_\Omega |\nabla u|^2 + \frac{d_2^2}{2}\int_\Omega (v-\bar{v})^2 \end{aligned}$$

Hence

$$\frac{d_2^2}{2}\int_\Omega |\nabla v|^2 \le \left(d_1^2+\frac{d_1}{\mu_1}+\frac{1}{2\mu_1^2}\right)\int_\Omega |\nabla u|^2$$

which yields

$$\frac{\int_\Omega |\nabla u|^2}{\int_\Omega |\nabla v|^2} \ge \frac{(\mu_1 d_2)^2}{2(\mu_1 d_1)^2 + 2\mu_1 d_1 + 1}.$$

This finishes the proof of (i).

(iii) Using the inequality (2.1) we have

$$uv \le \left(\frac{b+1}{a}\right)^{(p-1)/2} u^{(p+1)/2} v \quad \text{ in } \Omega.$$

Next, we multiply the second equation of (1.1) with v and integrate over Ω. We find

$$\begin{aligned} d_2\int_\Omega |\nabla v|^2 &= b\int_\Omega uv - \int_\Omega u^p v^2 \\ &\le b\left(\frac{b+1}{a}\right)^{(p-1)/2}\int_\Omega u^{(p+1)/2} v - \int_\Omega u^p v^2 \\ &\le b\left(\frac{b+1}{a}\right)^{(p-1)/2}\left(\int_\Omega u\right)^{1/2}\left(\int_\Omega u^p v^2\right)^{1/2} - \int_\Omega u^p v^2 \\ &= b\left(\frac{b+1}{a}\right)^{(p-1)/2} a^{1/2}|\Omega|^{1/2}\left(\int_\Omega u^p v^2\right)^{1/2} - \int_\Omega u^p v^2. \end{aligned}$$

In particular, the right-hand side of the above inequality is non-negative, so

$$\int_\Omega u^p v^2 \le ab^2\left(\frac{b+1}{a}\right)^{p-1}|\Omega|$$

and

$$d_2\int_\Omega |\nabla v|^2 \le \frac{ab^2}{4}\left(\frac{b+1}{a}\right)^{p-1}|\Omega|.$$

Now, (ii) follows from (iii) since

$$\int_\Omega |\nabla u|^2 \le \frac{d_2^2}{d_1^2}\int_\Omega |\nabla v|^2 \le \frac{ab^2 d_2}{(2d_1)^2}\left(\frac{b+1}{a}\right)^{p-1}|\Omega|.$$

3. Nonexistence results

3.1. Case $0 < p \le 1$

Theorem 3.1. *Assume that* $0 < p \le 1$. *Then,* $(u, v) = \big(a, ba^{1-p}\big)$ *is the unique solution of system* (1.1).

Proof. Let (u, v) be a classical solution of (1.1). Let also x_1 (resp. x_2) be a maximum point of u (resp. v) and x_3 (resp. x_4) be a minimum point of u (resp. v) in $\overline{\Omega}$. Using Lemma 2.1(i) in the first equation of (1.1) we have

$$(b+1)u(x_1) \le a + u(x_1)^p v(x_1). \tag{3.1}$$

Now, Lemma 2.1(i) applied to the second equation in (1.1) yields

$$bu(x_2) \ge u(x_2)^p v(x_2),$$

that is, $v(x_2) \le bu(x_2)^{1-p}$. Therefore

$$v(x_1) \le v(x_2) \le bu(x_2)^{1-p} \le bu(x_1)^{1-p}. \tag{3.2}$$

Therefore (3.1) and (3.2) imply $(b+1)u(x_1) \le a + bu(x_1)$, that is,

$$u \le u(x_1) \le a \qquad \text{in } \Omega \tag{3.3}$$

On the other hand, Lemma 2.1(ii) applied to the second equation of (1.1) leads us to $v(x_4) \ge bu(x_4)^{1-p}$. Further we have

$$v(x_3) \ge v(x_4) \ge bu(x_4)^{1-p} \ge bu(x_3)^{1-p}. \tag{3.4}$$

Next, Lemma 2.1(ii) applied to the first equation in (1.1) yields

$$(b+1)u(x_3) \ge a + u(x_3)^{1-p} v(x_3) \ge a + bu(x_3),$$

which implies

$$u \ge u(x_3) \ge a \qquad \text{in } \Omega. \tag{3.5}$$

Now (3.3) and (3.5) produce $u \equiv a$ in Ω and by (1.1) we also have $v \equiv ba^{1-p}$. This ends the proof. □

3.2. Case $p > 1$

Theorem 3.2. (i) *Let* $a, b, d_2 > 0$ *be fixed. There exists* $D = D(a, b, d_2) > 0$ *such that system* (1.1) *has no non-constant solutions for all* $d_1 > D$.
(ii) *Let* $a, d_1, d_2 > 0$ *be fixed. There exists* $B = B(a, d_1, d_2) > 0$ *such that system* (1.1) *has no non-constant solutions for all* $0 < b < B$.

Proof. We first prove the following useful result.

Lemma 3.3. *Let* $a, b, d_2 > 0$ *be fixed and let* $\{\delta_n\} \subset (0, \infty)$ *be such that* $\delta_n \to \infty$ *as* $n \to \infty$. *If* (u_n, v_n) *is a solution of* (1.1) *with* $d_1 = \delta_n$ *then*

$$(u_n, v_n) \to (a, ba^{1-p}) \qquad \text{in } C^2(\overline{\Omega}) \times C^2(\overline{\Omega}) \text{ as } n \to \infty. \tag{3.6}$$

Proof. By Theorem 2.5 the sequence $\{(u_n, v_n)\}$ is bounded in $C^3(\overline{\Omega}) \times C^3(\overline{\Omega})$. Hence, passing to a subsequence if necessary, $\{(u_n, v_n)\}$ converges in $C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$ to some $(u, v) \in C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$. We divide by δ_n in the corresponding equation to u_n and then we pass to the limit with $n \to \infty$. We obtain that (u, v) satisfies

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ -d_2 \Delta v = bu - u^p v & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

Also, u_n and u satisfy (2.6). Now, the first equation in (3.7) together with $\partial u/\partial \nu = 0$ on $\partial\Omega$ implies that u is constant. Combining this fact with (2.6) it follows that $u \equiv a$. Thus, from (3.7), v satisfies

$$-d_2 \Delta v = ab - a^p v \text{ in } \Omega, \quad \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega.$$

Multiplying the above equality with $ab - a^p v$ and then integrating over Ω we obtain

$$0 \le \frac{d_2}{a^p} \int_\Omega |\nabla(ab - a^p v)|^2 dx = -\int_\Omega (ab - a^p v)^2 dx \le 0.$$

Hence $v \equiv a^{1-p} b$ and the proof follows. □

We first introduce the function spaces

$$H^2_{\mathrm{n}}(\Omega) = \left\{ w \in W^{2,2}(\Omega) : \frac{\partial w}{\partial \nu} = 0 \right\}, \qquad L^2_0(\Omega) = \left\{ w \in L^2(\Omega) : \int_\Omega w = 0 \right\}.$$

Thus, letting $w = u - a$, by (2.6) and the standard elliptic regularity, system (1.1) is equivalent to

$$\begin{cases} -\Delta w = \delta\big[a - (b+1)(w+a) + (w+a)^p v\big] & \text{in } \Omega, \\ -d_2 \Delta v = b(w+a) - (w+a)^p v & \text{in } \Omega, \\ w \in H^2_{\mathrm{n}}(\Omega) \cap L^2_0(\Omega), \ v \in H^2_{\mathrm{n}}(\Omega), \end{cases} \tag{3.8}$$

where $\delta = 1/d_1$. Define

$$\mathcal{F} : \mathbb{R} \times (H^2_{\mathrm{n}}(\Omega) \cap L^2_0(\Omega)) \times H^2_{\mathrm{n}}(\Omega) \to L^2_0(\Omega) \times L^2(\Omega),$$

by

$$\mathcal{F}(\delta, w, v) = \begin{pmatrix} \Delta w + \delta \mathcal{P}(a - (b+1)(w+a) + (w+a)^p v) \\ \\ d_2 \Delta v + b(w+a) - (w+a)^p v \end{pmatrix},$$

where $\mathcal{P} : L^2(\Omega) \to L^2_0(\Omega)$ is the projection operator from $L^2(\Omega)$ onto $L^2_0(\Omega)$, namely,

$$\mathcal{P}(z) = z - \frac{1}{|\Omega|} \int_\Omega z(x) dx, \qquad \text{for all } z \in L^2(\Omega).$$

Now (3.8) is equivalent to

$$\mathcal{F}(\delta, w, v) = \mathbf{0}. \tag{3.9}$$

Indeed, if $\mathcal{F}(\delta, w, v) = \mathbf{0}$, then

$$d_2\Delta v + b(w+a) - (w+a)^p v = 0 \text{ in } \Omega, \qquad v \in H^2_{\mathrm{n}}(\Omega).$$

It is easy to see that the above relations imply

$$b(w+a) - (w+a)^p v \in L^2_0(\Omega).$$

Since $w \in L^2_0(\Omega)$, this yields

$$a - (b+1)(w+a) + (w+a)^p v \in L^2_0(\Omega),$$

so that

$$\mathcal{P}(a - (b+1)(w+a) + (w+a)^p v) = a - (b+1)(w+a) + (w+a)^p v.$$

Therefore (3.8) is satisfied.

With the same method as in the proof of Lemma 3.3 we have that the equation $\mathcal{F}(0, w, v) = \mathbf{0}$ has the unique solution $(w, v) = (0, ba^{1-p})$. Next it is easy to see that

$$D_{(w,v)}\mathcal{F}(0, 0, ba^{1-p}) : (H^2_{\mathrm{n}}(\Omega) \cap L^2_0(\Omega)) \times H^2_{\mathrm{n}}(\Omega) \to L^2_0(\Omega) \times L^2(\Omega),$$

is given by

$$D_{(w,v)}\mathcal{F}(0, 0, ba^{1-p}) = \begin{pmatrix} \Delta & 0 \\ b(1-p) & d_2\Delta - a^p \end{pmatrix}.$$

Thus $D_{(w,v)}\mathcal{F}(0, 0, ba^{1-p})$ is invertible and we are in the frame of the Implicit Function Theorem. It follows that there exists $\delta_0, r > 0$ such that $(0, 0, ba^{1-p})$ is the unique solution of

$$\mathcal{F}(\delta, w, v) = \mathbf{0} \qquad \text{in } [0, \delta_0] \times B_r\left(0, ba^{1-p}\right),$$

where $B_r(0, ba^{1-p})$ denotes the open ball in $(H^2_{\mathrm{n}}(\Omega) \cap L^2_0(\Omega)) \times H^2_{\mathrm{n}}(\Omega)$ centered at $(0, ba^{1-p})$ and having the radius $r > 0$.

Let now $\{\delta_n\}$ be a sequence of positive real numbers such that $\delta_n \to \infty$ as $n \to \infty$ and let (u_n, v_n) be an arbitrary solution of (1.1) for a, b, d_2 fixed and $d_1 = \delta_n$. Letting $w_n = u_n - a$, it follows that

$$\mathcal{F}\Big(\frac{1}{\delta_n}, w_n, v_n\Big) = \mathbf{0}.$$

According to Lemma 3.3 we have

$$(w_n, v_n) \to \left(0, ba^{1-p}\right) \qquad \text{in } C^2(\overline{\Omega}) \times C^2(\overline{\Omega}) \text{ as } n \to \infty.$$

This means that for $n \geq 1$ large enough there holds

$$\left(\frac{1}{\delta_n}, w_n, v_n\right) \in (0, \delta_0) \times B_r(0, ba^{1-p})$$

which yields $(w_n, v_n) = (0, ba^{1-p})$. Hence, for $d_1 = 1/\delta_n$ small enough, system (1.1) has only the constant solution (a, ba^{1-p}). The proof of (ii) is similar. □

4. Existence results

Throughout this section we will assume that $p > 1$ and we derive the existence of at least one non-constant solution to system (1.1). Let us introduce the space

$$\mathbf{X} = \left\{ \mathbf{w} = (u,v) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) : \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega \right\} \tag{4.1}$$

and decompose

$$\mathbf{X} = \bigoplus_{k \geq 0} \mathbf{X}_k, \tag{4.2}$$

where $\mathbf{X}_k$ denotes the eigenspace corresponding to μ_k, $k \geq 0$. Also, let

$$\mathbf{X}^+ = \{\mathbf{w} = (u,v) \in \mathbf{X} : \; u, v > 0 \text{ in } \Omega\}$$

and write the system (1.1) in the form

$$-\Delta \mathbf{w} = \mathcal{G}(\mathbf{w}) , \qquad \mathbf{w} \in \mathbf{X}^+, \tag{4.3}$$

where

$$\mathcal{G}(\mathbf{w}) = \begin{pmatrix} \frac{1}{d_1}(a - (b+1)u + u^p v) \\ \frac{1}{d_2}(bu - u^p v) \end{pmatrix}.$$

It is more convenient to write (4.3) in the form

$$\mathcal{F}(\mathbf{w}) = \mathbf{0} , \qquad \mathbf{w} \in \mathbf{X}^+, \tag{4.4}$$

where

$$\mathcal{F}(\mathbf{w}) = \mathbf{w} - (\mathbf{I} - \Delta)^{-1}(\mathcal{G}(\mathbf{w}) + \mathbf{w}) , \qquad \mathbf{w} \in \mathbf{X}^+. \tag{4.5}$$

Let $\mathbf{w}_0 = (a, ba^{1-p})$ be the uniform steady-state solution of (1.1). Then

$$\nabla\mathcal{F}(\mathbf{w}_0) = \mathbf{I} - (\mathbf{I} - \Delta)^{-1}(\mathbf{I} + A),$$

where

$$A := \nabla\mathcal{G}(\mathbf{w}_0) = \begin{pmatrix} \frac{b(p-1)-1}{d_1} & \frac{a^p}{d_1} \\ -\frac{b(p-1)}{d_2} & -\frac{a^p}{d_2} \end{pmatrix}.$$

If $\nabla\mathcal{F}(\mathbf{w}_0)$ is invertible, by [8, Theorem 2.8.1] the index of $\mathcal{F}$ at $\mathbf{w}_0$ is given by

$$\text{index}(\mathcal{F}, \mathbf{w}_0) = (-1)^\gamma, \tag{4.6}$$

where γ denotes the number of the negative eigenvalues of $\nabla\mathcal{F}(\mathbf{w}_0)$. On the other hand, using the decomposition (4.2) we have that $\mathbf{X}_i$ is an invariant space under $\nabla\mathcal{F}(\mathbf{w}_0)$ and $\xi \in \mathbb{R}$ is an eigenvalue of $\nabla\mathcal{F}(\mathbf{w}_0)$ in $\mathbf{X}_i$ if and only if ξ is an eigenvalue of $(\mu_i + 1)^{-1}(\mu_i\mathbf{I} - A)$. Therefore, $\nabla\mathcal{F}(\mathbf{w}_0)$ is invertible if and only if for any $i \geq 0$ the matrix $(\mu_i\mathbf{I} - A)$ is invertible.

Let us define

$$H(a, b, d_1, d_2, \mu) = \det(\mu\mathbf{I} - A). \tag{4.7}$$

Then, if $(\mu_i\mathbf{I} - A)$ is invertible for any $i \geq 0$, with the same arguments as in [9] we have

$$\gamma = \sum_{\substack{i\geq 0,\\ H(a,b,d_1,d_2,\mu_i)<0}} m(\mu_i). \tag{4.8}$$

A straightforward computation yields

$$H(a,b,d_1,d_2,\mu) = \mu^2 - \left(\frac{b(p-1)-1}{d_1a^p} - \frac{a^p}{d_2}\right)\mu + \frac{a^p}{d_1d_2}.$$

If

$$b(p-1) > \left(1 + \sqrt{\frac{d_1}{d_2}a^p}\,\right)^2, \tag{4.9}$$

then the equation $H(\mu) = 0$ has two positive solutions $\mu^{\pm}(a,b,d_1,d_2)$ given by

$$\mu^{\pm}(a,b,d_1,d_2) = \frac{1}{2}\Big(\theta(a,b,d_1,d_2) \pm \sqrt{\theta(a,b,d_1,d_2)^2 - 4a^p/(d_1d_2)}\,\Big),$$

where

$$\theta(a,b,d_1,d_2) = \frac{b(p-1)-1}{d_1a^p} - \frac{a^p}{d_2}.$$

With the same method as in [9] (see also [4, 10]) we have the following result.

Theorem 4.1. *Assume that condition* (4.9) *holds and there exist* $i > j \geq 0$ *such that*

(i) $\mu_i < \mu^+(a,b,d_1,d_2) < \mu_{i+1}$ *and* $\mu_j < \mu^-(a,b,d_1,d_2) < \mu_{j+1}$*;*
(ii) $\sum_{k=j+1}^{i} m(\mu_k)$ *is odd.*

Then (1.1) *has at least one non-constant solution.*

Proof. The proof uses some topological degree arguments. By Theorem 3.2(i) we can fix $D > d_1$ such that

(a) system (1.1) with diffusion coefficients D and d_2 has no non-constant solutions;
(b) $H(a,b,D,d_2,\mu) > 0$ for all $\mu \geq 0$.

Further, by Proposition 2.4 one can find $C_1, C_2 > 0$ depending on a, b, d_1, d_2 such that for any $d \geq d_1$, any solution (u,v) of (1.1) with diffusion coefficients d and d_2 satisfies

$$C_1 < u, v < C_2 \quad \text{ in } \overline{\Omega}.$$

Set

$$\mathcal{M} = \{(u,v) \in C(\overline{\Omega}) \times C(\overline{\Omega}) : C_1 < u, v < C_2 \text{ in } \overline{\Omega}\},$$

and define

$$\Psi : [0,1] \times \mathcal{M} \to C(\overline{\Omega}) \times C(\overline{\Omega}),$$

by

$$\Psi(t,\mathbf{w}) = (-\Delta + \mathbf{I})^{-1} \left(\begin{array}{c} u + \left(\frac{1-t}{D} + \frac{t}{d_1}\right)(a - (b+1)u + u^pv) \\ v + \frac{1}{d_2}(bu - u^pv) \end{array} \right).$$

It is easy to see that solving (1.1) is equivalent to find a fixed point of $\Psi(1,\cdot)$ in $\mathcal{M}$. Further, from the definition of $\mathcal{M}$ and Proposition 2.4, we have that $\Psi(t,\cdot)$ has no fixed points in $\partial\mathcal{M}$ for all $0 \le t \le 1$. Therefore, the Leray-Schauder topological degree $\deg(\mathbf{I}-\Psi(t,\cdot),\mathcal{M},0)$ is well defined.

Using (4.5) we have $\mathbf{I}-\Psi(1,\cdot)=\mathcal{F}$. Thus, if (1.1) has no other solutions except the constant one $\mathbf{w}_0$, then by (4.6) and (4.8) we have

$$\deg(\mathbf{I}-\Psi(1,\cdot),\mathcal{M},0)=\operatorname{index}(\mathcal{F},\mathbf{w}_0)=(-1)^{\sum_{k=j+1}^{i} m(\mu_k)}=-1. \tag{4.10}$$

On the other hand, from the invariance of the Leray-Schauder degree at the homotopy we deduce

$$\deg(\mathbf{I}-\Psi(1,\cdot),\mathcal{M},0)=\deg(\mathbf{I}-\Psi(0,\cdot),\mathcal{M},0). \tag{4.11}$$

Remark that by our choice of D, we have that $\mathbf{w}_0$ is the only fixed point of $\Psi(0,\cdot)$. Furthermore by (b) above we have

$$\deg(\mathbf{I}-\Psi(0,\cdot),\mathcal{M},0)=\operatorname{index}(\mathbf{I}-\Psi(\cdot,0),\mathbf{w}_0)=1. \tag{4.12}$$

Now, from (4.10)–(4.12) we reach a contradiction. Therefore, there exists a non-constant solution of (1.1). This ends the proof. □

Corollary 4.2. *Let $a,b,d_2>0$ be fixed. Assume that*

$$b(p-1)>1 \tag{4.13}$$

and all the eigenvalues μ_i have odd multiplicity. Then, there exists a sequence of intervals $\{(k_n,K_n)\}$ with $0<k_n<K_n<k_{n-1}\to 0$ (as $n\to\infty$) such that the steady-state system (1.1) *has at least one non-constant solution for all*

$$d_1 \in \bigcup_{n\ge 1}(k_n,K_n).$$

Proof. In view of (4.13), condition (4.9) holds for small values of $d_1>0$. Also for $a,b,d_2>0$ fixed we have

$$\mu^-(a,b,d_1,d_2)\to \frac{a^p}{d_2(b(p-1)-1)} \qquad \text{as } d_1\to 0.$$

$$\mu^+(a,b,d_1,d_2)\to\infty \qquad \text{as } d_1\to 0.$$

Therefore we can find a sequence of intervals $\{(k_n,K_n)\}_n$ such that

$$\sum_{\substack{i\ge 0,\\ \mu^-(a,b,d_1,d_2)<\mu_i<\mu^+(a,b,d_1,d_2)}} m(\mu_i) \quad \text{is odd} \tag{4.14}$$

for all $d_1\in\bigcup_{n\ge1}(k_n,K_n)$.Therefore, conditions (i)–(ii) in Theorem 4.1 are fulfilled. □

Corollary 4.3. *Let $a,b,d_1>0$ be fixed. Assume that* (4.13) *holds and*

$$\sum_{\substack{i\ge 0,\\ 0<\mu_i<\frac{b(p-1)-1}{d_1}}} m(\mu_i) \quad \textit{is odd.} \tag{4.15}$$

Then there exists $D > 0$ such that the steady-state system (1.1) *has at least one non-constant solution for any $d_2 > D$.*

Proof. By virtue of (4.13), for any $d_2 > 0$ large enough condition (4.9) holds. Also for any a, b, d_1 fixed we have

$$0 < \mu^-(a,b,d_1,d_2) < \mu^+(a,b,d_1,d_2) < \frac{b(p-1)-1}{d_1}$$

and

$$\mu^-(a,b,d_1,d_2) \to 0\ ,\quad \mu^+(a,b,d_1,d_2) \to \frac{b(p-1)-1}{d_1} \qquad \text{as } d_2 \to \infty.$$

Therefore, for $d_2 > 0$ large, condition (4.15) implies (i)–(ii) in Theorem 4.1. This concludes the proof. □

The next result provides existence of non-constant solutions to system (1.1) with respect to parameter b.

Corollary 4.4. *Let $a, d_1, d_2 > 0$ be fixed. Assume that all the eigenvalues μ_i have odd multiplicity. Then, there exists a sequence of intervals $\{(b_n, B_n)\}$ with $0 < b_n < B_n < b_{n+1} \to \infty$ (as $n \to \infty$) such that the steady-state system* (1.1) *has at least one non-constant solution for all $b \in \cup_{n\geq 1}(b_n, B_n)$.*

Proof. We proceed similarly. Since $p > 1$, for large values of b condition (4.9) is fulfilled. Also for $a, d_1, d_2 > 0$ fixed we have

$$\mu^-(a,b,d_1,d_2) \to 0\ ,\quad \mu^+(a,b,d_1,d_2) \to \infty \qquad \text{as } b \to \infty.$$

Hence, we can find a sequence of non-overlapping intervals $\{(b_n, B_n)\}$ such that $b_n \to \infty$ as $n \to \infty$ and (4.14) holds for all $b \in \bigcup_{n\geq 1}(b_n, B_n)$. □

Our last result in this section concerns the existence of non-constant solutions with respect to the parameter a.

Corollary 4.5. *Assume that $b(p-1) > 1$ and*

$$\sum_{\substack{i\geq 0,\\ 0<\mu_i<\frac{b(p-1)-1}{d_1}}} m(\mu_i) \quad \textit{is odd.} \tag{4.16}$$

Then there exists $A > 0$ such that the steady-state system (1.1) *has at least one non-constant solution for any $0 < a < A$.*

Proof. It is easy to see that (4.9) holds for small values of $a > 0$. As before

$$0 < \mu^-(a,b,d_1,d_2) < \mu^+(a,b,d_1,d_2) < \frac{b(p-1)-1}{d_1}$$

and

$$\mu^-(a,b,d_1,d_2) \to 0\ ,\quad \mu^+(a,b,d_1,d_2) \to \frac{b(p-1)-1}{d_1} \qquad \text{as } a \to 0.$$

Therefore, for $a > 0$ small, condition (4.16) implies (i)–(ii) in Theorem 4.1. This ends the proof. □

References

[1] K.J. Brown and F.A. Davidson, *Global bifurcation in the Brusselator system.* Nonlinear Anal. **24** (1995), 1713–1725.

[2] F.A. Davidson and B.P. Rynne, *A priori bounds and global existence of solutions of the steady-state Sel'kov model.* Proc. Royal Soc. Edinb. A **130** (2000), 507–516.

[3] M. Delgado, J. Lopez-Gomez and A. Suarez, *On the symbiotic Lotka–Volterra model with diffusion and transport effects.* J. Differential Equations **160** (2000), 175–262.

[4] M. Ghergu, *Non-constant steady states for Brusselator type systems.* Nonlinearity **21** (2008), 2331–2345.

[5] M. Ghergu, V. Radulescu, *Turing patterns in general reaction-diffusion systems of Brusselator type*, Commun. Contemp. Math. **12** (2010), 661–679.

[6] G.M. Lieberman, *Bounds for the steady-state Sel'kov model for arbitrary p in any number of dimensions.* SIAM J. Math. Anal. **36** (2005), 1400–1406.

[7] Y. Lou and W.-M. Ni, *Diffusion vs. cross-diffusion: an elliptic approach.* J. Differential Equations **154** (1999), 157–190.

[8] L. Nirenberg, *Topics in Nonlinear Functional Analysis*, 2nd Edition, Courant Institute, 2001.

[9] R. Peng, J. Shi and M. Wang, *On stationary patterns of a reaction-diffusion model with autocatalysis and saturation law.* Nonlinearity **21** (2008), 1471–1488.

[10] R. Peng and M. Wang, *Pattern formation in the Brusselator system.* J. Math. Anal. Appl. **309** (2005), 151–166.

[11] I. Prigogine and R. Lefever, *Symmetry breaking instabilities in dissipative systems II.* J. Chem. Phys. **48** (1968), 1665–1700.

[12] A. Turing, *The chemical basis of morphogenesis.* Philos. Trans. Roy. Soc. B **237** (1952), 37–72.

Marius Ghergu
School of Mathematical Sciences
University College Dublin
Belfield
Dublin 4, Ireland
e-mail: marius.ghergu@ucd.ie

Operator Theory:
Advances and Applications, Vol. 216, 167–185

Ordinary Differential Equations with Distributions as Coefficients in the Sense of the Theory of New Generalized Functions

Uladzimir U. Hrusheuski

Abstract. We consider nonautonomous differential equations with distributions as coefficients. Such equations are ill posed from the mathematical point of view since they contain a product of distributions. There are several approaches to formalize that sort of problems, however in general all these approaches lead to different solutions. In the paper the theory of new generalized functions is used. Such approach, on the one hand, makes possible to encompass the solutions in the sense of traditional approaches, and on the other hand it permits to formalize wider classes of equations. We use modification of *Lazakovich's algebra of generalized random processes* [12] and the notion of *generalized differential* $d_{\widetilde{h}}$. It allows us to get associated solutions of regularized problems which cannot be obtained by using another constructions.

Mathematics Subject Classification (2000). Primary 46F30, 34A37; Secondary 34A36.

Keywords. Algebra of new generalized functions; Differential equations with distributions; Differential equations with generalized coefficients; Associated solutions.

1. Introduction

In the middle of the XXth century the development of mathematical theory was significantly stimulated by physics necessity. New problems required to consider differential equations with singularities in their right-hand sides. And the appearance of distribution theory became well-timed event to overcome such problems. The fact that the space of locally integrable functions is embedded into $D'(\mathbb{R})$ made it possible to investigate, for example, differential equations of the form

$$\dot{x}(t) = f(t, x(t))\dot{L}(t), \tag{1.1}$$

where $\dot{L}$ is a distributional derivative of function L of bounded variation. Simultaneously, the solution in the sense of distribution theory is in accord with classic one if last one exists. But the usage of methods of distribution theory is limited due to Schwartz impossibility result concerning multiplication of distributions.

There are a lot of approaches to formalize an ordinary differential equation with distributional coefficients. They can be classified as follows.

The first approach [1, 2, 14, 15] is carried out in the framework of distribution theory and based on the fact that $C^\infty(\mathbb{R})$ is everywhere dense in $D'(\mathbb{R})$. In this case the product of distributions uv is defined as the limit of sequence $u_n v_n$ in $D'(\mathbb{R})$, where $u_n \to u$, $v_n \to v$ in $D'(\mathbb{R})$, $u_n, v_n \in C^\infty(\mathbb{R})$. Since the existence and the value of the limit depends on the choice of sequences u_n, v_n, the disadvantage of this approach is impossibility to define the product for all pairs of u and v. Let, for example, $u_n(x) = v_n(x) = a(x)cos(nx)$, where $a(x) \in C^\infty(\mathbb{R})$. Then $u_n = v_n \to 0$ in $D'(\mathbb{R})$, but $u_n v_n = u_n^2 \to \frac{a^2(x)}{2}$ in $D'(\mathbb{R})$.

According to the second approach [5, 19], the differential equation (1.1) is interpreted as the integral one

$$x(t) = x_0 + \int\limits_{[0,t]} f(s, x(s))dL(s), \tag{1.2}$$

where integral is understood in some sense. Due to Carathéodory theory solution of equation (1.1) exists if the coefficient $\dot{L}$ is a distributional derivative of absolutely continuous function L. Therefore the interpretation of differential equation as integral one is a natural idea. It should be emphasized however, that the values of jumps of solution $x(t)$ depend on the values of subintegral function f in the points of discontinuity of function L. Let consider the Cauchy problem

$$\dot{x}(t) = (t - x(t))\delta(t-1), \quad x(0) = x_0, \tag{1.3}$$

where $\delta(\cdot)$ is delta function. The solution of (1.3) is understood as the solution of the following integral equation with Lebesgue-Stieltjes integral

$$x(t) = x_0 + \int\limits_{[0,t]} (s - x(s\pm))dH(s-1), \tag{1.4}$$

where $H(\cdot) = 1_{[0,+\infty)}(\cdot)$ is Heaviside function. Then the functions $x_1(t) = x_0 + \frac{1-x_0}{2}H(t-1)$ and $x_2(t) = x_0 + (1-x_0)H(t-1)$ are solutions of (1.3).

In the framework of the third approach [22] the solution of equation (1.1) is defined as the limit of solutions of classical equations which are approximations of initial equation. Consider (1.3) and the sequence of approximate equations $\dot{x}_n(t) = (t - x_n(t))\dot{H}_n(t)$, $n \in \mathbb{N}$ with initial data $x_n(0) = x_0$, where the sequence of continuous functions $H_n(t) = H(t-1)$, $t \notin \left(1 - \frac{1}{n}, 1\right)$ converges to $H(t-1)$ point-by-point. Since the sequence of solutions

$$x_n(t) = t - e^{-H_n(t)}\left(-x_0 + \int_{[0,t]} e^{H_n(s)}ds\right)$$

converges, the limit

$$x(t) = t - e^{-H(t-1)}\left(-x_0 + \int_{[0,t]} e^{H(s-1)}ds\right) = x_0 + H(t-1)\frac{(1-x_0)(e-1)}{e}$$

is a solution of problem (1.3). It is worth emphasizing that in general case existence and the value of the limit $x(t)$ depend on approximation of distributional coefficient.

Generally speaking, different approaches applied to one and the same equation lead to different solutions.

The theory of new generalized function became a base to form a unique approach to formalize differential equations with distributional coefficients. According to this approach the initial equation is carried over to associative differential algebra by the regularization procedure. And the solution of (1.1) is defined as associated solution of regularized equation.

Since every distribution is associated with a number of new generalized functions, different ways of interpretation of initial problem as equation in new generalized functions lead to different associated solutions. In particular, it allows to encompass solutions in the sense of traditional approaches and also get new definitions of solution. Thus, the theory of new generalized functions theoretically grounds the possibility of existence of several different approaches. Simultaneously, it boils down the question about the choice of the most preferable approach to the question about the choice of new generalized function associated with given distribution. This choice has to be based on the refinement of the physical problem since the initial equation does not carry any information about it.

It should be noted that it is much easier to prove the existence of solution of regularized equation than to find function which associates this solution. The aim of the present paper is to find associated solutions of regularized one-dimensional nonautonomous differential equation of the form (1.1). We will use algebra $\widetilde{G}$ which is modification of *Lazakovich's algebra of generalized random processes*[1] and the notion of *generalized differential* $d_{\widetilde{h}}$ [12]. It allows us to get associated solutions of regularized problems which cannot be obtained by using another constructions. Another variants of algebras of new generalized functions can be found in [4, 18, 20, 6].

Let us note that problem (1.1) was investigated in algebra $\widetilde{G}$ mostly in the case of Lipschitz continuous function f. Thus, the autonomous one-dimensional equation of the form (1.1) was considered in [11], [21]. The necessary and sufficient conditions which shows when associated solution of regularized autonomous one-dimensional equation can be interpreted as ordinary function was presented in [3]. The autonomous one-dimensional equation of the form (1.1) in which function f has finite number of points of discontinuity was considered in [16, 17].

[1] Lazakovich's definition of generalized random process uses the notion of Egorov's new generalized function.

The paper is divided into four parts. The construction of algebra $\widetilde{G}$ and other preliminaries are given in part 2. In part 3 the equation (1.1) is considered when function f is Lipschitz continuous and L has bounded variation on T. The case of discontinuous function f and continuous function of bounded variation L is considered in part 4. And part 5 is devoted to investigation of problem (1.1) with discontinuous function f and piecewise constant function L.

2. Algebra of new generalized functions

Let us recall main notions from [6]. Define the set $G(\mathbb{R})$ of new generalized functions as quotient algebra $\overline{G}(\mathbb{R})/J(\mathbb{R})$, where $\overline{G}(\mathbb{R}) = \{\{f_n\} \mid f_n \in C^\infty(\mathbb{R}), n \in \mathbb{N}\}$ and $J(\mathbb{R}) = \{\{f_n\} \in \overline{G}(\mathbb{R}) \mid \exists n_0 : f_n(\cdot) = 0,\ n \geq n_0\}$.

Define also extended real line $\widetilde{\mathbb{R}}$ as quotient algebra $\overline{\mathbb{R}}/I$, where $\overline{\mathbb{R}} = \{\{y_n\} \mid y_n \in \mathbb{R},\ n \in \mathbb{N}\}$ and $I = \{\{y_n\} \in \overline{\mathbb{R}} \mid \exists n_0 : y_n = 0,\ n \geq n_0\}$. The elements of $\widetilde{\mathbb{R}}$ are called generalized real numbers and denoted by $\widetilde{y} = [\{y_n\}]$, $\widetilde{y}$ or $[\{y_n\}]$. The product of generalized numbers $\widetilde{y} = [\{y_n\}]$ $\widetilde{b} = [\{b_n\}]$ is defined as generalized number $[\{y_n b_n\}]$. Note, that $\widetilde{\mathbb{R}}$ is not a field (for example, generalized number $[\{\frac{(-1)^{n-1}+1}{2}\}] \neq \widetilde{0}$ is not a convertible), also $\mathbb{R} \subset \widetilde{\mathbb{R}}$. Fix $\alpha \in \mathbb{R}$ and select in $\widetilde{\mathbb{R}}$ the following subsets

$$\widetilde{T} = \left\{\widetilde{t} \in \widetilde{\mathbb{R}} \mid \forall \{t_n\} \in \widetilde{t} :\ 0 \leq t_n \leq \alpha,\ n \in \mathbb{N}\right\},$$

$$H = \left\{\widetilde{h} \in \widetilde{\mathbb{R}} \mid \forall \{h_n\} \in \widetilde{h} :\ h_n > 0,\ n \in \mathbb{N},\ \lim_{n\to\infty} h_n = 0\right\},$$

$$S = \left\{\widetilde{h} \in H \mid \forall \{h_n\} \in \widetilde{h} :\ h_n = o\left(\frac{1}{n}\right),\ n \to \infty\right\},$$

$$I = \left\{\widetilde{h} \in H \mid \forall \{h_n\} \in \widetilde{h} :\ \frac{1}{n} = o(h_n),\ n \to \infty\right\}.$$

Let $\widetilde{y} = [\{y_n\}] \in \widetilde{\mathbb{R}}$, $\widetilde{f} = [\{f_n\}] \in G(\mathbb{R})$. Consider algebra $\widetilde{G}(\widetilde{\mathbb{R}})$ of new generalized functions of the form $\widetilde{f}(\widetilde{y}) = [\{f_n(y_n)\}]$. Note, that algebra $\widetilde{G}(\widetilde{\mathbb{R}})$ is similar to Lazakovich's algebra of generalized random processes [12].

Assume $\widetilde{f}(\widetilde{y}), \widetilde{g}(\widetilde{y}) \in \widetilde{G}(\widetilde{\mathbb{R}})$, $\widetilde{y} \in \widetilde{\mathbb{R}}$. Define the composition $(\widetilde{f} \circ \widetilde{g})(\widetilde{y}) \in \widetilde{G}(\widetilde{\mathbb{R}})$ and the product $(\widetilde{f}\widetilde{g})(\widetilde{y}) \in \widetilde{G}(\widetilde{\mathbb{R}})$ as new generalized functions $[\{f_n(g_n(y_n))\}]$, $[\{(f_n g_n)(y_n)\}]$ respectively.

Let $\widetilde{y} = [\{y\}] \in \widetilde{\mathbb{R}}$, $\widetilde{h} \in H$. Introduce the notion of generalized differential

$$d_{\widetilde{h}} \widetilde{f}(\widetilde{y}) = [\{f_n(y + h_n) - f_n(y)\}] \in \widetilde{G}(\widetilde{\mathbb{R}}).$$

The generalized differential $d_{\widetilde{h}} \widetilde{f}(\widetilde{y})$ is called an I-generalized (S-generalized) and is denoted by $d^I_{\widetilde{h}} \widetilde{f}(\widetilde{y})$ ($d^S_{\widetilde{h}} \widetilde{f}(\widetilde{y})$), if $h \in I$ ($h \in S$).

Algebras of new generalized functions $\widetilde{G}(\widetilde{T})$ and $\widetilde{G}(\widetilde{T} \times \widetilde{\mathbb{R}})$ are constructed in an analogous way.

Consider in $D = T \times \mathbb{R}$, $T = [0, \alpha] \subset \mathbb{R}$ the Cauchy problem

$$\begin{cases} \dot{x}(t) = f(t, x(t))\dot{L}(t), \\ x(0) = x_0, \end{cases} \tag{2.1}$$

where $\dot{L}$ is a distributional derivative of function $L : T \to \mathbb{R}$ of bounded variation. Throughout this paper it is supposed that function L is right-continuous, μ_i, $i \in \mathbb{N}$ – the points of discontinuity of L and $L(0) = 0$.

Put in correspondence to the problem (2.1) an equation in differentials

$$\begin{cases} d_{\tilde{h}}\tilde{X}\left(\tilde{t}\right) = \tilde{f}\left(\tilde{t}, \tilde{X}\left(\tilde{t}\right)\right) d_{\tilde{h}}\tilde{L}\left(\tilde{t}\right), \\ \tilde{X}\Big|_{[\tilde{0},\tilde{h})}\left(\tilde{t}\right) = \tilde{X}_0, \end{cases} \tag{2.2}$$

which can be written in representatives' form

$$\begin{cases} X_n(t + h_n) - X_n(t) = f_n(t, X_n(t))[L_n(t + h_n) - L_n(t)], \\ X_n(t)|_{[0,h_n)} = X_{n0}(t),\ t \in T, \end{cases} \tag{2.3}$$

where

$$f_n(t, x) = (f * \rho_n)(t, x) = \int_{[0,\frac{1}{n}]^2} f(t + l, x + s)\rho_n(l, s)dlds,$$

$$L_n(t) = (L * \overline{\rho}_n)(t) = \int_{[0,\frac{1}{n}]} L(t + s)\overline{\rho}_n(s)ds,$$

ρ_n, $\overline{\rho}_n$ – standard δ-sequences, i.e., $\rho_n(l, s) = n^2\rho(nl, ns)$, $\overline{\rho}_n(s) = n\overline{\rho}(ns)$, $\rho \in C^\infty(\mathbb{R}^2)$, $\rho \geq 0$, supp $\rho(l, s) \subseteq [0, 1]^2$, $\int\limits_{[0,1]^2} \rho(l, s)dlds = 1$; $\overline{\rho} \in C^\infty(\mathbb{R})$, $\overline{\rho} \geq 0$, supp $\overline{\rho}(s) \subseteq [0, 1]$, $\int\limits_{[0,1]} \overline{\rho}(s)ds = 1$.

Let t be an arbitrary fixed point from segment T. Then t can be represented in the form $t = \tau_t + m_t h_n$, where $\tau_t \in [0, h_n)$, $m_t \in N$. Let $t_k = \tau_t + kh_n$. It is easy to show that solution of system (2.3) one can write in the form

$$X_n(t) = X_{n0}(\tau_t) + \sum_{k=0}^{m_t-1} f_n(t_k, X_n(t_k))[L_n(t_{k+1}) - L_n(t_k)].$$

We will use the following notations throughout this paper: $\Delta L(\mu_i) = L(\mu_i) - L(\mu_i-)$, $L^d(t) = \sum\limits_{\mu_i \leq t} \Delta L(\mu_i)$, $L^c(t) = L(t) - L^d(t)$, $\Delta L_n^k = L_n(t_{k+1}) - L_n(t_k)$, $\Delta L^k = L((k+1)h_n) - L(kh_n)$, C – absolute constant. Instead of symbol $\sum\limits_{k=0}^{m_t-1}$ we will write symbol $\sum$.

Definition 2.1. The element X of topological space Ω is called an associated solution of equation in differentials (2.2), if there exist the representatives $\{f_n\}$ and $\{L_n\}$ for which the solution X_n of problem (2.3) converges to X in topology of space Ω.

Remark 2.2. Replacing in (2.2) symbol $d_{\widetilde{h}}$ by $d^I_{\widetilde{h}}$ ($d^S_{\widetilde{h}}$), one can define also I-associated (S-associated) solution of equation in differentials (2.2).

Due to approach under study, we will understand an associated solution of problem (2.2) as solution of Cauchy problem (2.1).

The following result which is similar to analogous result from [13] gives necessary and sufficient condition for existence and uniqueness of solution of problem (2.2).

Theorem 2.3. *The equation* (2.2) *has unique solution if and only if for any representatives* $\{f_n\}$, $\{L_n\}$, $\{X_{n0}\}$, $\{h_n\}$ *following conditions hold*

$$X_{n0}(t) \in C^\infty[0, h_n), \tag{2.4}$$

$$\begin{aligned} \lim_{s\to 0+} \Big(& \tfrac{d^l}{dt^l} X_{n0}(h_n - s) - \tfrac{d^l}{dt^l} X_{n0}(s) \\ & - \tfrac{d^l}{dt^l}\left(f_n(s, X_{n0}(s))\left(L_n(s+h_n) - L_n(s)\right)\right)\Big) = 0, \quad l = 0, 1, \ldots \end{aligned} \tag{2.5}$$

3. The case of the continuous function f and the arbitrary function L of bounded variation

The following theorems show that associated solution of problem (2.2) considerably depends on the connection between n and h_n. It is known [21], that if a convolution of L with standard δ-sequence is taken as a representative of new generalized function $\widetilde{L}$, the associated solution of problem (2.2) exists only in two cases – either $h_n = o\left(\frac{1}{n}\right)$ or $\frac{1}{n} = o(h_n)$, $n \to \infty$.

Lemma 3.1. [11] *Let for any n the following inequality holds*

$$Z_{n+1} \le A + \sum_{k=1}^{n} A_k + \sum_{k=1}^{n} B_k Z_k,$$

where A, A_k, B_k, Z_k – some positive constants, $k \in \{1, \ldots, n\}$. Then

$$Z_{n+1} \le \left(A + \sum_{k=1}^{n} A_k\right) e^{\sum\limits_{k=1}^{n} B_k}.$$

Theorem 3.2. *Suppose function f is bounded and satisfies to Lipschitz condition with respect to both variables, function L has bounded variation on T and the following condition holds*

$$\nabla_n^1 := \sup_{t\in[0,h_n)} |X_{n0}(t) - x_0| \underset{\substack{n\to\infty \\ h_n \to 0}}{\longrightarrow} 0. \tag{3.1}$$

Then for any $t \in T$ $X_n(t) \to X(t)$ as $n \to \infty$, $h_n \to 0$, $\frac{1}{n} = o(h_n)$, where $X_n(t)$ – solution of problem (2.3), *$X(t)$ – solution of equation*

$$X(t) = x_0 + \int\limits_{[0,t]} f(s, X(s-))dL(s). \tag{3.2}$$

Remark 3.3. It is known [8] that there is a unique solution of equation (3.2) on T under conditions of the theorem (3.2).

Proof. The equation (3.2) is equivalent to equation

$$X(t) = x_0 + \int\limits_{[0,t]} f(s, X(s))dL^c(s) + \sum_{\mu_i \le t} f(\mu_i, X(\mu_i-))\Delta L(\mu_i).$$

Since the function L has bounded variation on T, it follows that

$$\sum_{i=1}^{\infty} |\Delta L(\mu_i)| = \mathrm{Var}_{t\in T} L^d(t) = \mathrm{Var}_{t\in T} L(t) - \mathrm{Var}_{t\in T} L^c(t) \le \mathrm{Var}_{t\in T} L(t) < +\infty.$$

Consequently, for any $\epsilon > 0$ there is $N_\epsilon \in \mathbb{N}$ such, that $\sum\limits_{i=N+1}^{\infty} |\Delta L(\mu_i)| < \epsilon$. Then

$$L^d(t) = \sum_{i=1}^{N} 1_{\{\mu_i \le t\}} \Delta L(\mu_i) + \sum_{i=N+1}^{\infty} 1_{\{\mu_i \le t\}} \Delta L(\mu_i) = L^{\le N}(t) + L^{>N}(t).$$

We have

$$\begin{aligned}
&|X_n(t) - X(t)| \\
&= \Big| X_{n0}(\tau_t) + \sum f_n(t_k, X_n(t_k))\Delta L_n^k \\
&\quad - x_0 - \int\limits_{[0,t]} f(s, X(s))dL^c(s) - \sum_{\mu_i \le t} f(\mu_i, X(\mu_i)-)\Delta L(\mu_i) \Big| \\
&\le +\nabla_n^1 + \Big| \sum (f_n(t_k, X_n(t_k)) - f(t_k, X_n(t_k)))\Delta L_n^k \Big| \\
&\quad + \Big| \sum (f(t_k, X_n(t_k)) - f(t_k, X(t_k)))\Delta L_n^k \Big| \\
&\quad + \Big| \sum f(t_k, X(t_k))(L_n^c(t_{k+1}) - L_n^c(t_k)) - \sum f(t_k, X(t_k))(L^c(t_{k+1}) - L^c(t_k)) \Big| \\
&\quad + \Big| \sum f(t_k, X(t_k))(L^c(t_{k+1}) - L^c(t_k)) - \int\limits_{[\tau_t, t]} f(s, X(s))dL^c(s) \Big| \\
&\quad + \Big| \int\limits_{[0,\tau_t]} f(s, X(s))dL^c(s) \Big| + \Big| \sum f(t_k, X(t_k))(L_n^d(t_{k+1}) - L_n^d(t_k)) \\
&\quad - \sum_{\mu_i \le t} f(\mu_i, X(\mu_i-))\Delta L^d(\mu_i) \Big| = \nabla_n^1 + \sum_{i=1}^{6} D_i.
\end{aligned}$$

Let $V_n = \sup\limits_{|u-v| \le 2h_n + \frac{2}{n}} \mathrm{Var}_{t\in[u,v]} L^c(t)$. Then by using definition of f_n, Lipschitz continuity of f and its boundedness, boundedness of variation of function L

it is easy to show the following estimates

$$D_1 \le \frac{C}{n}, \quad D_2 \le C \sum |X_n(t_k) - X(t_k)| \left|\Delta L_n^k\right|,$$
$$D_3 + D_4 + D_5 \le CV_n, \quad D_6 \le C\left(h_n + \frac{1}{n} + \epsilon + V_n\right).$$

Thus, we have

$$|X_n(t) - X(t)| \le \nabla_n^1 + C\left(h_n + \frac{1}{n} + \epsilon + V_n\right)$$
$$+ C \sum |X_n(t_k) - X(t_k)| \left|L_n(t_{k+1}) - L_n(t_k)\right|.$$

Employing a lemma (3.1) to last inequality we immediately obtain the inequality

$$|X_n(t) - X(t)| \le C\left(\nabla_n^1 + h_n + \frac{1}{n} + \epsilon + V_n\right).$$

The function $|L^c|(t) = \mathrm{Var}_{s\in[0,t]} L^c(s)$ is uniformly continuous on T. Therefore $V_n \to 0$ as $n \to \infty$. Tending $n \to \infty$, $h_n \to 0$, $\frac{1}{n} = o(h_n)$ and $\epsilon \to 0$ we come to the end of the proof. □

Theorem 3.4. *Suppose that all conditions from the previous theorem and conditions* (2.4)*,* (2.5) *hold. Then I-associated solution of the problem* (2.2) *is a solution of equation* (3.2)*.*

Proof. The truth of the theorem follows from the definition of I-associated solution and theorems (2.3) and (3.2). □

Remark 3.5. It is obvious, that I-associated solution of the regularized problem (1.3) coincides with solution in the sense of the papers [5, 19].

Theorem 3.6. *Suppose function f is bounded and satisfies to Lipschitz condition with respect to both variables, function L has bounded variation on T and the condition* (3.1) *holds. Then for any $t \in T$ $X_n(t) \to X(t)$ as $n \to \infty$, $h_n \to 0$, $h_n = o\left(\frac{1}{n}\right)$, where $X_n(t)$ is a solution of the problem* (2.3)*, $X(t)$ is a solution of the equation*

$$\begin{aligned} X(t) = x_0 &+ \int\limits_{[0,t]} f(s, X(s))dL^c(s) \\ &+ \sum_{\mu_i \le t} (\varphi(\Delta L(\mu_i) f(\mu_i, \cdot), X(\mu_i-), 1) - X(\mu_i-)), \end{aligned} \tag{3.3}$$

where $\varphi(z, x, u)$ is the solution of auxiliary integral equation

$$\varphi(z, x, u) = x + \int\limits_{[0,u]} z(\varphi(z, x, s))ds. \tag{3.4}$$

Remark 3.7. Due to classical theorems of the theory of differential equations there is a unique solution of integral equation (3.4) on T.

Remark 3.8. The equation (3.3) can be boiled down to equation of the form (3.2) but with another subintegral function. Therefore there is a unique solution of equation (3.3) on T.

Proof. We have

$$\begin{aligned}
&|X_n(t) - X(t)| \\
&\quad \le \left|X_{n0}(\tau_t) - x_0\right| + \Bigg|\sum f_n(t_k, X_n(t_k))(L_n^c(t_{k+1}) - L_n^c(t_k)) \\
&\qquad - \int\limits_{[0,t]} f(s, X(s))dL^c(s)\Bigg| + \Bigg|\sum f_n(t_k, X_n(t_k))(L_n^d(t_{k+1}) - L_n^d(t_k)) \\
&\qquad - \sum_{\mu_i \le t} (\varphi(\Delta L(\mu_i) f(\mu_i, \cdot), X(\mu_i -), 1) - X(\mu_i -))\Bigg| \\
&\quad \le \nabla_n^1 + D_1 + D_2.
\end{aligned}$$

By using representation $L^d(t) = L^{\le N}(t) + L^{>N}(t)$, definition of f_n, Lipschitz continuity of f and its boundedness, boundedness of variation of function L it can be shown the following inequalities

$$D_1 \le C(\frac{1}{n} + V_n) + C\sum |X_n(t_k) - X(t_k)| \, |L_n^c(t_{k+1}) - L_n^c(t_k)| \, ,$$

$$D_2 \le C\left(\frac{1}{n} + h_n n + h_n + \epsilon + V_n\right) + C\sum_{\mu_i \le t} |X_n(t_{j_i}) - X(t_{j_i})| \left|\Delta L^{\le N}(\mu_i)\right| ,$$

where t_{j_i} is such an index, that $t_{j_i} < \mu_i - \frac{1}{n} \le t_{j_i+1}$.

Employing a lemma (3.1) to inequality

$$\begin{aligned}
|X_n(t) - X(t)| \le \nabla_n^1 &+ C\left(\frac{1}{n} + h_n n + h_n + \epsilon + V_n\right) \\
&+ C\sum |X_n(t_k) - X(t_k)| \, |L_n^c(t_{k+1}) - L_n^c(t_k)| \\
&+ \sum_{\mu_i \le t} |X_n(t_{j_i}) - X(t_{j_i})| \left|\Delta L^{\le N}(\mu_i)\right| ,
\end{aligned}$$

we obtain the estimate

$$|X_n(t) - X(t)| \le C\left(\nabla_n^1 + \frac{1}{n} + h_n n + h_n + \epsilon + V_n\right).$$

Tending $n \to \infty$, $h_n \to 0$, $h_n = o\left(\frac{1}{n}\right)$ and $\epsilon \to 0$ we come to the end of the proof. □

Theorem 3.9. *Suppose that all conditions from the previous theorem and conditions* (2.4), (2.5) *hold. Then* S*-associated solution of the problem* (2.2) *is a solution of equation* (3.3).

Proof. The truth of the theorem follows from the definition of S-associated solution and theorems (2.3) and (3.6). □

Example. Let us consider the equation (1.3). The function $\varphi(f(t,x)=t-x,x_0,y)=(x_0-1)e^{-y}+1$ is a solution of the equation (3.4). Then the function $X(t)=x_0+H(t-1)\frac{(1-x_0)(e-1)}{e}$ is a S-associated solution of the regularized problem (1.3). Note, that $X(t)$ coincides with solution in the sense of the monograph [22].

Example. The solutions of the equation (1.3) in the sense of the papers [1, 2, 14, 15] can be obtained as associated solutions of corresponding equation (2.2) if sequences $\rho_n, \overline{\rho}_n$ of another type are taken.

4. The case of the discontinuous function *f* and the continuous function *L*

The function f is required to satisfy the following main assumptions throughout next sections:

(I) f is bounded by constant M, the set of points of discontinuity has the form $\{(t,x)\,|\,x=\psi(t),\ \psi\in C^1(T)\}$,

(II) f is continuable from any domain of continuity to one's boundary and satisfies Lipschitz condition with respect to both variables in any domain of continuity. Let

$$f^+(t,x):=\begin{cases} f(t,x),\ x\neq\psi(t),\\ \lim f(t,x^*),\ x^*\to x,\ x^*>\psi(t),\ x=\psi(t),\end{cases}$$

$$f^-(t,x):=\begin{cases} f(t,x),\ x\neq\psi(t),\\ \lim f(t,x^*),\ x^*\to x,\ x^*<\psi(t),\ x=\psi(t).\end{cases}$$

Theorem 4.1. *Suppose the condition* (3.1) *and the following conditions hold*

(I) *function L is continuous and there is a constant $\gamma>0$ such that for any $t_2>t_1,\ t_1,t_2\in T$ the inequality $L(t_2)-L(t_1)\geq\gamma(t_2-t_1)$ holds,*

(II) *function f satisfies main assumptions and boundary condition $f^-(t,\psi(t))>\frac{K}{\gamma},\ f^+(t,\psi(t))<-\frac{K}{\gamma},\ t\in T$, where $K=\max\limits_{t\in T}|\psi'(t)|$.*

Then for any $t\in T$ $X_n(t)\to X(t)$ as $n\to\infty$, $h_n\to 0$, where $X_n(t)$ is a solution of problem (2.3)*, $X(t)$ is a solution of equation*

$$X(t)=x_0+\int\limits_{[0,t]}u(s)dL(s), \tag{4.1}$$

where $u(t)\in F(t,x(t))$ for almost all $t\in T$ in the sense of measure $\nu_{|L|}$ ($\nu_{|L|}$ is a measure, generated by function $|L|(t)=\mathrm{Var}_{s\in[0,t]}L(s)$) and $F(t,x)$ is the least convex closed set which contains the limiting values of function $f(t,x^*)$, as $x^*\to x,\ x^*\neq\psi(t)$.

Remark 4.2. It is known [9, 10] that there is a unique solution of equation (4.1) on T under conditions of the theorem (4.1).

Proof. Let $U_t^n(R_n) = \{x \mid \left|x - \left(\varphi(t) - \frac{1}{2n}\right)\right| \leq R_n\}$. By using induction on k we have that the inclusion $X_n(t_{k^*}) \in U_{t_{k^*}}^n(R_n)$, which is true for some k^* and large enough n, implies inclusion $X_n(t_k) \in U_{t_k}^n(R_n)$, which is true for all $k > k^*$ and same n, if $R_n = \frac{1}{2n} + \frac{K}{n} + MV_n + Kh_n$.

It is worth emphasizing that the similar assertion is also true for functional sequence $\widetilde{X}_n(t)$: $\widetilde{X}_n(0) = x_0$,

$$\widetilde{X}_n(t) = \widetilde{X}_n(kh_n) + u_{nk}(L(t) - L(kh_n)),\ t \in (kh_n, (k+1)h_n],$$

where $u_{nk} \in F(kh_n, \widetilde{X}_n(kh_n))$.

We have

$$\begin{aligned} |X_n(t) - X(t)| &\leq \left|X_n(t) - \widetilde{X}_n(t-\tau_t)\right| + \left|\widetilde{X}_n(t-\tau_t) - \widetilde{X}_n(t)\right| + \left|\widetilde{X}_n(t) - X(t)\right| \\ &= H_1 + H_2 + H_3. \end{aligned}$$

It is shown in [9] that $\widetilde{X}_n(t) \to X(t)$, where $X(t)$ is a solution of equation (4.1). It is easy to see that $H_2 = |u_{nm_t}(L(t) - L(m_t h_n))| \leq CV_n$. Let us consider several cases to get the estimate for H_1.

It is supposed that $x_0 \neq \psi(0)$. Let $k_1 \in \{0, \ldots, m_t - 1\}$ be such index that $X_n(t_k) \notin U_{t_k}^n(R_n)$ for any $k \leq k_1$ ($k \in \{0, \ldots, m_t\}$), but $X_n(t_{k_1+1}) \in U_{t_{k_1+1}}^n(R_n)$. Let $k_2 \in \{0, \ldots, m_t - 1\}$ be such index that $\widetilde{X}_n(kh_n) \notin U_{kh_n}^n(R_n)$ for any $k \leq k_2$ ($k \in \{0, \ldots, m_t\}$), but $\widetilde{X}_n((k_2+1)h_n) \in U_{(k_2+1)h_n}^n(R_n)$.

Case A1. Suppose indexes k_1 and k_2 are defined both. Then for any $k > k_1$ the inclusion $X_n(t_k) \in U_{t_k}^n(R_n)$ holds. In particular, it implies inclusion $X_n(t) \in U_t^n(R_n)$. Similarly, for any $k > k_2$ the inclusion $\widetilde{X}_n(kh_n) \in U_{kh_n}^n(R_n)$ holds and, in particular, we have $\widetilde{X}_n(t-\tau_t) \in U_{t-\tau_t}^n(R_n)$.

Then

$$\begin{aligned} H_3 \leq &\left|X_n(t) - \left(\varphi(t) - \frac{1}{2n}\right)\right| + \left|\widetilde{X}_n(t-\tau_t) - \left(\varphi(t-\tau_t) - \frac{1}{2n}\right)\right| \\ &+ \left(\left(\varphi(t) - \frac{1}{2n}\right) - \left(\varphi(t-\tau_t) - \frac{1}{2n}\right)\right) \leq C\left(\frac{1}{n} + V_n + h_n\right). \end{aligned}$$

Case A2. Suppose the indexes k_1 and k_2 are not defined both. Then the negations of definitions of indexes k_1 and k_2 imply that for any $k \in \{0, \ldots, m_t - 1\}$ the points $(t_k, X_n(t_k))$ and $(kh_n, \widetilde{X}_n(kh_n))$ belong to the same domain of continuity of function f and $X_n(t_k) \notin U_{t_k}^n(R_n)$, $\widetilde{X}_n(kh_n) \notin U_{kh_n}^n(R_n)$.

By using Lipschitz continuity of f we obtain the following inequality

$$\begin{aligned} |X_n(t) - \widetilde{X}_n(t-\tau_t)| \leq \nabla_n^1 + C\left(\frac{1}{n} + h_n + V_n\right) \\ + C\sum \left|X_n(t_k) - \widetilde{X}_n(kh_n)\right| \left|\Delta L_n^k\right|. \end{aligned}$$

Employing a lemma (3.1) we have

$$\left|X_n(t) - \widetilde{X}_n(t-\tau_t)\right| \le C\left(\nabla_n^1 + \frac{1}{n} + h_n + V_n\right).$$

Let ε_n denotes the right-hand side of this inequality.

Case A3. Suppose that only one of the indexes k_1, k_2 is defined. Let it be the index k_1. Then for any $k \in \{0, \ldots, m_t - 1\}$ $\widetilde{X}_n(kh_n) \notin U^n_{kh_n}(R_n)$. By using an induction on k it is easy to show that

$$\left|\widetilde{X}_n(t-\tau_t) - \left(\varphi(t-\tau_t) - \frac{1}{2n}\right)\right| \le \left|\widetilde{X}_n((k_1+1)h_n) - \left(\varphi((k_1+1)h_n) - \frac{1}{2n}\right)\right|.$$

Note, that the final estimate from *case* A2 is true for expression

$$\left|\widetilde{X}_n((k_1+1)h_n) - X_n(t_{k_1+1})\right|.$$

Hence

$$H_3 \le \left|X_n(t) - \left(\varphi(t) - \frac{1}{2n}\right)\right| + \left|\widetilde{X}_n((k_1+1)h_n) - \left(\varphi((k_1+1)h_n) - \frac{1}{2n}\right)\right|$$
$$+ \left|\left(\varphi(t) - \frac{1}{2n}\right) - \left(\varphi(t-\tau_t) - \frac{1}{2n}\right)\right| \le \left|X_n(t) - \left(\varphi(t) - \frac{1}{2n}\right)\right|$$
$$+ \left|\widetilde{X}_n((k_1+1)h_n) - X_n(t_{k_1+1})\right| + \left|X_n(t_{k_1+1}) - \left(\varphi((k_1+1)h_n) - \frac{1}{2n}\right)\right|$$
$$+ \left|\left(\varphi(t) - \frac{1}{2n}\right) - \left(\varphi(t-\tau_t) - \frac{1}{2n}\right)\right| \le C\left(\frac{1}{n} + h_n n + h_n +\right) + \varepsilon_n.$$

Tending $n \to \infty$, $h_n \to 0$ we come to the end of the proof. The proof boils down to the *case* A1 if suppose that $x_0 = \psi(0)$. □

Theorem 4.3. *Suppose that all conditions from the previous theorem and conditions* (2.4)*,* (2.5) *hold. Then the associated solution of the problem* (2.2) *is a solution of equation* (4.1)*.*

Proof. The truth of the theorem follows from the definition of associated solution and theorems (2.3) and (4.1). □

Remark 4.4. It is should be noted that if $L(t) = t$ then the methods of the theory of differential equations with discontinuous right-hand parts are applicable to equation (2.1) also (see [7]). In this case the associated solution of the regularized problem (1.1) coincides with solution in the sense of Filippov definition. Moreover, it is a solution whose trajectory is called sliding motion. Let us pay attention that sliding motion is a main mode of operation of systems with varying structure.

Example. Let us consider the equation $\dot{x}(t) = -3\,\mathrm{sgn}(x + t^2 - 2t)$, $t \in [0,2]$, $x(0) = 2\frac{1}{4}$. Suppose that

$$X(t) = \begin{cases} -3t + 2\frac{1}{4}, & t \in \left[0, \frac{1}{2}\right), \\ -t^2 + 2t, & t \in \left[\frac{1}{2}, 2\right], \end{cases} \qquad u(t) = \begin{cases} -3, & t \in \left[0, \frac{1}{2}\right), \\ 2(1-t), & t \in \left[\frac{1}{2}, 2\right]. \end{cases}$$

Then the integral equality (4.1) holds. Moreover, the inclusion $2(1-t) \in [-2,1]$, $t \in \left[\frac{1}{2}, 2\right]$ implies the inclusion

$$u(t) \in F(t, X(t)) = \begin{cases} -3, & t \in \left[0, \frac{1}{2}\right), \\ [-3,3], & t \in \left[\frac{1}{2}, 2\right], \end{cases} \qquad t \in [0,2].$$

5. The case of the discontinuous functions f and L

During investigation of equation (2.1) with continuous function f it was noted that if a convolution of function L with standard δ-sequence is taken as a representative of new generalized function $\widetilde{L}$, the associated solution of problem (2.2) exists only in two cases – either $\frac{1}{n} = o(h_n)$ or $h_n = o\left(\frac{1}{n}\right)$, $n \to \infty$. Therefore it is natural to investigate the associated solutions of regularized equation (2.1) with discontinuous functions f and L namely in this cases.

The function L is required to be piecewise constant function with finite number of points of discontinuity μ_i, $i \in \{1, \dots, n_0\}$ throughout this section.

Theorem 5.1. *Suppose that L is a piecewise constant and nondecreasing (nonincreasing) function, f is a nondecreasing (nonincreasing) with respect to variable x and nonincreasing with respect to variable t function, which satisfies main assumptions. Moreover, suppose that the condition* (3.1) *holds and inequality*

$$X_{n0}(t) \geq x_0 + Kh_n \quad \forall\, t \in [0, h_n)$$

is true for large enough n. Then

$$\int_T |X_n(t) - X(t)| dt \to 0$$

as $n \to \infty$, $h_n \to 0$, $\frac{1}{n} = o(h_n)$, where $X_n(t)$ is a solution of problem (2.3), *$X(t)$ is a solution of equation*

$$X(t) = x_0 + \sum_{\mu_i \leq t} f^+(\mu_i, X(\mu_i-))\Delta L(\mu_i). \tag{5.1}$$

Proof. Let us consider the set

$$T_n^i = T \bigcap \left(\coprod_{k=0}^{m_\alpha+1} \left[\mu_i - \frac{1}{n} + kh_n, \mu_i + kh_n\right] \right)$$

for any $i \in \{1, \dots n_0\}$ and $n \in \mathbb{N}$. Due to relation $\frac{1}{n} = o(h_n)$ the union of sets in T_n^i is disjoint. It is obviously that $\nu_T(\bigcup_i T_n^i) \to 0$ as $n \to \infty$ (ν_T is a Lebesgue measure on T).

We will use an induction on the points of discontinuity of function L. Suppose that $X_n(t)$ converges to $X(t)$ in $L_1[0, \mu_i]$ and there is a numerical sequence λ_n^i such that inequality

$$X(\mu_i-) + Kh_n \le X_n(t_{j_i}) \le X(\mu_i-) + \lambda_n^i \quad \forall\, t \in [\mu_i, \mu_{i+1}) \setminus \bigcup_i T_n^i \tag{5.2}$$

holds for large enough n. Let us show that similar assertion is true for $i+1$. Note, that the basis of induction will hold if set up $\lambda_n^1 = \nabla_n^1$.

Since the inequality

$$\begin{aligned} &|X_n(t) - X(t)| \\ &\quad = \left|X_n(t_{j_i}) + f_n(t_{j_i}, X_n(t_{j_i}))\Delta L(\mu_i) - X(\mu_i-) - f^+(\mu_i, X(\mu_i-))\Delta L(\mu_i)\right| \\ &\quad \le |X_n(t_{j_i}) - X(\mu_i-)| + |\Delta L(\mu_i)| \left|f_n(t_{j_i}, X_n(t_{j_i})) - f^+(\mu_i, X(\mu_i-))\right| \end{aligned}$$

implies the uniform convergence $X_n(t)$ to $X(t)$ on $t \in [\mu_i, \mu_{i+1}) \setminus \bigcup_i T_n^i$, we have

$$\begin{aligned} \int\limits_{[\mu_i,\mu_{i+1}]} |X_n(t) - X(t)|\, dt = &\int\limits_{[\mu_i,\mu_{i+1})\setminus \bigcup_i T_n^i} |X_n(t) - X(t)|\, dt \\ &+ \int\limits_{[\mu_i,\mu_{i+1})\bigcap\left(\bigcup_i T_n^i\right)} |X_n(t) - X(t)|\, dt \to 0 \end{aligned}$$

as $n \to \infty$. Consequently, $X_n(t)$ converges to $X(t)$ in $L_1[0, \mu_{i+1}]$.

Let $\lambda_n^{i+1} = \sup\limits_{t\in[\mu_i,\mu_{i+1})\setminus \bigcup_i T_n^i} |X_n(t) - X(\mu_{i+1}-)|$. Then the truth of inequality

$$X(\mu_{i+1}-) + Kh_n \le X_n(t_{j_{i+1}}) \le X(\mu_{i+1}-) + \lambda_n^{i+1} \quad \forall\, t \in [\mu_{i+1}, \mu_{i+2}) \setminus \bigcup_i T_n^i$$

follows from the definition of the set $\bigcup_i T_n^i$ and monotony of function f. □

Theorem 5.2. *Suppose that all conditions from the previous theorem and conditions* (2.4), (2.5) *hold. Then the* I^+*-associated solution of the problem* (2.2) *is a solution of equation* (5.1).

Proof. The truth of the theorem follows from the definition of I-associated solution and theorems (2.3) and (5.1). □

Theorem 5.3. *Suppose that* L *is a piecewise constant and nonincreasing* (*nondecreasing*) *function,* f *is a nondecreasing* (*nonincreasing*) *with respect to variable* x *and nonincreasing with respect to variable* t *function, which satisfies main assumptions. Moreover, suppose that the condition* (3.1) *holds and inequality*

$$X_{n0}(t) \le x_0 - \frac{1}{n} - Kh_n \quad \forall\, t \in [0, h_n)$$

is true for large enough n. *Then*

$$\int\limits_T |X_n(t) - X(t)|dt \to 0$$

as $n \to \infty,\ h_n \to 0,\ \frac{1}{n} = o(h_n)$, *where* $X_n(t)$ *is a solution of problem* (2.3), $X(t)$ *is a solution of equation*

$$X(t) = x_0 + \sum_{\mu_i \le t} f^-(\mu_i, X(\mu_i-))\Delta L(\mu_i). \tag{5.3}$$

Proof. The proof of this theorem is similar to the proof of the theorem (5.1). □

Theorem 5.4. *Suppose that all conditions from the previous theorem and conditions* (2.4), (2.5) *hold. Then the* I^-*-associated solution of the problem* (2.2) *is a solution of equation* (5.3).

Proof. The truth of the theorem follows from the definition of I-associated solution and theorems (2.3) and (5.3). □

Theorem 5.5. *Suppose that the condition* (3.1) *holds,* L *is a piecewise constant function,* f *satisfies to main assumptions and in any point* μ_i, $i \in \{1, \ldots, n_0\}$ *one of the following conditions holds*

(I) $f^-(\mu_i, \psi(\mu_i)) > 0,\ f^+(\mu_i, \psi(\mu_i)) < 0,\ \Delta L(\mu_i) > 0,$

(II) $f^-(\mu_i, \psi(\mu_i)) < 0,\ f^+(\mu_i, \psi(\mu_i)) > 0,\ \Delta L(\mu_i) < 0,$

(III) $f^-(\mu_i, \psi(\mu_i)) > 0,\ f^+(\mu_i, \psi(\mu_i)) > 0,$

(IV) $f^-(\mu_i, \psi(\mu_i)) < 0,\ f^+(\mu_i, \psi(\mu_i)) < 0.$

Then for any $t \in T$ $X_n(t) \to X(t)$ *as* $n \to \infty,\ h_n \to 0,\ h_n = o\left(\frac{1}{n}\right)$, *where* $X_n(t)$ *is a solution of the problem* (2.3), $X(t)$ *is a solution of the equation*

$$X(t) = x_0 + \sum_{\mu_i \le t} (\varphi_i(1) - \varphi_i(0)). \tag{5.4}$$

Here, $\varphi_i(z)$ *is a solution of the auxiliary integral equation*

$$\varphi_i(z) = X(\mu_i-) + \Delta L(\mu_i) \int\limits_{[0,z]} u_i(s)ds, \quad i \in \{1, \ldots, n_0\} \tag{5.5}$$

and function $u_i(t)$ *satisfies to the inclusion* $u_i(s) \in F(\mu_i, \varphi_i(s))$ *for almost all* $s \in [0,1]$ *in the sense of Lebesgue measure* $\nu_{[0,1]}$ *on* $[0,1]$.

Remark 5.6. It is known [7], that there is a unique solution of equation (5.5) on T under conditions of the theorem (5.5).

Proof. It follows from (2.1) that $X_n(t) \to X(t)$ for any $t \in [0, \mu_1)$. Moreover, it is a uniform convergence on $[0, \mu_1 - \xi]$ for some $\xi > 0$. Take, for example, $\xi := \frac{1}{4} \min_i (\mu_{i+1} - \mu_i)$. Then $\frac{1}{n} < \xi$ for large enough n. Consequently, $X_n(t) = X_{n0}(\tau_t)$ for any $t \in [0, \mu_1 - \xi]$ and $\sup_{t \in [0, \mu_1 - \xi]} |X_n(t) - X(t)| = \sup_{t \in [0, h_n)} |X_{n0}(t) - x_0| \to 0$ as $n \to \infty$, $h_n \to 0$. Simultaneously, for fixed $t \in [0, \mu_1)$ and large enough n the inequality $t < \mu_1 - \frac{1}{n}$ holds. Then $X_n(t) = X_{n0}(\tau_t)$ and $X_n(t) \to X(t)$ as $n \to \infty$ due to (2.1).

We will use an induction on points of discontinuity of function L to prove the assertion of the theorem. Suppose that condition (I) holds in μ_i, $X_n(t) \to X(t)$ for any $t \in [\mu_{i-1}, \mu_i)$ and the convergence is uniform on $[\mu_{i-1}, \mu_i - \xi]$. Let us show that $X_n(t) \to X(t)$ for any $t \in [\mu_i, \mu_{i+1})$ and the convergence is uniform on $[\mu_i, \mu_{i+1} - \xi]$.

Let $X(\mu_i-) > \psi(\mu_i)$ (the case when $X(\mu_i-) < \psi(\mu_i)$ is considered similarly). Then there are only four variants for $\varphi_i(1)$.

(A1) $\varphi_i(1) < \psi(\mu_i)$. This inequality corresponds to the case when solution of the equation (5.5) goes through the point of discontinuity $\psi(\mu_i)$ of function $f(\mu_i, \cdot)$.

(A2) $\varphi_i(1) = \psi(\mu_i)$. The equality corresponds to the case when solution of the equation (5.5) comes into the point of discontinuity $\psi(\mu_i)$ of function $f(\mu_i, \cdot)$.

(A3) $\psi(\mu_i) < \varphi_i(1) < X(\mu_i-)$. The inequality corresponds to the case when solution of the equation (5.5) does not reach the point of discontinuity $\psi(\mu_i)$ of function $f(\mu_i, \cdot)$.

(A4) $\varphi_i(1) \geq X(\mu_i-)$. In this case the solution of the equation (5.5) goes away from the point of discontinuity $\psi(\mu_i)$ of function $f(\mu_i, \cdot)$.

Case A1. The inequality $\varphi_i(1) < \psi(\mu_i)$ is not accord with condition (I) (see [7], p. 42).

Case A2. Since function L is piecewise constant, it follows that

$$X_n(t) = \sum f_n\left(t_k, X_n\left(t_k\right)\right) \Delta L_n^k$$
$$= \sum_{\mu_i \leq t} \sum_{l=0}^{p+1} f_n\left(t_{j_i+l}, X_n\left(t_{j_i+l}\right)\right) \left(L_n(t_{j_i+l+1}) - L_n(t_{j_i+l})\right),$$

where $p = W\left(\frac{1}{nh_n}\right)$, $W(y)$ is an integer part of y.

Let

$$\widetilde{V}_n := \max_{i,l} \ |L_n(t_{j_i+l+1}) - L_n(t_{j_i+l})|, \quad \nabla_n^i := \sup_{t \in [\mu_{i-1}, \mu_i - \xi]} |X_n(t) - X(t)|.$$

Note, that $\widetilde{V}_n \to 0$ as $n \to \infty$ since $h_n = o\left(\frac{1}{n}\right)$.

By using definition of f_n, Lipschitz continuity of f and its boundedness, boundedness of variation of function L and induction on l, we have that $\exists\, N : \forall\, n \geq N \quad \exists\, \overline{l}(n) : \forall\, l > \overline{l},\ l \in \{0, \ldots, p+2\}$ the following estimate holds

$$|X_n\left(t_{j_i+l}\right) - \psi(\mu_i)| \leq C\left(\widetilde{V}_n + nh_n + \nabla_n^i + h_n + \frac{1}{n}\right).$$

In particular, we have

$$|X_n(t_{j_i+p+2}) - \psi(\mu_i)| \leq C\left(\widetilde{V}_n + nh_n + \nabla_n^i + h_n + \frac{1}{n}\right).$$

Hence, for any $t \in [\mu_i, \mu_{i+1})$

$$|X_n(t) - X(t)| \leq |X_n(t) - X(\mu_i-) - (\varphi_i(1) - \varphi_i(0))|$$
$$= |X_n(t_{j_i+p+2}) - \psi(\mu)| \leq C\left(\widetilde{V}_n + nh_n + \nabla_n^i + h_n + \frac{1}{n}\right).$$

Tending $n \to \infty$, $h_n \to 0$, $h_n = o\left(\frac{1}{n}\right)$ we come to the end of the proof.

In the *cases* A3 and A4 the sets $\{(t, X_n(t)|t \in (\mu_i - h_n - \frac{1}{n}, \mu_i]\}$, $\{(t, \varphi_i(z))|z \in [0,1], t \in (\mu_i - h_n - \frac{1}{n}, \mu_i]\}$ are subsets of the same domain of continuity of function f where it satisfies Lipschitz condition. Therefore the proof boils down to the proof of the theorem (3.6) in both cases.

Suppose now that condition (III) holds in μ_i instead of (I). Let $X(\mu_i-) > \psi(\mu_i)$ (the case when $X(\mu_i-) < \psi(\mu_i)$ is considered similarly). Then there are the same four variants for $\varphi_i(1)$. Moreover, the proofs are kept in the Cases A2, A3, A4. If $\varphi_i(1) < \psi(\mu_i)$, it can be established the estimate

$$|X_n(t) - X(t)| = |X_n(t) - \varphi_i(1)| \leq C\left(nh_n + \frac{1}{n} + \widetilde{V}_n + \nabla_n^i\right), \; t \in [\mu_i, \mu_{i+1}),$$

but with another constant C than above.

If the condition (II) (condition (IV)) holds in μ_i then for functions $f_1(t,x) = -f(t,x)$ and $L_1(t) = -L(t)$ the condition (I) (condition (III)) holds. It boils down these cases to already considered ones. □

Remark 5.7. In contrast to I and II, conditions

$$(\text{I})^* \;\; f^-(\mu_i, \psi(\mu_i)) < 0, \; f^+(\mu_i, \psi(\mu_i)) > 0, \; \Delta L(\mu_i) > 0,$$
$$(\text{II})^* \;\; f^-(\mu_i, \psi(\mu_i)) > 0, \; f^+(\mu_i, \psi(\mu_i)) < 0, \; \Delta L(\mu_i) < 0$$

and condition

$$f^-(\mu_i, \psi(\mu_i)) f^+(\mu_i, \psi(\mu_i)) = 0$$

do not guarantee the uniqueness of solution of equation (5.5).

Example. Let $f(\mu_{i_0}, x) = \text{sign}(x)$, $X(\mu_{i_0}-) = 0 = \psi(\mu_{i_0})$, $\Delta L(\mu_{i_0}) = 1$. Then the set of solutions of equation (5.5) one can present in the form

$$X_\zeta^1(t) = \begin{cases} 0, & t \in [0,\zeta], \\ t - \zeta, & t \in (\zeta, 1], \end{cases} \qquad X_\zeta^2(t) = \begin{cases} 0, & t \in [0,\zeta], \\ -t + \zeta, & t \in (\zeta, 1], \end{cases} \qquad \zeta \in [0,1].$$

Theorem 5.8. *Suppose that all conditions from the previous theorem and conditions* (2.4), (2.5) *hold. Then the S-associated solution of the problem* (2.2) *is a solution of equation* (5.4).

Proof. The truth of the theorem follows from the definition of S-associated solution and theorems (2.3) and (5.5). □

Example. Consider the equation $\dot{x}(t) = (1 - 2\,\mathrm{sgn}(x - \cos t))\delta(t - \frac{\pi}{3})$, $t \in [0, \pi]$, $x(0) = \frac{1}{2}$. Then the functions $X_S(t) = \frac{1}{2}$, $X_I^+(t) = \frac{1}{2} - H(t - \frac{\pi}{3})$, and $X_I^-(t) = \frac{1}{2} + 3H(t - \frac{\pi}{3})$ are S-, I^+-, I^--associated solutions respectively. It is should be noted that the usage an inclusion in the equality (5.5) is essential. Let φ_1 be the solution of integral equation

$$\varphi_1(z) = X(\mu_1-) + \Delta L(\mu_1) \int\limits_{[0,z]} f(\mu_1, \varphi_1(s))ds,$$

which can be rewritten due to given equation in the form

$$\varphi_1(z) = \frac{1}{2} + \int\limits_{[0,z]} \left(1 - 2\,\mathrm{sgn}\left(\varphi_1(s) - \frac{1}{2}\right)\right) ds. \tag{5.6}$$

It is easy to see that equation (5.6) does not have any solutions on $[0, 1]$.

Acknowledgment

The author expresses his heartfelt gratitude to Functional Analysis Department, Belarusian State University and Mathematical Analysis Department, Belarusian State Pedagogical University, for their encouragement and friendly atmosphere.

References

[1] P. Antosik, J. Ligeza, *Products of measures and functions of finite variations.* Generalized functions and operational calculus: Proc. Conf. – Varna (1975), 20–26.

[2] P. Antosik, J. Mikusinski, R. Sikorski, *The theory of generalized functions.* Mir, 1976.

[3] N.V. Bedyuk, A.L. Yablonski, *Differential equations with generalized coefficients in the algebra of new generalized functions.* (Russian) Dokl. Naz. Akad. Nauk Belarusi **53** (2009), N. 2, 11–15.

[4] J.F. Colombeau, *A multiplication of distributions.* J. Math. Anal. and Appl. **94** (1983), N. 1, 96–115.

[5] P.C. Das, R.R. Sharma, *Existence and stability of measure differential equations.* Czech. Math. J. **22** (1972), N. 1, 145–158.

[6] Yu.V. Egorov, *A contribution to the theory of generalized functions.* Russian Math. Surveys **45** (1990), 1–49.

[7] A.F. Filippov, *Differential equations with discontinuous right-hand sides.* Mathematics and its Applications (Soviet Series), 18. Kluwer Academic Publishers Group, Dordrecht, 1988.

[8] J. Groh, *A nonlinear Volterra-Stieltjes integral equation and a Gronwall inequality in one dimension.* Illinis J. Math. **24** (1980), N. 2, 244–263.

[9] U.U. Hrusheuski, *The theorem of existence for nonautonomous differential equations with generalized coefficients.* (Russian) Vestnik BGU (2007), N. 1, 87–90.

[10] U.U. Hrusheuski, D.S. Paulic, *Sufficient condition for uniqueness of solution of nonautonomous differential equation with generalized coefficients.* Proc. of the IV International Conference "Analytic Methods in Analysis and Differential Equations" (AMADE-2006), 13–19 September, 2006, Minsk, Republic of Belarus, **3** (2006), 48–51.

[11] A.N. Kavalchuk, V.G. Navakhrost, A.L. Yablonskij, *Approximation of differential equations with generalized coefficients by finite-difference equations with averaging.* (English, Russian) Russ. Math. **49** (2005), N. 3, 21–29; translation from Izv. Vyssh. Uchebn. Zaved., Mat. (2005), N. 3, 23–31.

[12] N.V. Lazakovich, *Stochastic differentials in the algebra of generalized random processes.* (Russian) Dokl. Akad. Nauk Belarusi **38** (1994), N. 5, 23–27.

[13] N.V. Lazakovich, S.P. Stashulenok, I.V. Yufereva, *Stochastic differential equations in the algebra of generalized random processes.* (Russian) Differential Equations **31** (1995), N. 12, 2056–2058.

[14] J. Ligeza, *On generalized solutions of some differential nonlinear equations of order n.* Ann. Polon. Math. **31** (1975), N. 2, 115–120.

[15] J. Ligeza, *The existence and uniqueness of some systems of nonlinear differential equations.* Časopis Pěst. Mat. **102** (1977), N. 1, 30–36.

[16] V.G. Navakhrost, *Differential equations with generalized coefficients in the algebra of new generalized functions.* (Russian) Vesti Naz. Akad. Nauk Belarusi (2006), N. 3, 51–56.

[17] V.G. Navakhrost, *Differential equations with generalized right-hand side in the algebra of new generalized functions.* (Russian) Dokl. Akad. Nauk Belarusi **51** (2007), N. 1, 16–21.

[18] M. Oberguggenberger, *Multiplication of distributions.* Z. reine angew. Math. Bd. **265** (1986), 1–11.

[19] S.G. Pandit, S.G. Deo, *Differential systems involving impulses.* Lect. Notes in Math. 954, Springer, Berlin, 1982.

[20] E.E. Rosinger, *Generalized solutions of nonlinear partial differential equations* Amsterdam, Oxford, New York: North-Holland 1987.

[21] A. Yablonski, *Differential equation with generalized coefficients.* J. Nonlinear Analysis **63** (2005), 171–197.

[22] S.T. Zavalishchin, A.N. Sesekin, *Dynamic impulse systems. Theory and applications.* Mathematics and its Applications, 394. Kluwer Academic Publishers Group, Dordrecht, 1997.

Uladzimir U. Hrusheuski
Faculty of Mathematics
Belarusian State Pedagogical University
Sovetskaja Str., 18
220050 Minsk, Republic of Belarus
e-mail: GrushevskyVladimir@gmail.com

Operator Theory:
Advances and Applications, Vol. 216, 187–210

A Boundary Condition and Spectral Problems for the Newton Potential

T.Sh. Kalmenov and D. Suragan

Abstract. In this paper we give a boundary condition on the Newton(volume) potential for a bounded domain Ω and find its eigenvalues and eigenfunctions for the 2-disk and the 3-ball. We also extend these results in different directions.

Mathematics Subject Classification (2000). 31B20; 35P20.

Keywords. Boundary conditions, the Newton potential, eigenvalues, eigenfunctions, polyharmonic equations.

1. Introduction

The Newton potential is an operator in vector calculus that acts as the inverse to the negative Laplacian, on functions that are smooth and decay rapidly enough at infinity. As such, it is a fundamental object of study in potential theory.

In the statement of the law of gravitation given by I. Newton [1] (1687) the only forces considered are the forces of mutual attraction acting upon two material particles of small size or two material points. After first partial achievements by Newton and others, studies carried out by J.L. Lagrange (1773), A. Legendre (1784–1794) and P.S. Laplace (1782–1799) became of major importance. Lagrange [2] has established that a field of gravitational forces, as it is called now, is a potential field and has introduced a function which was later called by G. Green (1828) a potential function and by C.F. Gauss (1840) – just a potential.

Already Gauss [3] and his contemporaries discovered that the method of potentials can be applied not only to solve problems in the theory of gravitation but, in general, to solve a wide range of problems in mathematical physics, in particular in electrostatics and magnetism. The principal boundary value problems were defined, such as the Dirichlet problem and the Neumann problem, the electrostatic problem of the static distribution of charges on conductors or the Robin problem. To solve the above-mentioned problems in the case of domains with sufficiently smooth boundaries certain types of potentials turned out to be efficient, i.e., special classes of parameter-dependent integrals such as the Newton potential of distributed mass, single-layer and double-layer potentials, logarithmic potentials ($n = 2$), Green potentials, [4]–[9] etc.

In a bounded simply connected domain Ω, in the n-dimensional Euclidean space $R^n (n > 1)$, with sufficiently smooth boundary S, consider the following integral

$$u(x) = \varepsilon_n * f \equiv \int\limits_{\Omega} \varepsilon_n(x-y) f(y) dy, \tag{1.1}$$

where

$$\varepsilon_2(x-y) = -\frac{1}{2\pi} \ln |x-y|,$$
$$\varepsilon_n(x-y) = \frac{1}{(n-2)\sigma_n} |x-y|^{2-n}, \qquad n \geq 3$$

is a fundamental solution of the negative Laplace equation, i.e., $-\Delta_x \varepsilon_n(x-y) \equiv -\sum_{i=1}^{n} \frac{\partial^2 \varepsilon_n(x-y)}{\partial x_i^2} = \delta(x-y)$ and δ is the delta function, $\sigma_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$ is the surface area of the unit sphere in R^n, Γ is the gamma-function, $|x-y| = [\sum_{k=1}^{n} (x_k - y_k)^2]^{\frac{1}{2}}$ is the distance between two points $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in R^n.

The following three integrals, which depend on x as a parameter,

$$u(x) = \int\limits_{\Omega} \varepsilon_n(x-y) f(y) dy, \tag{1.2}$$

$$V(x) = \int\limits_{S} \varepsilon_n(x-y) \mu(y) dS_y, \tag{1.3}$$

$$W(x) = \int\limits_{S} \frac{\partial \varepsilon_n(x-y)}{\partial n_y} \rho(y) dS_y, \tag{1.4}$$

are called the volume potential, the single-layer potential and the double-layer potential, respectively. The functions $f(y)$, $\mu(y)$ and $\rho(y)$ are called the densities of the corresponding potentials; hereafter they are assumed to be absolutely integrable over Ω or S, respectively. For $n = 3$ (and sometimes for $n \geq 3$) the integrals (1.1), (1.2) and (1.3) are called the Newton potential and the Newton single- and double-layer potentials; for $n = 2$ they are called logarithmic mass, single-layer or double-layer potentials, respectively.

First, we discuss some properties of potentials (1.2)–(1.4) as we need. Let $f(y)$ be of class $C^2(\Omega) \bigcap C^1(S)$. Then the Newton potential and its first derivatives are continuous everywhere on R^n; moreover, they can be calculated by differentiation under the integral sign, i.e., $u \in C^1(R^n)$. Further,

$$\lim_{|x| \to \infty} \frac{u(x)}{\varepsilon_n(x)} = M, \qquad M = \int_{\Omega} f(y) dy. \tag{1.5}$$

The second derivatives are continuous everywhere outside S, but they have a discontinuity when passing across the surface S; moreover, in Ω they satisfy the Poisson equation

$$-\Delta u(x) = f(x), x \in \Omega, \tag{1.6}$$

and in $R^n \backslash \Omega$ – the Laplace equation $\Delta u = 0$, $x \in R^n \backslash \Omega$. The above-mentioned properties characterize the Newton potential (1.2).

Let $\mu \in C^1(S)$. The single-layer potential $V(x)$ is a harmonic function when $x \in \Omega$; moreover,

$$\lim_{|x|\to\infty} \frac{V(x)}{\varepsilon_n(x)} = M, \qquad M = \int_\Omega \mu(y) dS_y, \tag{1.7}$$

in particular, $\lim_{|x|\to\infty} V(x) = 0$ for $n \geq 3$, but $\lim_{|x|\to\infty} V(x) = 0$ when $n = 2$ if and only if $\int_S \mu(y) dS_y = 0$. A single-layer potential is continuous everywhere on R^n, $V \in C(R^n)$, moreover, $V(x)$ and its tangential derivatives are continuous when passing across the surface S. The normal derivative of a single-layer potential has a discontinuity when passing across the surface S:

$$\left(\frac{\partial V}{\partial n_x}\right)^+ = \frac{1}{2}\mu(x) + \frac{\partial V(x)}{\partial n_x}, \qquad x \in S, \tag{1.8}$$

$$\left(\frac{\partial V}{\partial n_x}\right)^- = -\frac{1}{2}\mu(x) + \frac{\partial V(x)}{\partial n_x}, \qquad x \in S, \tag{1.9}$$

where $(\frac{\partial V}{\partial n_x})^+$ and $(\frac{\partial V}{\partial n_x})^-$ are the limit values of the normal derivative from Ω and $R^n \backslash \Omega$, respectively, i.e.,

$$\left(\frac{\partial V}{\partial n_x}\right)^+ = \lim_{x' \to x, x' \in \Omega} \frac{\partial V(x')}{\partial n_x}, \qquad x \in S, \tag{1.10}$$

$$\left(\frac{\partial V}{\partial n_x}\right)^- = \lim_{x' \to x, x' \in R^n \backslash \Omega} \frac{\partial V(x')}{\partial n_x}, \qquad x \in S, \tag{1.11}$$

$\frac{\partial V(x)}{\partial n_x}$ denotes the so-called direct value of the normal derivative of a single-layer potential calculated over the surface S, i.e.,

$$\frac{\partial V(x)}{\partial n_x} = \int_S \frac{\partial \varepsilon_n(x-y)}{\partial n_x} \mu(y) dS_y, \qquad x \in S. \tag{1.12}$$

It is a continuous function of the points $x \in S$, and the kernel $\frac{\partial}{\partial n_x}\varepsilon_n(x-y)$ has a weak singularity on S,

$$\left| \frac{\partial}{\partial n_x}\varepsilon_n(x-y) \right| \leq \frac{\text{const}}{|x-y|^{n-2}}, \qquad x, y \in S. \tag{1.13}$$

These properties characterize single-layer potential (1.3).

Let $\rho \in C^1(S)$. The double-layer potential $W(x)$ is a harmonic function for x; moreover,

$$\lim_{|x|\to\infty} \sigma_n |x|^{n-1} W(x) = M, \qquad M = \int_S \rho(y) dS_y. \tag{1.14}$$

When passing across the surface S the double-layer potential has a discontinuity (whence its name):

$$W^+(x) = -\frac{1}{2}\rho + W(x), \quad W^-(x) = \frac{1}{2}\rho + W(x), \qquad x \in S, \tag{1.15}$$

where $W^+(x)$ and $W^-(x)$ are the limit values of the double-layer potential from Ω and $R^n\backslash\Omega$, respectively, that is,

$$W^+(x) = \lim_{x'\to x, x\in\Omega} W(x'), W^-(x) = \lim_{x'\to x, x\in R^n\backslash\Omega} W(x'). \tag{1.16}$$

$W(x)$ when $x \in S$ denotes the so-called direct value of the double-layer potential calculated over the surface S, that is,

$$W(x) = \int_S \frac{\partial \varepsilon_n(x-y)}{\partial n_y} \rho(y) dS_y, \qquad x \in S. \tag{1.17}$$

It is a continuous function of the points $x \in S$, and the kernel $\frac{\partial}{\partial n_y}\varepsilon_n(x-y)$ has a weak singularity on S,

$$\left| \frac{\partial}{\partial n_y} \varepsilon_n(x-y) \right| \le \frac{\text{const}}{|x-y|^{n-2}}, \qquad x, y \in S. \tag{1.18}$$

The tangential derivatives of a double-layer potential also have a discontinuity when passing across the surface S, but the normal derivative $\frac{\partial W(x)}{\partial n_x}$ retains its value when passing across S:

$$\left(\frac{\partial W(x)}{\partial n_x} \right)^+ = \left(\frac{\partial W(x)}{\partial n_x} \right)^-, \qquad x \in S. \tag{1.19}$$

These properties characterize double-layer potential (1.4).

Below, certain properties of potentials under weaker restrictions on the densities and the surface are given.

- If $f \in L_1(\Omega)$, then $u(x)$ is a harmonic function for $x \in R^n\backslash\Omega$ and $u(x)$ is summable on Ω.
- If $f \in L_p, 1 \le p \le \frac{n}{2}$, then $u \in L_q(R^n), \frac{1}{p} + \frac{1}{p} = 1, 1 < q < \frac{np}{(n-2p)}$; if $f \in L_p(\Omega), p > \frac{n}{2}$, then $u \in C(R^n)$.
- If $f \in L_p(\Omega), 1 \le p \le n$, then $u \in W_q^1(R^n), 1 < q < \frac{np}{n-p}$; if $f \in L_p(\Omega), p > n$, then $u \in C^1(R^n)$.
- If $f \in L_2(\Omega)$, then the generalized second derivatives of $u(x)$ exist, they are also of class $L_2(\Omega)$ and are expressed by singular integrals:

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = -\frac{1}{n}\delta_{ij} f(x) + \int_\Omega \frac{\partial^2}{\partial x_i \partial x_j} \varepsilon_n(x-y) f(y) dy, \qquad i, j = 1, \dots, n, \tag{1.20}$$

where $\delta_{ij} = 1$ for $i = j$, $\delta_{ij} = 0$ for $i \ne j$; if $f \in L_p(\Omega), 1 \le p < +\infty$, then all generalized derivatives $\frac{\partial^2 u}{\partial x_i \partial x_j}$ also exist and belong to $L_p(R^n)$.

- If $f \in L_p(\Omega), 1 \le p < +\infty$, then $u(x)$ is a generalized solution of the Poisson equation $-\Delta u = f(x), x \in \Omega$. If $f \in C^{(0,\alpha)}(\Omega)$ and $S \in C^{(1,\alpha)}, 0 < \alpha < 1$, then $u \in C^{(2,\alpha)}$ in Ω.
- Let $S \in C^{1,\alpha}$, $0 < \alpha < 1$, let $\overline{D}$ be a closed bounded domain such that $\Omega \cup S \subset D \subset \overline{D} \subset R^n$. If $\mu \in L_p$ then $V \in L_p(\overline{D})$, $V \in L_p(S)$, $\frac{\partial V}{\partial x_i} \in L_p(\overline{D}), p = 1, 2$, $i = 1, \ldots, n$. If the density is bounded and summable, then $V \in C^{(0,\lambda)}$ for all $\lambda \in (0, 1)$.
- If $\mu \in C^{(0,\alpha)}(S), 0 < \alpha < 1$, then $V \in C^{1,\alpha}$ in Ω. If $\rho \in C^{(0,\alpha)}(S)$, then $W \in C^{(0,\alpha)}$ in Ω.
- If $\mu \in C^{(l,\alpha)}(S)$ and $S \in C^{(k+1,\alpha)}$, $0 < \alpha < 1$, l, k integers, $0 \le l \le k$, then $V \in C^{l+1,\alpha}$ in Ω. If $\rho \in C^{(l,\alpha)}(S)$, and $S \in C^{(k+1,\alpha)}$, $0 < \alpha < 1$, l, k integers, $0 \le l \le k+1$, then $W \in C^{(l,\alpha)}$ in Ω.

For potentials and their derivatives extended by continuity on S the above-described properties of smoothness are also valid under the corresponding smoothness conditions on the density and the surface S ([4]–[9] etc.).

In this paper, compared to all earlier works discussed above, we follow an entirely different approach in potential theory and we extend several results, concerning the Newton potential.

The paper is organized as follows. In Section 2 we give a boundary condition on the Newton potential. In Section 3 a boundary condition on polyharmonic volume potential is given. In Section 4 we find eigenvalues and eigenfunctions of the Newton potential (1.1) in an explicit form for a ball. In Section 5 some its applications are shown.

2. A boundary condition on the Newton (volume) potential

Theorem 2.1. *For any function* $f \in L_2(\Omega)$*, the Newton potential* (1.1) *satisfies the boundary condition*

$$-\frac{u(x)}{2} - \int_S \varepsilon_n(x-y)\frac{\partial u(y)}{\partial n_y} dS_y + \int_S \frac{\partial \varepsilon_n(x-y)}{\partial n_y} u(y) dS_y = 0, \qquad x \in S. \quad (2.1)$$

Conversely, if a function $u \in W_2^2(\Omega)$ *satisfies* (1.6) *and boundary condition* (2.1)*, then it determines the Newton potential* (1.1)*, where* $\frac{\partial}{\partial n_y}$ *denotes the outer normal derivative on the boundary.*

Proof of Theorem 2.1. First, we assume that $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. A direct calculation shows that, for any $x \in \Omega$, we have

$$u(x) = \varepsilon_n * f = -\int_\Omega \varepsilon_n(x-y)\Delta_y u(y) dy$$
$$= \int_S \left(-\frac{\partial u(y)}{\partial n_y}\varepsilon_n(x-y) + \frac{\partial \varepsilon_n(x-y)}{\partial n_y} u(y) \right) dS_y - \int_\Omega \Delta_y \varepsilon_n(x-y) u(y) dy$$

$$= u(x) + \int_S \left(\frac{\partial \varepsilon_n(x-y)}{\partial n_y} u(y) - \frac{\partial u(y)}{\partial n_y} \varepsilon_n(x-y) \right) dS_y,$$

where $\frac{\partial}{\partial n_y} = n_1 \frac{\partial}{\partial y_1} + \cdots + n_n frac\partial\partial y_n$ is the normal derivative and $n_1, \ldots, n_n$ are the components of the unit normal. This implies

$$I_u(x) = \int_S \left(\frac{\partial \varepsilon_n(x-y)}{\partial n_y} u(y) - \frac{\partial u(y)}{\partial n_y} \varepsilon_n(x-y) \right) dS_y \equiv 0, \qquad x \in \Omega. \qquad (2.2)$$

Since $\Delta_x \varepsilon_n(x-y) = 0$ and $\Delta_x \frac{\partial \varepsilon_n(x-y)}{\partial n_y} = 0$ for $x \neq y$, it follows that $\Delta_x I_u(x) \equiv 0$.

Applying properties of the double-layer potential and single-layer potential to (2.2) with $x \to S$, we obtain

$$I_u(x) = -\frac{u(x)}{2} + \int_S \left(\frac{\partial \varepsilon_n(x-y)}{\partial n_y} u(y) - \frac{\partial u(y)}{\partial n_y} \varepsilon_n(x-y) \right) dS_y = 0,\ x \in S. \quad (2.3)$$

Since $I_u(x)$ is a solution of the homogeneous Laplace equation for $x \in \Omega$, it follows from the uniqueness of a solution to the Dirichlet problem that the identity $I_u(x) = 0, x \in \Omega$ is equivalent to (2.3), i.e., $I_u(x)|_{x \in S} = 0$ is a boundary condition for the Newton potential (1.1). Next, it is easy to show by passing to the limit that relation (2.3) remains valid for all $u \in W_2^2(\Omega)$. Thus, the Newton potential (1.1) satisfies boundary condition (2.3).

Conversely, if a function $u_1 \in W_2^2(\Omega)$ satisfies the equation $-\Delta u_1 = f$ and boundary condition (2.3), then it coincides with the Newton potential (1.1).

Indeed, if this is not so, then the function $v = u - u_1 \in W_2^2(\Omega)$, where $u = \varepsilon_n * f(x)$ is the Newton potential, satisfies the homogeneous equation $\Delta v = 0$ and the homogeneous condition

$$I_v(x) = -\frac{v(x)}{2} + \int_S \left(\frac{\partial \varepsilon_n(x-y)}{\partial n_y} v(y) - \frac{\partial v(y)}{\partial n_y} \varepsilon_n(x-y) \right) dS_y = 0,\ x \in S. \quad (2.4)$$

As above, applying the Green formula to $v \in W_2^2(\Omega)$, we see that

$$\int_\Omega \varepsilon_n(x-y) \Delta_y v(y) dy = v(x) + \int_S \left(\frac{\partial \varepsilon_n(x-y)}{\partial n_y} v(y) - \frac{\partial v(y)}{\partial n_y} \varepsilon_n(x-y) \right) dS_y$$
$$= v(x) + I_v(x) \equiv 0, \ \ \forall x \in \Omega.$$

Passing to the limit as $x \to S$, we obtain

$$v(x) - \frac{v(x)}{2} + \int_S \left(\frac{\partial \varepsilon_n(x-y)}{\partial n_y} v(y) - \frac{\partial v(y)}{\partial n_y} \varepsilon_n(x-y) \right) dS_y$$
$$= v(x)|_{x \in S} + I_v(x)|_{x \in S} = 0. \qquad (2.5)$$

Condition (2.3) implies $I_v(x)|_{x \in S} = 0$; therefore, it follows from (2.5) that $v(x) \equiv 0$ for any $x \in S$. By virtue of the uniqueness of a solution to the Dirichlet problem for the Laplace equation, we have $v(x) = u(x) - u_1(x) \equiv 0$ for any $x \in \Omega$, i.e., $u_1 \equiv u$, u_1 coincides with the Newton potential. This completes the proof of Theorem 2.1. □

Remark 2.2. It follows from Theorem 2.1 that the kernel of the Newton potential (1.1), i.e., fundamental solution of the Laplace equation $\varepsilon_n(x-y)$ is the Green function for boundary value problem (1.6), (2.1) in Ω.

Example. (Theorem 2.1 for ODE) Consider the one-dimensional Newton potential $(n = 1)$

$$u(x) = \frac{1}{2}\int_0^1 |x-t|f(t)dt$$

in $\Omega = (0,1)$. This function satisfies the one-dimensional Poisson equation $u''(x) = f(x)$. Integrating by part, we obtain

$$\begin{aligned} u(x) &= \frac{1}{2}[-\int_0^x (x-t)u''(t)dt - \int_x^1 (t-x)u''(t)dt] \\ &= u(x) - x\frac{u'(0)+u'(1)}{2} - \frac{-u'(1)+u(0)+u(1)}{2}. \end{aligned}$$

Therefore, self-adjoint boundary conditions for the one-dimensional Newton potential are $u'(0)+u'(1) = 0$, $-u'(1)+u(0)+u(1) = 0$; hence if we solve the equation $u''(x) = f(x)$ with these boundary conditions in $\Omega = (0,1)$, then we find unique solutions of this problem in the form (the Newton potential)

$$u(x) = \frac{1}{2}\int_0^1 |x-t|f(t)dt.$$

3. Boundary conditions on the polyharmonic volume potential

On a bounded simply connected domain Ω, in the n-dimensional Euclidean space $R^n (n > 1)$, with sufficiently smooth boundary S, consider the polyharmonic volume potential

$$u(x) = \varepsilon_{m,n} * f \equiv \int\limits_\Omega \varepsilon_{m,n}(x-y)f(y)dy, \tag{3.1}$$

where

$$\begin{aligned} &\varepsilon_{m,n} = d_{m,n}|x-y|^{2m-n} \qquad \text{for } n\text{-odd and } 2m < n,\ n\text{-even}, \\ &\varepsilon_{m,n} = d_{m,n}|x-y|^{2m-n}\ln|x-y| \qquad \text{for } 2m \geq n,\ n\text{-even}, \end{aligned}$$

is a fundamental solution of the polyharmonic equation, i.e.,

$$(-\Delta_x)^m \varepsilon_{m,n}(x-y) = \delta(x-y), \qquad m = 1,2,\ldots, \tag{3.2}$$

here

$$d_{m,n} = \frac{1}{(m-1)!2^{m-1}(2m-n)(2(m-1)-n)\ldots(2-n)} \cdot \frac{\Gamma(\frac{n}{2})}{2^m \pi^{\frac{n}{2}}}.$$

It is easy to show that the polyharmonic potential (3.1) satisfies the inhomogeneous polyharmonic equation

$$(-\Delta_x)^m u(x) = f(x). \tag{3.3}$$

The following theorem is valid.

Theorem 3.1. *For any function $f \in L_2(\Omega)$, the polyharmonic volume potential* (3.1) *satisfies boundary conditions*

$$\begin{aligned}
&-\frac{1}{2}(-\Delta_x)^i u(x)\\
&+\sum_{j=0}^{m-i-1}\int_S \frac{\partial}{\partial n_y}(-\Delta_y)^{m-i-1-j}\varepsilon_{m-i,n}(x-y)(-\Delta_y)^j(-\Delta_y)^i u(y)dS_y\\
&-\sum_{j=0}^{m-i-1}\int_S (-\Delta_y)^{m-i-1-j}\varepsilon_{m-i,n}(x-y)\frac{\partial}{\partial n_y}(-\Delta_y)^j(-\Delta_y)^i u(y)dS_y = 0
\end{aligned} \tag{3.4}$$

for $i = \overline{0, m-1}$, $x \in S$. Conversely, if a function $u \in W_2^{2m}(\Omega)$ satisfies (3.3) *and boundary condition* (3.4)*, then it determines the polyharmonic volume potential* (3.1)*, where $\frac{\partial}{\partial n_y}$ denotes the outer normal derivative on the boundary.*

Proof of Theorem 3.1. First, we assume that $u \in C^{2m}(\Omega) \cap C^{2m-1}(\overline{\Omega})$. A direct calculation shows that, for any $x \in \Omega$, we have

$$\begin{aligned}
u(x) &= \varepsilon_{m,n} * f\\
&= \int_\Omega \varepsilon_{m,n}(x-y)(-\Delta_y)^m u(y)dy\\
&= \int_\Omega (-\Delta_y)\varepsilon_{m,n}(x-y)(-\Delta_y)^{m-1}u(y)dy\\
&\quad + \int_S \frac{\partial \varepsilon_{m,n}(x-y)}{\partial n_y}(-\Delta_y)^{m-1}u(y))dS_y\\
&\quad - \int_S \varepsilon_{m,n}(x-y)\frac{\partial}{\partial n_y}(-\Delta_y)^{m-1}u(y)dS_y\\
&= \int_\Omega (-\Delta_y)^2\varepsilon_{m,n}(x-y)(-\Delta)^{m-2}u(y)dy\\
&\quad + \int_S \frac{\partial(-\Delta)\varepsilon_{m,n}(x-y)}{\partial n_y}(-\Delta_y)^{m-2}u(y))dS_y\\
&\quad - \int_S (-\Delta_y)\varepsilon_{m,n}(x-y)\frac{\partial}{\partial n_y}(-\Delta_y)^{m-2}u(y)dS_y
\end{aligned}$$

$$
\begin{aligned}
&\quad + \int_S \frac{\partial \varepsilon_{m,n}(x-y)}{\partial n_y}(-\Delta_y)^{m-1} u(y)) dS_y \\
&\quad - \int_S \varepsilon_{m,n}(x-y) \frac{\partial}{\partial n_y}(-\Delta_y)^{m-1} u(y) dS_y \\
&= \int_\Omega (-\Delta_y)^3 \varepsilon_{m,n}(x-y)(-\Delta)^{m-3} u(y) dy \\
&\quad + \int_S \frac{\partial (-\Delta_y)^2 \varepsilon_{m,n}(x-y)}{\partial n_y}(-\Delta_y)^{m-3} u(y)) dS_y \\
&\quad - \int_S (-\Delta_y)^2 \varepsilon_{m,n}(x-y) \frac{\partial}{\partial n_y}(-\Delta_y)^{m-3} u(y) dS_y \\
&\quad + \int_S \frac{\partial (-\Delta_y) \varepsilon_{m,n}(x-y)}{\partial n_y}(-\Delta_y)^{m-2} u(y)) dS_y \\
&\quad - \int_S (-\Delta_y) \varepsilon_{m,n}(x-y) \frac{\partial}{\partial n_y}(-\Delta_y)^{m-2} u(y) dS_y \\
&\quad + \int_S \frac{\partial \varepsilon_{m,n}(x-y)}{\partial n_y}(-\Delta_y)^{m-1} u(y)) dS_y \\
&\quad - \int_S \varepsilon_{m,n}(x-y) \frac{\partial}{\partial n_y}(-\Delta_y)^{m-1} u(y) dS_y \\
&= u(x) + \sum_{j=0}^{m-1} \int_S \frac{\partial (-\Delta_y)^{m-1-j} \varepsilon_{m,n}(x-y)}{\partial n_y}(-\Delta_y)^j u(y) dS_y \\
&\quad - \sum_{j=0}^{m-1} \int_S (-\Delta_y)^{m-1-j} \varepsilon_{m,n}(x-y) \frac{\partial}{\partial n_y}(-\Delta_y)^j u(y) dS_y,
\end{aligned}
$$

where

$$\frac{\partial}{\partial n_y} = n_1 \frac{\partial}{\partial y_1} + \cdots + n_n \frac{\partial}{\partial y_n}$$

is the normal derivative and $n_1, \dots, n_n$ are the components of the unit normal. This implies

$$
\begin{aligned}
I_0(u(x)) = &\sum_{j=0}^{m-1} \int_S \frac{\partial (-\Delta_y)^{m-1-j} \varepsilon_{m,n}(x-y)}{\partial n_y}(-\Delta_y)^j u(y) dS_y \\
&- \sum_{j=0}^{m-1} \int_S (-\Delta_y)^{m-1-j} \varepsilon_{m,n}(x-y) \frac{\partial}{\partial n_y}(-\Delta_y)^j u(y) dS_y \equiv 0, \quad x \in \Omega.
\end{aligned}
\tag{3.5}
$$

Hereafter

$$\sum_{j=0}^{m-1}\int_S \frac{\partial(-\Delta_y)^{m-1-j}\varepsilon_{m,n}(x-y)}{\partial n_y}(-\Delta_y)^j u(y)dS_y \\ -\sum_{j=0}^{m-1}\int_S (-\Delta_y)^{m-1-j}\varepsilon_{m,n}(x-y)\frac{\partial}{\partial n_y}(-\Delta_y)^j u(y)dS_y$$

is denoted by $I_0(u(x))$. It is easy to see

$$(-\Delta_x)^m I_0(u(x)) = 0, \qquad x \in \Omega. \tag{3.6}$$

Applying properties of the double-layer potential and the single-layer potential to (3.5) with $x \to S$, we obtain that

$$I_0(u(x)) = -\frac{u(x)}{2} + \sum_{j=0}^{m-1}\int_S \frac{\partial(-\Delta_y)^{m-1-j}\varepsilon_{m,n}(x-y)}{\partial n_y}(-\Delta_y)^j u(y)dS_y \\ -\sum_{j=0}^{m-1}\int_S (-\Delta_y)^{m-1-j}\varepsilon_{m,n}(x-y)\frac{\partial}{\partial n_y}(-\Delta_y)^j u(y)dS_y \equiv 0, \\ x \in S, \tag{3.7}$$

is a boundary condition for the polyharmonic potential (3.1).

Now we find other boundary conditions for the polyharmonic potential (3.1), consider

$$(-\Delta_x)^{m-i}(-\Delta_x)^i u(x) = f(x), i = \overline{0, m-1}. \tag{3.8}$$

As above, a direct calculation shows that, for any $x \in \Omega$, we have

$$\begin{aligned}(-\Delta_x)^i u(x) &= \varepsilon_{m-i,n} * f \\ &= \int_\Omega \varepsilon_{m-i,n}(x-y)(-\Delta_y)^{m-i}(-\Delta_y)^i u(y)dy \\ &= \int_\Omega (-\Delta_y)\varepsilon_{m-i,n}(x-y)(-\Delta_y)^{m-i-1}(-\Delta_y)^i u(y)dy \\ &\quad + \int_S \frac{\partial\varepsilon_{m-i,n}(x-y)}{\partial n_y}(-\Delta_y)^{m-i-1}(-\Delta_y)^i u(y)dS_y \\ &\quad - \int_S \varepsilon_{m-i,n}(x-y)\frac{\partial}{\partial n_y}(-\Delta_y)^{m-i-1}(-\Delta_y)^i u(y)dS_y \\ &= \int_\Omega (-\Delta_y)^2\varepsilon_{m-i,n}(x-y)(-\Delta_y)^{m-i-2}(-\Delta_y)^i u(y)dy\end{aligned}$$

$$+\int_S \frac{\partial(-\Delta_y)\varepsilon_{m-i,n}(x-y)}{\partial n_y}(-\Delta_y)^{m-i-2}(-\Delta_y)^i u(y)dS_y$$
$$-\int_S (-\Delta_y)\varepsilon_{m-i,n}(x-y)\frac{\partial}{\partial n_y}(-\Delta_y)^{m-i-2}(-\Delta_y)^i u(y)dS_y$$
$$+\int_S \frac{\partial\varepsilon_{m-i,n}(x-y)}{\partial n_y}(-\Delta_y)^{m-i-1}(-\Delta_y)^i u(y)dS_y$$
$$-\int_S \varepsilon_{m-i,n}(x-y)\frac{\partial}{\partial n_y}(-\Delta_y)^{m-i-1}(-\Delta_y)^i u(y)dS_y$$

$$=\int_\Omega (-\Delta_y)^3\varepsilon_{m-i,n}(x-y)(-\Delta_y)^{m-i-3}(-\Delta_y)^i u(y)dy$$
$$+\int_S \frac{\partial(-\Delta_y)^2\varepsilon_{m-i,n}(x-y)}{\partial n_y}(-\Delta_y)^{m-i-3}(-\Delta_y)^i u(y)dS_y$$
$$-\int_S (-\Delta_y)^2\varepsilon_{m-i,n}(x-y)\frac{\partial}{\partial n_y}(-\Delta_y)^{m-i-3}(-\Delta_y)^i u(y)dS_y$$
$$+\int_S \frac{\partial(-\Delta_y)\varepsilon_{m-i,n}(x-y)}{\partial n_y}(-\Delta_y)^{m-i-2}(-\Delta_y)^i u(y)dS_y$$
$$-\int_S (-\Delta_y)\varepsilon_{m-i,n}(x-y)\frac{\partial}{\partial n_y}(-\Delta_y)^{m-i-2}(-\Delta_y)^i u(y)dS_y$$
$$+\int_S \frac{\partial\varepsilon_{m-i,n}(x-y)}{\partial n_y}(-\Delta_y)^{m-i-1}(-\Delta_y)^i u(y)dS_y$$
$$-\int_S \varepsilon_{m-i,n}(x-y)\frac{\partial}{\partial n_y}(-\Delta_y)^{m-i-1}(-\Delta_y)^i u(y)dS_y$$

$$=(-\Delta_x)^i u(x)$$
$$+\sum_{j=0}^{m-i-1}\int_S \frac{\partial(-\Delta)^{m-i-1-j}\varepsilon_{m-i,n}(x-y)}{\partial n_y}(-\Delta_y)^j(-\Delta_y)^i u(y)dS_y$$
$$-\sum_{j=0}^{m-i-1}\int_S (-\Delta_y)^{m-i-1-j}\varepsilon_{m-i,n}(x-y)\frac{\partial}{\partial n_y}(-\Delta_y)^j(-\Delta_y)^i u(y)dS_y,$$

where $\varepsilon_{m-i,n}(x-y)$ is a fundamental solution of $(-\Delta_x)^{m-i}$, i.e.,

$$(-\Delta_x)^{m-i}\varepsilon_{m-i,n}(x-y)=\delta(x-y), \qquad i=\overline{0,m-1}.$$

This implies

$$I_i(u(x)) = \sum_{j=0}^{m-i-1} \int_S \frac{\partial(-\Delta_y)^{m-i-1-j}\varepsilon_{m-i,n}(x-y)}{\partial n_y}(-\Delta_y)^j(-\Delta_y)^i u(y)dS_y$$
$$- \sum_{j=0}^{m-i-1} \int_S (-\Delta_y)^{m-i-1-j}\varepsilon_{m-i,n}(x-y)\frac{\partial}{\partial n_y}(-\Delta_y)^j(-\Delta_y)^i u(y)dS_y \equiv 0 \quad (3.9)$$

for all $x \in \Omega$. Hereafter

$$\sum_{j=0}^{m-i-1} \int_S \frac{\partial(-\Delta_y)^{m-i-1-j}\varepsilon_{m-i,n}(x-y)}{\partial n_y}(-\Delta_y)^j(-\Delta_y)^i u(y)dS_y$$
$$- \sum_{j=0}^{m-i-1} \int_S (-\Delta_y)^{m-i-1-j}\varepsilon_{m-i,n}(x-y)\frac{\partial}{\partial n_y}(-\Delta_y)^j(-\Delta_y)^i u(y)dS_y$$

is denoted by $I_i(u(x))$. It is easy to see

$$(-\Delta_x)^{m-i} I_i(u(x)) = 0, x \in \Omega. \quad (3.10)$$

Applying properties of the double layer potential to (3.9) with $x \to S$, we obtain

$$I_i(u(x)) = -\frac{1}{2}(-\Delta_x)^i u(x)$$
$$+ \sum_{j=0}^{m-i-1} \int_S \frac{\partial(-\Delta_y)^{m-i-1-j}\varepsilon_{m-i,n}(x-y)}{\partial n_y}(-\Delta_y)^j(-\Delta_y)^i u(y)dS_y$$
$$- \sum_{j=0}^{m-i-1} \int_S (-\Delta_y)^{m-i-1-j}\varepsilon_{m-i,n}(x-y)\frac{\partial}{\partial n_y}(-\Delta_y)^j(-\Delta_y)^i u(y)dS_y$$
$$\equiv 0, \qquad x \in S,\ i = \overline{0, m-1}. \quad (3.11)$$

Next, it is easy to show by passing to the limit that relation (3.11) remains valid for all $u \in W_2^{2m}(\Omega)$. Thus, the polyharmonic potential potential (3.1) satisfies boundary condition (3.4).

Conversely, if a function $u_1 \in W_2^{2m}(\Omega)$ satisfies the equation $-\Delta u_1(x) = f(x)$ and boundary condition (3.4), then it coincides with the polyharmonic potential (3.1). Indeed, if this is not so, then the function $v = u - u_1 \in W_2^{2m}(\Omega)$, where $u = \varepsilon_{m,n} * f$ is the Newton potential, satisfies the homogeneous equation

$$(-\Delta_x)^m v = 0, \quad (3.12)$$

and homogeneous boundary conditions

$$\begin{aligned} I_i(v(x)) = &-\frac{1}{2}(-\Delta_x)^i v(x) \\ &+ \sum_{j=0}^{m-i-1} \int_S \frac{\partial(-\Delta_y)^{m-i-1-j}\varepsilon_{m,n}(x-y)}{\partial n_y}(-\Delta_y)^j(-\Delta_y)^i v(y)dS_y \\ &- \sum_{j=0}^{m-i-1} \int_S (-\Delta_y)^{m-i-1-j}\varepsilon_{m,n}(x-y)\frac{\partial}{\partial n_y}(-\Delta_y)^j(-\Delta_y)^i v(y)dS_y \\ \equiv 0, &\qquad x \in S, i = \overline{0, m-1}. \end{aligned} \tag{3.13}$$

As above, applying the Green formula to $v \in W_2^{2m}(\Omega)$, we see that

$$\begin{aligned} 0 &\equiv \int_\Omega \varepsilon_{m-i,n}(x-y)(-\Delta_y)^{m-i}(-\Delta_y)^i v(y)dy \\ &= (-\Delta_x)^i v(x) + \sum_{j=0}^{m-i-1} \int_S \frac{\partial(-\Delta_y)^{m-i-1-j}\varepsilon_{m,n}(x-y)}{\partial n_y}(-\Delta_y)^j(-\Delta_y)^i v(y)dS_y \\ &\quad - \sum_{j=0}^{m-i-1} \int_S (-\Delta_y)^{m-i-1-j}\varepsilon_{m,n}(x-y)\frac{\partial}{\partial n_y}(-\Delta_y)^j(-\Delta_y)^i v(y)dS_y \\ &= (-\Delta_x)^i v(x) + I_i(v(x)), \qquad x \in \Omega, i = \overline{0, m-1}, \end{aligned} \tag{3.14}$$

and

$$(-\Delta_x)^i v(x) = -I_i(v(x)), \qquad i = \overline{0, m-1}, x \in S. \tag{3.15}$$

By virtue of the uniqueness of a solution to the problem

$$(-\Delta_x)^m v = 0, \tag{3.16}$$

with boundary conditions

$$(-\Delta_x)^i v(x)|_{x\in S} = 0, i = \overline{0, m-1}, \tag{3.17}$$

we have $v(x) = u(x) - u_1(x) \equiv 0$ for any $x \in \Omega$, i.e., $u_1 \equiv u$, u_1 coincides with the polyharmonic potential. □

Remark 3.2. It follows from Theorem 3.1 that the kernel of the polyharmonic volume potential (3.1), i.e., fundamental solution of polyharmonic equation $\varepsilon_{m,n}(x-y)$, is the Green function for the boundary value problem (3.3)–(3.4).

Example. (Theorem 3.1 for ODE) Consider the one-dimensional biharmonic potential $(n = 1, m = 2)$

$$u(x) = \frac{1}{12}\int_{-1}^{1} |x-t|^3 f(t)dt,$$

in $\Omega = (-1,1)$. This function satisfies the one-dimensional biharmonic equation $u^{iv}(x) = f(x)$. As above (see Example 1.1), integrating by part implies

$$\begin{aligned} u(x) &= \frac{1}{12}[-\int_{-1}^{x}(x-t)^3 u^{IV}(t)dt - \int_{x}^{1}(t-x)^3 u^{IV}(t)dt] \\ &= u(x) + \frac{1}{12}[(x+1)^3 u'''(-1) + 3(x+1)^2 u''(-1) + 6(x+1)u'(-1) + 6u(-1) \\ &\qquad + (x-1)^3 u'''(1) + 3(x-1)^2 u''(1) + 6(x-1)u'(1) + 6u(1)] \\ &= 0, \qquad x \in \Omega = (-1,1). \end{aligned}$$

It follows

$$\begin{aligned} &u'''(-1) + u'''(1) = 0, \\ &u'''(-1) - u'''(1) + u''(-1) + u''(1) = 0, \\ &u'''(-1) + u'''(1) + 2(u''(-1) - u''(1)) + 2(u'(-1) + u'(1)) = 0, \\ &u'''(-1) - u'''(1) + 3(u''(-1) + u''(1)) + 6(u'(-1) - u'(1)) + 6(u(-1) + u(1)) = 0. \end{aligned}$$

It is equivalent to

$$\begin{aligned} &u'''(-1) + u'''(1) = 0, \\ &u'''(-1) - u'''(1) + u''(-1) + u''(1) = 0, \\ &u''(-1) - u''(1) + u'(-1) + u'(1) = 0, \\ &u''(-1) + u''(1) + 3(u'(-1) - u'(1)) + 3(u(-1) + u(1)) = 0. \end{aligned}$$

Therefore, the one-dimensional biharmonic volume potential satisfies these boundary conditions, i.e., if we solve the equation $u^{iv}(x) = f(x)$ with these boundary conditions in $\Omega = (-1,1)$, then we find a unique solution of this problem in the form (the one-dimensional biharmonic potential)

$$u(x) = \frac{1}{12}\int_{-1}^{1} |x-t|^3 f(t)dt.$$

4. Spectral problems for the Newton potential

First, for example, consider the one-dimensional spectral problem for the Newton potential in $\Omega = (0,1)$

$$u(x) = -\lambda \int_0^1 \frac{1}{2}|x-y|u(y)dy, \tag{4.1}$$

According to the example in Section 2, it is equivalent to

$$u''(x) = -\lambda u(x), \tag{4.2}$$

$$u'(0) + u'(1) = 0, \tag{4.3}$$

$$-u'(1) + u(0) + u(1) = 0. \tag{4.4}$$

We search solution of problem (4.2)–(4.4) in the form

$$u(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x. \tag{4.5}$$

Putting in (4.3) and (4.4), we obtain

$$C_1 \sin\sqrt{\lambda} - C_2(1 + \cos\sqrt{\lambda}) = 0, \tag{4.6}$$

$$C_1(1 + \cos\sqrt{\lambda} - \sqrt{\lambda}\sin\sqrt{\lambda}) + C_2(\sin\sqrt{\lambda} + \sqrt{\lambda}\cos\sqrt{\lambda}) = 0. \tag{4.7}$$

Let $\sin\sqrt{\lambda} = 0$ then we get from (4.6)–(4.7) $C_2 = 0$ and $\cos\sqrt{\lambda} = -1$. Thus, we obtain eigenvalues and eigenfunctions corresponding to each eigenvalue in the following form

$$\lambda_{1k} = (2k-1)^2\pi^2, \tag{4.8}$$

$$u_{1k} = C_1 \cos(2k-1)\pi x, \tag{4.9}$$

$k = 0, \pm 1, \pm 2, \dots$.

Now consider when $\sin\sqrt{\lambda} \neq 0$ then we have from (4.6) $C_1 = C_2 \frac{1+\cos\sqrt{\lambda}}{\sin\sqrt{\lambda}}$. Putting this in (4.7) gives

$$C_2[(1+\cos\sqrt{\lambda})(1+\cos\sqrt{\lambda} - \sqrt{\lambda}\sin\sqrt{\lambda}) + \sin\sqrt{\lambda}(\sin\sqrt{\lambda} + \sqrt{\lambda}\cos\sqrt{\lambda})] = 0.$$

A direct calculation shows that, for z_k the roots of equation $\cot z_k = z_k$, we have eigenvalues and eigenfunctions corresponding to each eigenvalue in the following form

$$\lambda_{2k} = 4z_k^2, \tag{4.10}$$

$$u_{2k} = C_1 \cos 2z_k x + C_2 \sin 2z_k x, \tag{4.11}$$

$k = 0, \pm 1, \pm 2, \dots$. Thus, we solved the one-dimensional spectral problem for the Newton potential in $(0,1)$, i.e., we found eigenvalues and eigenfunctions for the Newton potential (4.1) in $(0,1)$.

According to Theorem 2.1, we can easily solve the following boundary value problem

$$-\Delta u(x) = f(x), \tag{4.12}$$

$$-\frac{u(x)}{2} - \int\limits_S \varepsilon_n(x-y)\frac{\partial u(y)}{\partial n_y}dS_y + \int\limits_S \frac{\partial \varepsilon_n(x-y)}{\partial n_y}u(y)dS_y = 0, \qquad x \in S, \tag{4.13}$$

in any bounded domain $\Omega \in R^n$, so it is not similar to the principal boundary value problems, such as the Dirichlet problem, Neumann problem or Robin problem (it is difficult to find the Green function and to solve in any bounded domain Ω for these problems). Therefore, authors think computing eigenvalues and eigenfunctions of the boundary value problem (4.12)–(4.13) is an interesting problem.

Let consider the spectral problem on eigenvalues and eigenfunctions of the Newton potential in the 2-disk $\Omega = \{x : |x| < \delta\} \subset R^2$ with boundary $S = \{x : |x| = \delta\} \subset R^2$

$$u(x) = -\lambda \int_\Omega -\frac{1}{2\pi} \ln|x-y| u(y)dy. \tag{4.14}$$

It is equivalent to the spectral problem

$$-\Delta u(x) = \lambda u(x), \tag{4.15}$$

$$-\frac{u(x)}{2} - \int\limits_S \varepsilon_2(x-y)\frac{\partial u(y)}{\partial n_y}dS_y + \int\limits_S \frac{\partial \varepsilon_2(x-y)}{\partial n_y}u(y)dS_y = 0, \quad x \in S, \tag{4.16}$$

where $\varepsilon_2(x-y) = -\frac{1}{2\pi}\ln|x-y|$. The following theorem is valid.

Theorem 4.1. *The eigenvalues λ_{kj} of the two-dimensional Newton potential in the 2-disk are given by*

$$\lambda_{kj} = \frac{[\mu_j^{(k)}]^2}{\delta^2}, \qquad k = 0, 1, \dots, \; j = 1, 2, \dots, \tag{4.17}$$

where $\mu_j^{(k)}$-the are the roots of the transcendental equation

$$kJ_k(\mu_j^{(k)}) + \frac{\mu_j^{(k)}}{2}(J_{k-1}(\mu_j^{(k)}) - J_{k+1}(\mu_j^{(k)})) = 0, \qquad k = 1, 2, \dots, \tag{4.18}$$

$$J_0(\mu_j^{(0)}) + \mu_j^{(0)}\ln(\tfrac{1}{\delta})(J_{-1}(\mu_j^{(0)}) - J_1(\mu_j^{(0)})) = 0.$$

The eigenfunctions corresponding to each eigenvalue λ_{kj}, form a complete orthogonal system and can be represented in the form

$$u_{kj} = J_k(\mu_j^{(k)}\frac{r}{\delta})e^{ik\varphi}, \tag{4.19}$$

in which the J_k are the Bessel functions and (r,φ) are polar coordinates.

Proof of Theorem 4.1. Applying the Fourier method to (4.15) and setting $u(r,\varphi) = R(r)\Phi(\varphi)$, we obtain the two one-dimensional boundary value problems

$$-\Phi'' = \mu\Phi, \Phi(\varphi) = \Phi(\varphi + 2\pi), \tag{4.20}$$

$$r(rR')' + (\lambda r^2 - \mu)R = 0, \qquad |R(0)| < \infty. \tag{4.21}$$

The eigenvalues and the eigenfunctions of problem (4.20) (which are trigonometric functions) are easy to calculate:

$$\mu_k = k^2, \quad \Phi_k(\varphi) = \frac{1}{\sqrt{2\pi}}e^{ik\varphi}, \qquad k = 0, 1, \dots. \tag{4.22}$$

Passing to the polar coordinate system, we rewrite the boundary condition (4.16) in the form

$$\begin{aligned} -\frac{u(\delta,\varphi)}{2} &+ \frac{1}{4\pi}\int_0^{2\pi} \delta\ln(2\delta^2(1-\cos(\psi-\varphi)))\frac{\partial u(\rho,\psi)}{\partial\rho}|_{\rho=\delta}d\psi \\ &- \frac{1}{4\pi}\int_0^{2\pi} \delta\frac{\partial\ln((\delta^2+\rho^2-2\delta\rho\cos(\psi-\varphi)))}{\partial\rho}|_{\rho=\delta}u(\delta,\psi)d\psi = 0. \end{aligned} \tag{4.23}$$

Using this condition and the formula $\int_0^{2\pi}\ln(1-\cos\psi)d\psi = -2\pi\ln 2$, $\int_0^{2\pi}\ln(1-\cos\psi)e^{ik\psi}d\psi = -\frac{2\pi}{k}$, $k \neq 0$, $\int_0^{2\pi}\Phi_k(\psi)d\psi = 0$ and performing direct calculations, we obtain

$$kR_k(r) + rR_k'(r)|_{r=\delta} = 0, \quad k = 1, 2, \dots, \qquad R_0(r) - r\ln rR_0'(r)|_{r=\delta} = 0. \tag{4.24}$$

Thus, we have the following self-adjoint problem with respect to $R_k(r)$:

$$r(rR_k')' + (\lambda r^2 - k^2)R_k = 0, \qquad |R_k(0)| < \infty, \tag{4.25}$$

$$kR_k(r) + rR_k'(r)|_{r=\delta} = 0, \quad k \neq 0, \qquad R_0(r) - r\ln rR_0'(r)|_{r=\delta} = 0. \tag{4.26}$$

Note that the solution $R_k = J_k(\sqrt{\lambda}r)$ of problem (4.25), (4.26) has a complete orthogonal system in $L_2(r, 0, \delta)$ (see for example [5]), where the λ_{kj} are found from the transcendental equation

$$kJ_k(\mu_j^{(k)}) + \frac{\mu_j^{(k)}}{2}(J_{k-1}(\mu_j^{(k)}) - J_{k+1}(\mu_j^{(k)})) = 0, k = 1, 2, \dots\ . \tag{4.27}$$

Thus, the $\mu_j^{(k)} = \sqrt{\lambda_{kj}}\delta$ are the roots of Eq. (4.27).

Therefore, the eigenfunctions $u_{kj} = J_k(\mu_j^{(k)}\frac{r}{\delta})e^{ik\varphi}$, form a complete orthogonal system in $L_2(\Omega)$, and hence, problem (4.15), (4.16) has no other eigenvalues and eigenfunctions. □

Now, consider the problem on the eigenvalues of the Newton potential in the 3-ball $\Omega = \{x : |x| < \delta\} \subset R^3$ with boundary $S = \{x : |x| = \delta\} \subset R^3$.

Theorem 4.2. *The eigenvalues λ_{lj} of the three-dimensional Newton potential in the ball are given by*

$$\lambda_{lj} = \frac{[\mu_j^{(l+\frac{1}{2})}]^2}{\delta^2}, \qquad l = 0, 1, \dots,\ j = 1, 2, \dots, \tag{4.28}$$

where the $\mu_j^{(l+\frac{1}{2})}$ are the roots of the transcendental equation

$$(2l+1)J_{l+\frac{1}{2}}(\mu_j^{(l+\frac{1}{2})}) + \frac{\mu_j^{(l+\frac{1}{2})}}{2}(J_{l-\frac{1}{2}}(\mu_j^{(l+\frac{1}{2})}) - J_{l+\frac{3}{2}}(\mu_j^{(l+\frac{1}{2})})) = 0. \tag{4.29}$$

The eigenfunctions corresponding to each eigenvalue λ_{lj}, form a complete orthogonal system and can be represented in the form

$$u_{kj} = J_{l+\frac{1}{2}}(\sqrt{\lambda_{lj}}r)Y_l^m(\varphi, \theta), \tag{4.30}$$

where

$$Y_l^m(\varphi, \theta) = P_l^m(\cos\theta)\cos m\varphi, \qquad m = 0, 1, \dots, l,$$
$$Y_l^m(\varphi, \theta) = P_l^{|m|}(\cos\theta)\sin|m|\varphi, \qquad m = -1, \dots, -l$$

for $l = 0, 1, \dots$ are spherical functions, the P_l^m the associated Legendre polynomials and (r, θ, φ) are the spherical coordinates.

Proof of Theorem 4.2. According to Theorem 2.1, the spectral problem on the eigenvalues of the Newton potential in the 3-ball $\Omega = \{x : |x| < \delta\} \subset R^3$,

$$u(x) = -\lambda \int_\Omega \varepsilon_3(x-y)u(y)dy, \tag{4.31}$$

is equivalent to the following spectral problem

$$-\Delta u_k(x) = \lambda_k u_k(x), x \in \Omega, \tag{4.32}$$

$$-\frac{u_k(x)}{2} - \int\limits_S \varepsilon_3(x-y)\frac{\partial u_k(y)}{\partial n_y} dS_y + \int\limits_S \frac{\partial \varepsilon_3(x-y)}{\partial n_y} u_k(y) dS_y = 0, \quad x \in S. \tag{4.33}$$

where $\varepsilon_3(x-y) = \frac{1}{4\pi}\frac{1}{|x-y|}$.

This problem is convenient for solving in spherical coordinates, i.e.,

$$x_1 = r\sin\theta\cos\varphi, x_2 = r\sin\theta\sin\varphi, x_3 = r\cos\theta,$$

with $0 \le r < \delta, 0 \le \theta < \pi, 0 \le \varphi < 2\pi$, and

$$y_1 = \rho\sin\vartheta\cos\psi, y_2 = \rho\sin\vartheta\sin\psi, y_3 = r\cos\vartheta,$$

for $0 \le \rho < \delta, 0 \le \vartheta < \pi, 0 \le \psi < 2\pi$. In these coordinates the problem (4.32)–(4.33) for the function $\widetilde{u}(r,\theta,\varphi) := u(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)$, becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial \widetilde{u}}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\widetilde{u}}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\widetilde{u}}{\partial\varphi^2} = -\lambda\widetilde{u} \tag{4.34}$$

$$\frac{1}{2}u(r,\theta,\varphi) + \frac{1}{4\pi}\int_0^\pi\int_0^{2\pi}\frac{\rho^2}{\sqrt{\rho^2 - 2r\rho\Psi + r^2}}\frac{\partial u(\rho,\vartheta,\psi)}{\partial\rho}|_{\rho=\delta}d\vartheta d\psi \tag{4.35}$$

$$-\frac{1}{4\pi}\int_0^\pi\int_0^{2\pi}\rho^2\frac{1}{\partial\rho}\frac{1}{\sqrt{\rho^2 - 2r\rho\Psi + r^2}}u(\rho,\vartheta,\psi)|_{\rho=\delta}d\vartheta d\psi = 0,$$

the latter for $r = \delta$, where $\Psi = \sin\theta\cos\varphi\sin\vartheta\cos\psi + \sin\theta\sin\varphi\sin\vartheta\sin\psi + \cos\theta\cos\vartheta$. To a boundary condition at $r = \delta$, it is necessary to add also a boundary condition at $r = 0$. This condition consists that function $\widetilde{u}$, obviously, should be bounded and 2π-periodic corresponding φ, i.e.,

$$|\widetilde{u}(0,\theta,\varphi)| < \infty, \qquad \widetilde{u}(\delta,\theta,\varphi) = \widetilde{u}(\delta,\theta,\varphi+2\pi). \tag{4.36}$$

According to the general scheme Fourier method for eigenfunctions of the problem (4.35)–(4.36), we search in the form of product $\Re(r)Y(\theta,\varphi)$.

Separating variables, for functions Y and $\Re$ we obtain boundary value problems:

$$\frac{1}{r^2\sin\theta\frac{\partial}{\partial\theta}}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2} + \mu Y = 0, \qquad Y \in C^\infty(S), \tag{4.37}$$

$$(r^2\Re')' + (\lambda r^2 - \mu)\Re = 0, \qquad |\Re(0)| < \infty, \tag{4.38}$$

$$\Re(r)Y(\theta,\varphi) + \frac{1}{2\pi}\int_0^\pi\int_0^{2\pi}\frac{\rho^2}{\sqrt{\rho^2 - 2r\rho\Psi + r^2}}\frac{\partial\Re(\rho)Y(\vartheta,\psi)}{\partial\rho}|_{\rho=\delta}d\vartheta d\psi$$

$$-\frac{1}{2\pi}\int_0^\pi\int_0^{2\pi}\rho^2\frac{1}{\partial\rho}\frac{1}{\sqrt{\rho^2 - 2r\rho\Psi + r^2}}\Re(\rho)Y(\vartheta,\psi)|_{\rho=\delta}d\vartheta d\psi = 0, r = \delta \tag{4.39}$$

As $\mu = l(l+1), l = 0, 1, \ldots,$ the problem (4.38) has solutions and these solutions are spherical functions $Y_l^m, m = 0, \pm 1, \ldots, \pm l$.

Now we use the following expansions [4]

$$\frac{1}{\sqrt{1-2\nu\Psi+\nu^2}}=\sum_{l=0}^{\infty}P_l(\Psi)\nu^l, \nu<1, \tag{4.40}$$

$$\frac{1}{\sqrt{\rho^2-2r\rho\Psi+r^2}}|_{\rho=\delta}=\frac{1}{\sqrt{1-2\frac{r}{\delta}\Psi+\frac{r^2}{\delta^2}}}=\sum_{k=0}^{\infty}P_k(\Psi)\frac{r^k}{\delta^{k+1}},$$

$$\frac{\partial}{\partial\rho}\frac{1}{\sqrt{\rho^2-2r\rho\Psi+r^2}}|_{\rho=\delta}=\frac{\partial}{\partial\rho}\sum_{k=0}^{\infty}P_k(\Psi)\frac{r^k}{\delta^{k+1}}=\sum_{k=0}^{\infty}(k+1)P_k(\Psi)\frac{r^k}{\delta^{k+2}},$$

where P_l is the Legendre polynomial. Then the following lemma is valid.

Lemma 4.3 ([5]).

$$\int_0^{\pi}\int_0^{2\pi}(Y_l(\theta,\varphi))^{-1}Y_l(\vartheta,\psi)P_k(\Psi)d\vartheta d\psi=\frac{4\pi}{2l+1}\delta_{lk} \tag{4.41}$$

where $\delta_{lk}=0$ *for* $k\neq l, \delta_{lk}=1$ *for* $k=l$.

Using the previous discussions we can present (4.39) in the following form:

$$\Re(\delta)+\frac{2\delta}{4l+3}\Re'(\delta)=0. \tag{4.42}$$

Putting $\Re_1=\sqrt{r}\Re$ in the equation (4.38), we obtain the Bessel equation

$$r^2\Re_1''+r\Re_1'+(\lambda r^2-(l+\frac{1}{2})^2)\Re_1=0. \tag{4.43}$$

So we get the solution of the equation (4.38)

$$\Re(r)=\frac{1}{\sqrt{r}}J_{l+\frac{1}{2}}(\sqrt{\lambda}r). \tag{4.44}$$

To satisfy to a boundary condition (4.42), we have

$$J_{l+\frac{1}{2}}(\mu_j^{l+\frac{1}{2}})+\frac{\mu_j^{l+\frac{1}{2}}}{l+1}J'_{l+\frac{1}{2}}(\mu_j^{l+\frac{1}{2}})=0.$$

It follows from properties of the Bessel function

$$(2l+1)J_{l+\frac{1}{2}}(\mu_j^{(l+\frac{1}{2})})+\frac{\mu_j^{(l+\frac{1}{2})}}{2}(J_{l-\frac{1}{2}}(\mu_j^{(l+\frac{1}{2})})-J_{l+\frac{3}{2}}(\mu_j^{(l+\frac{1}{2})}))=0. \tag{4.45}$$

where $\mu_j^{(l+\frac{1}{2})}=\sqrt{\lambda_{lj}}\delta,\ l,j=1,2,\ldots,$ are positive roots of the equation (4.45). Thus, we find

$$\begin{aligned}\lambda_{lj}&=\frac{[\mu_j^{(l+\frac{1}{2})}]^2}{\delta^2},\\ u_{kjm}&=J_{l+\frac{1}{2}}(\sqrt{\lambda_{lj}}r)Y_l^m(\varphi,\theta),\\ &\quad l=0,1,\ldots,\ j=1,2,\ldots,\ m=0,\pm1,\ldots,\pm l,\end{aligned} \tag{4.46}$$

as corresponding eigenvalues and eigenfunctions of the boundary value problem (4.32), (4.33). Therefore, the eigenfunctions $\{u_{ljm}\}$ form a complete orthogonal system in $L_2(\Omega)$, and hence, problem (4.32), (4.33) has no other eigenvalues and eigenfunctions. This proves Theorem 4.2. □

Remark 4.4. It follows from the asymptotics of the roots of Eqs. (4.18) and (4.29) that the eigenvalues of problems (4.15), (4.16) and (4.32), (4.33) have the same asymptotics as those of the Dirichlet problem for the Poisson operator.

5. Some applications

5.1. On an inhomogeneous boundary condition of the Newton potential

In a bounded simply connected domain Ω, in the n-dimensional Euclidean space R^n $(n > 1)$, with sufficiently smooth boundary S, consider the following Poisson equation

$$-\Delta u(x) = f(x), \qquad x \in \Omega, \tag{5.1}$$

with an inhomogeneous boundary condition

$$-\frac{u(x)}{2} - \int\limits_S \varepsilon_n(x-y)\frac{\partial u(y)}{\partial n_y}dS_y + \int\limits_S \frac{\partial \varepsilon_n(x-y)}{\partial n_y}u(y)dS_y = q(x), \qquad x \in S, \tag{5.2}$$

where $f \in L_2(\Omega)$ and $q(x) \in W_2^{\frac{1}{2}}(S)$ are given functions.

Theorem 5.1. *A unique solution of problem* (5.1)–(5.2) *in* $W_2^2(\Omega)$ *is*

$$u(x) = \int_\Omega \varepsilon_n(x-y)f(y)dy - \int_S \frac{\partial G(x,y)}{\partial n_y}q(y)dS_y, \tag{5.3}$$

where $G(x,y)$ *is the Green function of the Dirichlet problem for the Laplace operator, i.e.,*

$$-\Delta G(x,y) = \delta(x-y), \qquad G(x,y)|_{x\in S} = 0.$$

Proof of Theorem 5.1. If $u(x)$ is a solution of problem (5.1)–(5.2) in $W_2^2(\Omega)$, then it is easy to show that it is unique.

Let $u(x)$ be sum of two functions

$$u(x) = u_1(x) + u_2(x), \tag{5.4}$$

where

$$u_1(x) = \varepsilon_n * f = \int_\Omega \varepsilon_n(x-y)f(y)dy, \tag{5.5}$$

is the Newton (volume) potential and

$$u_2(x) = u(x) - u_1(x). \tag{5.6}$$

According to Theorem 2.1, the Newton potential $u_1(x)$ satisfies the following boundary condition

$$\frac{u_1(x)}{2} - \int\limits_S \varepsilon_n(x-y)\frac{\partial u_1(y)}{\partial n_y}dS_y + \int\limits_S \frac{\partial \varepsilon_n(x-y)}{\partial n_y}u_1(y)dS_y = u_1(x), \quad x \in S. \quad (5.7)$$

Since the Newton potential $u_1(x)$ is a solution of the Poisson equation (5.1) in Ω,

$$-\Delta u_2(x) = 0, \qquad x \in \Omega. \quad (5.8)$$

Using the above, we get

$$-\frac{u(x)}{2} - \int\limits_S \varepsilon_n(x-y)\frac{\partial u(y)}{\partial n_y}dS_y + \int\limits_S \frac{\partial \varepsilon_n(x-y)}{\partial n_y}u(y)dS_y$$
$$= -\frac{u_2(x)}{2} - \int\limits_S \varepsilon_n(x-y)\frac{\partial u_2(y)}{\partial n_y}dS_y + \int\limits_S \frac{\partial \varepsilon_n(x-y)}{\partial n_y}u_2(y)dS_y = q(x) \quad (5.9)$$

for all $x \in S$. As $u_2(x)$ is a harmonic function in Ω, the boundary condition (5.9) is equivalent to

$$u_2(x) = q(x), \qquad x \in S. \quad (5.10)$$

Solving (5.8) with the Dirichlet boundary condition (5.10), we obtain

$$u_2(x) = -\int_S \frac{\partial G(x,y)}{\partial n_y}q(y)dS_y, \qquad x \in \Omega. \quad (5.11)$$

Finally, we get

$$u(x) = u_1(x)+u_2(x) = \int_\Omega \varepsilon_n(x-y)f(y)dy - \int_S \frac{\partial G(x,y)}{\partial n_y}q(y)dS_y, \quad x \in \Omega. \quad (5.12)$$

□

5.2. A problem outside a ball

Now, consider the following problem.

Problem 5.2. *In $R^3 \backslash \Omega_\delta$, $\Omega_\delta = \{x : |x| < \delta\} \subset R^3$, solve the Helmholtz equation*

$$\Delta u(x) + k^2 u(x) = 0, \quad (5.13)$$

with boundary conditions

$$-\frac{u(x)}{2} - \int\limits_S \varepsilon_n(x-y)\frac{\partial u(y)}{\partial n_y}dS_y + \int\limits_S \frac{\partial \varepsilon_n(x-y)}{\partial n_y}u(y)dS_y = q(x), \quad x \in S, \quad (5.14)$$

and Sommerfield radiation conditions

$$u = O\left(\frac{1}{|x|}\right), \qquad \frac{\partial u}{\partial |x|} - iku = o\left(\frac{1}{|x|}\right), \qquad |x| \to \infty. \quad (5.15)$$

As shown in [5] for the Dirichlet problem, this problem has a unique solution. This problem is convenient for solving in spherical coordinates. Therefore, to construct the desired solution, we use the following decompositions in spherical coordinates

$$u(r,\theta,\varphi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\Re_{lm}(r)Y_l^m(\theta,\varphi), \tag{5.16}$$

$$q(\theta,\varphi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\ a_{lm}Y_l^m(\theta,\varphi). \tag{5.17}$$

Rewriting (5.14) in spherical coordinates,we obtain

$$\begin{aligned}\frac{1}{2}u(r,\theta,\varphi)+\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\frac{\rho^2}{\sqrt{\rho^2-2r\rho\Psi+r^2}}\frac{\partial u(\rho,\vartheta,\psi)}{\partial\rho}|_{\rho=\delta}d\vartheta d\psi\\ -\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\rho^2\frac{1}{\partial\rho}\frac{1}{\sqrt{\rho^2-2r\rho\Psi+r^2}}u(\rho,\vartheta,\psi)|_{\rho=\delta}d\vartheta d\psi\\ =q(\theta,\varphi)\end{aligned} \tag{5.18}$$

for $r=\delta$, where $\Psi=\sin\theta\cos\varphi\sin\vartheta\cos\psi+\sin\theta\sin\varphi\sin\vartheta\sin\psi+\cos\theta\cos\vartheta$. From (5.16) and (5.18),we get

$$\begin{aligned}&\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\Re_{lm}(\delta)Y_l^m(\theta,\varphi)\\ &\quad+\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{\Re'_{lm}(\delta)\delta^2}{2\pi}\int_0^{\pi}\int_0^{2\pi}\frac{1}{\sqrt{\rho^2-2r\rho\Psi+r^2}}|_{\rho=\delta}Y_l^m(\vartheta,\psi)d\vartheta d\psi\\ &\quad-\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{\Re_{lm}(\delta)\delta^2}{2\pi}\int_0^{\pi}\int_0^{2\pi}\frac{\partial}{\partial\rho}\frac{1}{\sqrt{\rho^2-2r\rho\Psi+r^2}}|_{\rho=\delta}Y_l^m(\vartheta,\psi)d\vartheta d\psi\\ &=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\ a_{lm}Y_l^m(\theta,\varphi).\end{aligned} \tag{5.19}$$

As above, from (5.19) we obtain

$$(4l+3)\Re_{lm}(r)+r\Re'_{lm}(r)|_{r=\delta}=a_{lm}. \tag{5.20}$$

Thus, unknown coefficients of the decomposition (5.16) should satisfy

$$(4l+3)\Re''_{lm}+\frac{2}{r}\Re'_{lm}+\left(k^2-\frac{l(l+1)}{r^2}\right)\Re_{lm}=0, \tag{5.21}$$

With the boundary condition (5.20) and the radiation condition

$$u = O\left(\frac{1}{r}\right), \qquad \frac{\partial u}{\partial r} - iku = o\left(\frac{1}{r}\right), \qquad r \to \infty. \tag{5.22}$$

Without the boundary conditions, the general solution to equation (5.21) is

$$\Re_{lm}(r) = \frac{c_1}{\sqrt{r}} H^{(1)}_{l+\frac{1}{2}}(kr) + \frac{c_2}{\sqrt{r}} H^{(2)}_{l+\frac{1}{2}}(kr) \tag{5.23}$$

where H is the Hankel function. Now consider the following asymptotic formulas

$$H^{(1)}_{\nu}(x) = \sqrt{\frac{2}{\pi x}} e^{i(x-\frac{\pi}{2}\nu)-\frac{\pi}{4}} + O(x^{-\frac{3}{2}}),$$

$$H^{(2)}_{\nu}(x) = \sqrt{\frac{2}{\pi x}} e^{-i(x-\frac{\pi}{2}\nu)-\frac{\pi}{4}} + O(x^{-\frac{3}{2}}).$$

Using these asymptotic of the Hankel functions, it is easy to understand that only function $\frac{c_1}{\sqrt{r}} H^{(1)}_{l+\frac{1}{2}}(kr)$ satisfies the condition (5.22), it means $c_2 = 0$.

Putting $c_2 = 0$ and

$$c_1 = \frac{\sqrt{\delta} a_{lm}}{(4l+3) H^{(1)}_{l+\frac{1}{2}}(k\delta) + \delta H'^{(1)}_{l+\frac{1}{2}}(k\delta)}$$

(it is found from (5.20)) in (5.16), we obtain the desired solution in the form

$$u(r,\varphi,\theta) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} \sqrt{\frac{\delta}{r}} \frac{H^{(1)}_{l+\frac{1}{2}}(kr)}{(4l+3) H^{(1)}_{l+\frac{1}{2}}(k\delta) + \delta H'^{(1)}_{l+\frac{1}{2}}(k\delta)}, \tag{5.24}$$

Remark 5.3. The Problem 5.2 is similarly considered outside a circle.

References

[1] I. Newton, The Principia, Mathematical principles of natural philosophy, a new translation, University of California Press (1999).

[2] Lagrange's works (in French) Oeuvres de Lagrange, edited by Joseph Alfred Serret, digitized by Guttinger Digitalisierungszentrum, Mécanique analytique is in volumes 11 and 12 (Paris, 1867).

[3] G.M. Rassias (ed.), The mathematical heritage of C.F. Gauss (Singapore, 1991).

[4] O.D. Kellogg, Foundations of Potential Theory, Ungar, New York, 4th printing (1970).

[5] V.S. Vladimirov, Equations of Mathematical Physics, Nauka, (Moscow, 1981) [in Russian].

[6] G.C. Hsiao, W.L. Wendland, Boundary Integral Equations, Springer (2008).

[7] N.S. Landkof, "Foundations of modern potential theory", Springer (1972) (translated from Russian).

[8] V.I. Fabrikant, "Applications of potential theory in mechanics. Selection of new results", Kluwer (1989).

[9] C. Constantinescu, A. Cornea, “Potential theory on harmonic spaces”, Springer (1972)

[10] T.Sh. Kal'menov, D. Suragan, To Spectral Problems for Volume Potential, Doklady Mathematics, 80 (2), pp. 646–649 (2009).

T.Sh. Kalmenov and D. Suragan
Institute of Mathematics Computer Science and Mechanics
Ministry of Education and Science of Kazakhstan
str. Shevchenko 28
Almaty, 050010, Kazakhstan
e-mail: kalmenov.t@mail.ru
suragan@list.ru

Operator Theory:
Advances and Applications, Vol. 216, 211–231

An Extremum Principle for a Class of Hyperbolic Type Equations and for Operators Connected with Them

I.U. Khaydarov, M.S. Salakhitdinov and A.K. Urinov

Abstract. In the present paper an extremum principle for a class of hyperbolic type equations is established using an explicit form of the Darboux problem's solution for this class of equations. Moreover, extremum principles for integral-differential operators, connected with those equations are obtained.

Mathematics Subject Classification (2000). Primary 35L10; Secondary 35L80.

Keywords. Hyperbolic equations; extremum principle; Darboux problem.

1. Introduction

It is known that in the process of proving the uniqueness and the stability of solutions of boundary value problems for elliptic, parabolic and mixed type equations, the extremum principle (EP) plays an essential role (e.g., [1, 2, 3]). Establishment of an EP for mixed elliptic-hyperbolic and parabolic-hyperbolic type equations is based on the EP for hyperbolic equations. In some principal cases, an EP for the hyperbolic type equations was established by a method of Agmon-Nirenberg-Protter [4]. Note also works by O.M. Jokhadze [5], M. Usanetashvili [6], where an EP for some classes of second-order elliptic and parabolic systems was studied. Also using an integral representation of a solution of the Darboux problem, an EP for some hyperbolic equations can be established. In this work an EP for the class of equations

$$(-y)^m u_{xx} - u_{yy} + \lambda^2 (-y)^m u = 0, \tag{1.1}$$

is established by the aforementioned method. In equation (1.1), λ is a real or complex number and $m \geq 0$.

Let Ω be a finite simply-connected domain of half-plane $y<0$, bounded by characteristics

$$\overline{AC}:\xi\equiv x-\frac{2}{m+2}(-y)^{\frac{m+2}{2}}=a,\quad \overline{BC}:\eta\equiv x+\frac{2}{m+2}(-y)^{\frac{m+2}{2}}=b$$

of the equation (1.1) and by segment $\overline{AB}=\{(x,y):y=0,a\le x\le b\}$, where $a<b$.

To obtain an EP for equation (1.1) in the domain Ω, one needs a representation of a solution of the Darboux problem for equation (1.1), which vanishes on one of its characteristics. From the results of the work [7] it follows that the unique solution of the Darboux problem for equation (1.1), satisfying conditions $u|_{\overline{AC}}=0$ or $u|_{\overline{BC}}=0$, at $m>0$, respectively has a form

$$u(x,y)=\gamma(\eta-\xi)^{1-2\beta}\int_a^\xi\frac{\overline{J}_{\beta-1}[\lambda\sqrt{(\xi-t)(\eta-t)}]}{[(\xi-t)(\eta-t)]^{1-\beta}}\tau(t)dt, \tag{1.2}$$

$$u(x,y)=\gamma(\eta-\xi)^{1-2\beta}\int_\eta^b\frac{\overline{J}_{\beta-1}[\lambda\sqrt{(t-\xi)(t-\eta)}]}{[(t-\xi)(t-\eta)]^{1-\beta}}\tau(t)dt, \tag{1.3}$$

where $\tau(x)=u(x,0)$, $\beta=m/(2m+4)$, $\gamma=\Gamma(1-\beta)/[\Gamma(\beta)\Gamma(1-2\beta)]$, $\overline{J}_s(z)=\Gamma(s+1)(z/2)^{-s}J_s(z)$, and $J_s(z)$ is Bessel's function of the first kind, of order s [8]; $\Gamma(z)$ is Euler's Gamma-function [9].

Assuming that $\tau(x)$ has a bounded first-order derivative in (a,b), passing to the limit at $\beta\to 0$ $(m\to 0)$ from (1.2) and (1.3), we can obtain a formula for solution of the Darboux problem for equation (1.1) at $m=0$, i.e., the telegraph equation

$$u_{xx}-u_{yy}+\lambda^2u=0, \tag{1.4}$$

satisfying conditions $u|_{\overline{AC}}=0$ or $u|_{\overline{BC}}=0$, respectively. We rewrite (1.2) in the form

$$\begin{aligned}u(x,y)=&\gamma(\eta-\xi)^{1-2\beta}\tau(\xi)\int_a^\xi[(\xi-t)(\eta-t)]^{\beta-1}dt\\&+\gamma(\eta-\xi)^{1-2\beta}\int_a^\xi\frac{\overline{J}_{\beta-1}\left[\lambda\sqrt{(\xi-t)(\eta-t)}\right]-\overline{J}_{\beta}\left[\lambda\sqrt{(\xi-t)(\eta-t)}\right]}{[(\xi-t)(\eta-t)]^{1-\beta}}\tau(t)dt\\&+\gamma(\eta-\xi)^{1-2\beta}\int_a^\xi\left\{\overline{J}_\beta\left[\lambda\sqrt{(\xi-t)(\eta-t)}\right]\tau(t)-\tau(\xi)\right\}[(\xi-t)(\eta-t)]^{\beta-1}dt\\=&\,l_1+l_2+l_3\end{aligned} \tag{1.5}$$

to get these formulas.

First we consider l_1. Replacing variables by the formula $t = a + (\xi - a)z$ and taking an integral representation of Gauss' hypergeometric function $F(a, b, c; z)$ into account, and also using autotransformer formula [9], we find

$$l_1 = \frac{\Gamma(1-\beta)}{\Gamma(1+\beta)\Gamma(1-2\beta)} \left(\frac{\xi - a}{\eta - a}\right)^{\beta} F\left(\beta, 2\beta, 1+\beta; \frac{\xi - a}{\eta - a}\right) \tau(\xi). \tag{1.6}$$

Considering an identity

$$\overline{J}_{\beta-1}(z) - \overline{J}_{\beta}(z) = -[z^2/4\beta(\beta+1)]\overline{J}_{\beta+1}(z), \tag{1.7}$$

we have

$$l_2 = -\frac{\lambda^2\Gamma(1-\beta)(\eta-\xi)^{1-2\beta}}{4\Gamma(2+\beta)\Gamma(1-2\beta)} \int_a^{\xi} [(\xi - t)(\eta - t)]^{\beta} \overline{J}_{\beta+1}\left[\lambda\sqrt{(\xi - t)(\eta - t)}\right] \tau(t)dt. \tag{1.8}$$

Since $\tau(x)$ has a first-order bounded derivative, then

$$\left|\overline{J}_{\beta}\left[\lambda\sqrt{(\xi - t)(\eta - t)}\right] \tau(t) - \tau(\xi)\right| = (\xi - t)O(1),$$

from where it follows that

$$l_3 = \frac{\Gamma(1-\beta)}{\Gamma(\beta)\Gamma(1-2\beta)} \int_a^{\xi} \frac{(\xi - t)^{\beta} O(1)}{(\eta - t)^{1-\beta}} dt. \tag{1.9}$$

Substituting (1.6), (1.8), (1.9) into (1.5) and passing to the limit at $\beta \to 0$, considering $\Gamma(1) = \Gamma(2) = 1$, $\lim\limits_{\beta \to +0} \Gamma(\beta) \to +\infty$, $F(0, 0, 1; z) = 1$, we obtain a solution of equation (1.4), satisfying condition $u|_{\overline{AC}} = 0$:

$$u(x, y) = \tau(\xi) - \frac{\lambda^2}{4}(\eta - \xi) \int_a^{\xi} \tau(t)\overline{J}_1[\lambda\sqrt{(\xi - t)(\eta - t)}]dt. \tag{1.10}$$

Analogously, from (1.3) at $\beta \to +0$ we get a formula for the solution of equation (1.4), satisfying condition $u|_{\overline{BC}} = 0$:

$$u(x, y) = \tau(\eta) - \frac{\lambda^2}{4}(\eta - \xi) \int_{\eta}^{b} \tau(t)\overline{J}_1[\lambda\sqrt{(t - \xi)(t - \eta)}]dt. \tag{1.11}$$

Upon considering equation (1.4) and formulas (1.10), (1.11) by Ω, we imply a domain, bounded by lines $x + y = a$, $x - y = b$, $y = 0$, and also $\xi = x + y$, $\eta = x - y$.

At $m > 0$ from (1.2) it follows [7] that

$$\lim_{y \to 0} u_y(x, y) = \gamma_0 C_{ax}^{1,\lambda}[\tau(x)], \tag{1.12}$$

where $\gamma_0 = (m+2)^{2\beta}\Gamma(\beta+1/2)/\Gamma(-\beta+1/2)$,

$$C_{sx}^{1,\lambda}[\tau(x)] \equiv \frac{1}{\Gamma(2\beta)}\frac{d}{dx}\int_s^x |x-t|^{2\beta-1}\overline{J}_\beta[\lambda(x-t)]\tau(t)dt$$
$$+\frac{\lambda^2 \text{sign}(x-s)}{4\beta(1+\beta)\Gamma(2\beta)}\int_s^x |x-t|^{2\beta}\overline{J}_{\beta+1}[\lambda(x-t)]\tau(t)dt. \quad (1.13)$$

Similarly, using formulas (1.3), (1.10) and (1.11) one can prove that equalities

$$\lim_{y\to 0} u_y(x,y) = \gamma_0 C_{bx}^{1,\lambda}[\tau(x)], \quad (1.14)$$

$$\lim_{y\to 0} u_y(x,y) = C_{ax}^{0,\lambda}[\tau(x)], \quad (1.15)$$

$$\lim_{y\to 0} u_y(x,y) = C_{bx}^{0,\lambda}[\tau(x)] \quad (1.16)$$

are true, where

$$C_{sx}^{0,\lambda}[\tau(x)] = \text{sign}(x-s)\left\{\tau'(x) + \frac{1}{2}\lambda^2\int_s^x \tau(t)\overline{J}_1[\lambda(x-t)]dt\right\}. \quad (1.17)$$

Here we must note that $C_{sx}^{0,\lambda}[\tau(x)]$ and $C_{sx}^{1,\lambda}[\tau(x)]$ are operators, introduced and studied in [7]; moreover there it was proved that $\lim_{\beta\to 0} C_{sx}^{\beta,\lambda}[\tau(x)] = C_{sx}^{0,\lambda}[\tau(x)]$. From (1.13) it follows that $C_{sx}^{1,0}[\tau(x)] = D_{sx}^{1-2\beta}[\tau(x)]$, where $D_{sx}^{1-2\beta}[\tau(x)]$ is a fractional differential operator [8].

Besides, using the equality (1.7) and

$$\overline{J}\,'_\beta[z] = -[z/(2(\beta+1))]\overline{J}_{\beta+1}(z), \quad (1.18)$$

it is not difficult to determine that

$$C_{ax}^{1,\lambda}[\tau(x)] \equiv \left(A_{a+}^{1-2\beta,\lambda}\right)^{-1}\tau(x) \equiv \left(\frac{d^2}{dx^2}+\lambda^2\right)\int_a^x \frac{(x-t)^{2\beta}}{\Gamma(1+2\beta)}\overline{J}_\beta[\lambda(x-t)]\tau(t)dt,$$

where $\left(A_{a+}^{1-2\beta,\lambda}\right)^{-1}$ is an inverse operator of [8, pp. 530–533]

$$A_{a+}^{1-2\beta,\lambda}f(x) \equiv \int_a^x \frac{(x-t)^{-2\beta}}{\Gamma(1-2\beta)}\overline{J}_{-\beta}[\lambda(x-t)]f(t)dt.$$

Nevertheless, in [5, pp. 32–36] it was also proved that, if $g(a)=0$, $g(x)\in C^{(0,\alpha)}[a,b]$, $\alpha > 1-2\beta$, then integral equation

$$\int_a^x (x-t)^{-2\beta}\overline{J}_{-\beta}[\lambda(x-t)]f(x)dt = g(x)$$

has a unique solution of the form $f(x) = \Gamma^{-1}(1-2\beta)C_{ax}^{1,\lambda}[g(x)]$.

It is known that Zaremba-Giraud's principle for uniformly elliptic equations [9, 10] determines the sign of the normal derivative of the solution for this equation in the boundary points of the considered domain, where the solution achieves its positive maximum (negative minimum).

If we determine the sign of $C_{sx}^{k,\lambda}[\tau(x)]$ on the point $x = x_0$, on which the function $\tau(x) = u(x, 0)$ achieves its positive maximum (negative minimum), then by using (1.12), (1.14), (1.15) and (1.16), the sign of $\lim\limits_{y\to 0} u_y(x_0, y)$, will be known. This fact is an analogy of Zarembo-Giraud's principle for equation (1.1) in the domain Ω. Therefore in this work, along with the establishment of an EP for equation (1.1), a similar result is obtained for the operator $C_{sx}^{k,\lambda}$, when $k = 0$ or $k = 1$ and $s = a$ or $s = b$.

Upon installing an EP for equation (1.1) and for operators $C_{sx}^{k,\lambda}$, we can use the equalities

$$\int\limits_0^{+\infty} [z(z+1)]^{\beta-1} e^{-c\left(\frac{1}{2}+z\right)} \overline{I}_{\beta-1}\left[c\sqrt{z(z+1)}\right] dz = B(\beta, 1-2\beta), \quad c \geq 0, \quad (1.19)$$

$$\int\limits_0^{+\infty} [z(z+1)]^{-\frac{1}{2}} e^{-c\left(\frac{1}{2}+z\right)} \overline{I}_{1}\left[c\sqrt{z(z+1)}\right] dz = \frac{2}{c}(1-e^{-\frac{c}{2}}), \quad c > 0, \quad (1.20)$$

$$\int\limits_0^{x} z^{\alpha+\gamma-1}(x-z)^{\delta-1} e^{-cz} \overline{I}_{\gamma}(cz) dz = x^{\alpha+\delta+\gamma-1} \Gamma\left[\begin{array}{c} \alpha+\gamma, \quad \delta \\ \alpha+\delta+\gamma \end{array}\right] \quad (1.21)$$

$$\times {}_2F_2(\gamma+\frac{1}{2}, \alpha+\gamma; 2\gamma+1, \alpha+\delta+\gamma; -2cx), \qquad x, \mathrm{Re}\delta, \mathrm{Re}(\alpha+\delta) > 0,$$

$$\int\limits_x^{+\infty} z^{2\beta-2} e^{-cz} \overline{I}_{\beta-1}(cz) dz = \frac{x^{2\beta-1}}{1-2\beta} {}_1F_1\left(\beta-\frac{1}{2}; 2\beta; -2cx\right), \ c \geq 0, \ x > 0, \ (1.22)$$

which can be proved using formulas 6, 8 and 1, 2 in pages 309 and 305 of handbook [13], respectively. Here $\overline{I}_s(z) = \Gamma(s+1)(z/2)^{-s} I_s(z)$, and $I_s(z)$ is a modified Bessel function of the order s [8]; $B(\alpha, \delta)$ is Euler's beta-function [7]; ${}_1F_1(\alpha; \delta; z)$, ${}_2F_2(\alpha, \delta; \gamma, \theta; z)$ are generalized hypergeometric functions [7]

$${}_1F_1(\alpha; \delta; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\delta)_n} \frac{z^n}{n!}, \qquad {}_2F_2(\alpha, \delta; \gamma, \theta; z) \sum_{n=0}^{\infty} \frac{(\alpha)_n (\delta)_n}{(\gamma)_n (\theta)_n} \frac{z^n}{n!},$$

where $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$ is Pohgammer's symbol [7].

2. Extremum principle for the class of equations (1.1)

Theorem 2.1. *Let $u(x, y)$ be a non-trivial and continuous in $\overline{\Omega}$ solution of equation (1.1) at $m > 0$, $\lambda \in R$, vanishing on $\overline{AC}$ (or on $\overline{BC}$). Then, the function $|u(x, y)|$ attains its maximum in $\overline{\Omega}$ on $\overline{AB}$.*

Proof. Consider, for example, case $u|_{\overline{AC}} = 0$. In this case function $u(x,y)$ has a form (1.2). We make up a difference $|u(x_0,0)| - |u(x,y)| = L_1(x,y)$, where $x_0 \in (a,b], (x,y) \in \overline{\Omega} \setminus \overline{AB}$ and investigate it. By virtue of (1.2) and designation $u(x,0) = \tau(x)$ we have

$$L_1(x,y) = |\tau(x_0)| - \gamma(\eta-\xi)^{1-2\beta}\left|\int\limits_a^{\xi} \frac{\tau(t)\overline{J}_{\beta-1}[\lambda\sqrt{(\xi-t)(\eta-t)}]}{[(\xi-t)(\eta-t)]^{1-\beta}}dt\right|. \qquad (2.1)$$

Replacing variables by formula $t = \xi - (\eta-\xi)z$ in the integral of (2.1), we have

$$L_1(x,y) \geq |\tau(x_0)|$$
$$-\gamma \int\limits_0^{(\xi-a)/(\eta-\xi)} \left|\tau[\xi-(\eta-\xi)z]\overline{J}_{\beta-1}\left[\lambda(\eta-\xi)\sqrt{z(z+1)}\right]\right| [z(z+1)]^{\beta-1}dz.$$

Using an equality, obtained from (1.19) at $c = 0$:

$$\int\limits_0^{+\infty} [z(z+1)]^{\beta-1}dz = B(\beta, 1-2\beta), \qquad (2.2)$$

the last inequality we can rewrite in the form

$$L_1(x,y) \geq \gamma|\tau(x_0)| \int\limits_{(\xi-a)/(\eta-\xi)}^{+\infty} [z(z+1)]^{\beta-1}dz \qquad (2.3)$$
$$+\gamma \int\limits_0^{(\xi-a)/(\eta-\xi)} \left\{|\tau(x_0)| - \left|\tau[\xi-(\eta-\xi)z]\overline{J}_{\beta-1}\left[\lambda(\eta-\xi)\sqrt{z(z+1)}\right]\right|\right\}[z(z+1)]^{\beta-1}dz.$$

Since $u(x,y) \not\equiv 0$ in $\overline{\Omega}$, then $u(x,0) = \tau(x) \not\equiv 0$ on $[a,b]$ (in the opposite case from (1.2) it follows that $u(x,y) \equiv 0$ in $\overline{\Omega}$). Therefore $\max\limits_{[a,b]}|\tau(t)| > 0$. Let $\max\limits_{[a,b]}|\tau(t)| = |\tau(x_0)|$. Then, by virtue of $\lambda \in \mathrm{R}$ and $|\overline{J}_{\beta-1}(\lambda x)| \leq 1$, the inequality

$$|\tau(x_0)| - \left|\tau[\xi-(\eta-\xi)z]\overline{J}_{\beta-1}\left[\lambda(\eta-\xi)\sqrt{z(z+1)}\right]\right| \geq 0, \quad (x,y) \in \overline{\Omega}\backslash\overline{AB}. \qquad (2.4)$$

is true.

Besides, for the remainder of convergent positive integral, (2.2)

$$\int\limits_{(\xi-a)/(\eta-\xi)}^{+\infty} [z(z+1)]^{\beta-1}dz > 0, \quad (x,y) \in \overline{\Omega}\backslash\overline{AB} \qquad (2.5)$$

is valid. Taking into account inequalities (2.4), (2.5) and $|\tau(x_0)| > 0$, $\gamma > 0$, from (2.3) we found that $L_1(x,y) > 0$, $(x,y) \in \overline{\Omega}\backslash\overline{AB}$, hence the statement of Theorem 2.1 follows. □

The case $u|_{\overline{BC}} = 0$, can be proved similarly, but only using formula (1.3) instead of (1.2).

Theorem 2.2. *Let $u(x, y)$ be a non-trivial and continuous in $\overline{\Omega}$ solution of (1.1) at $m > 0$, $i\lambda \in R$, vanishing on $\overline{AC}$. Then, if the maximum (minimum) in $\overline{\Omega}$ of the function $e^{-|\lambda|x}u(x, y)$ is positive (negative), then it is achieved on $\overline{AB}$.*

Proof. As λ is a pure imaginary number, according formula (1.2) and $\overline{J}_s(iz) = \overline{I}_s(|z|)$, the function $u(x, y)$ has the form

$$u(x,y) = \gamma(\eta-\xi)^{1-2\beta}\int_a^{\xi}\frac{\tau(t)\overline{I}_{\beta-1}[|\lambda|\sqrt{(\xi-t)(\eta-t)}]}{[(\xi-t)(\eta-t)]^{1-\beta}}dt. \tag{2.6}$$

Compose a difference $e^{-|\lambda|x_0}u(x_0, 0) - e^{-|\lambda|x}u(x, y) = L_2(x, y)$, where $x_0 \in (a, b]$, $(x, y) \in \overline{\Omega}\backslash\overline{AB}$, and investigate it.

Introducing the designation $T(x) = e^{-|\lambda|x}\tau(x)$ and taking (2.6) into account, we get

$$L_2(x,y) = T(x_0) - \gamma(\eta-\xi)^{1-2\beta}\int_a^{\xi}\frac{T(t)\overline{I}_{\beta-1}[|\lambda|\sqrt{(\xi-t)(\eta-t)}]}{[(\xi-t)(\eta-t)]^{1-\beta}}e^{|\lambda|(t-x)}dt.$$

Replacing the variables $t = \xi - (\eta - \xi)z$ in the integral, we have

$$L_2(x,y) = T(x_0) - \gamma\int_0^{(\xi-a)/(\eta-\xi)} T[\xi-(\eta-\xi)z]e^{-|\lambda|(\eta-\xi)(\frac{1}{2}+z)}\frac{\overline{I}_{\beta-1}\left[|\lambda|(\eta-\xi)\sqrt{z(z+1)}\right]}{[z(z+1)]^{1-\beta}}dz.$$

Using equality, obtained from (1.19) at $c = |\lambda|(\eta - \xi)$, the function $L_2(x, y)$ can be rewritten in the form

$$\begin{aligned} L_2(x,y) = \gamma T(x_0)&\int_{(\xi-a)/(\eta-\xi)}^{+\infty} e^{-|\lambda|(\eta-\xi)(\frac{1}{2}+z)}\frac{\overline{I}_{\beta-1}\left[|\lambda|(\eta-\xi)\sqrt{z(z+1)}\right]}{[z(z+1)]^{1-\beta}}dz \\ &+\gamma\int_0^{(\xi-a)/(\eta-\xi)}\{T(x_0)-T[\xi-(\eta-\xi)z]\}\,e^{-|\lambda|(\eta-\xi)(\frac{1}{2}+z)} \\ &\times\overline{I}_{\beta-1}\left[|\lambda|(\eta-\xi)\sqrt{z(z+1)}\right][z(z+1)]^{\beta-1}dz. \end{aligned} \tag{2.7}$$

Suppose that $\max\limits_{\overline{\Omega}} e^{-|\lambda|x}u(x, y) > 0$. Then the inequality $\max\limits_{[a,b]} T(t) > 0$ holds. Otherwise, from (2.6) it follows that $e^{-|\lambda|x}u(x, y) \leq 0$, $(x, y) \in \overline{\Omega}$, which is impossible.

Let $\max\limits_{[a,b]} T(t) = T(x_0)$. Then an inequality

$$T(x_0) - T[\xi-(\eta-\xi)z] \geq 0, \quad \forall(x,y) \in \overline{\Omega} \tag{2.8}$$

is true. Besides, using the remainder of convergent positive integral (1.19) we get

$$\int\limits_{(\xi-a)/(\eta-\xi)}^{+\infty} e^{-|\lambda|(\eta-\xi)(\frac{1}{2}+z)} \frac{\overline{I}_{\beta-1}\left[|\lambda|(\eta-\xi)\sqrt{z(z+1)}\right]}{[z(z+1)]^{1-\beta}}dz > 0,$$

$$\forall(x,y) \in \overline{\Omega}\backslash\overline{AB}. \tag{2.9}$$

By virtue of inequalities (2.8), (2.9) and $T(x_0) > 0$, $\gamma > 0$, from (2.7) it follows that $L_2(x,y) > 0$, $(x,y) \in \overline{\Omega}\backslash\overline{AB}$, hence the statement of Theorem 2.2 follows. In the case $\min\limits_{\overline{\Omega}} e^{-|\lambda|x}u(x,y) < 0$, Theorem 2 can be proved similarly. □

Theorem 2.3. *Let $u(x,y)$ be a non-trivial and continuous in $\overline{\Omega}$ solution of* (1.1) *at $m > 0$, $i\lambda \in R$, vanishing on $\overline{BC}$. Then, if the maximum* (*minimum*) *in $\overline{\Omega}$ of the function $e^{|\lambda|x}u(x,y)$ is positive* (*negative*), *then it is achieved on $\overline{AB}$.*

The proof of Theorem (1.3) can be obtained in a similar way using formula (1.3) instead of (1.2).

Theorem 2.4. *Let $u(x,y)$ be a non-trivial and continuous in $\overline{\Omega}$ solution of* (1.1) *at $m > 0$, $\lambda = \lambda_1 + i\lambda_2$, $\lambda_1\lambda_2 \neq 0$, λ_1, $\lambda_2 \in R$, vanishing on $\overline{AC}$* (*or on $\overline{BC}$*). *Then the maximum in $\overline{\Omega}$ of the function $e^{-|\lambda|x}|u(x,y)|$ $\left[e^{|\lambda|x}|u(x,y)|\right]$ will be achieved on $\overline{AB}$.*

Proof. We consider, for example, case $u(x,y)|_{\overline{BC}} = 0$. In this case function $u(x,y)$ is determined by formula (1.3). We make up the difference

$$e^{|\lambda|x_0}|u(x_0,0)| - e^{|\lambda|x}|u(x,y)| = L_3(x,y), \quad x_0 \in (a,b],\ (x,y) \in \overline{\Omega} \setminus \overline{AB}.$$

Using formula (1.3) and the designation $u(x,0) = \tau(x)$ we get

$$L_3(x,y) = e^{|\lambda|x_0}|\tau(x_0)| - \gamma(\eta-\xi)^{1-2\beta}e^{|\lambda|x}\left|\int\limits_{\eta}^{b} \frac{\tau(t)\overline{J}_{\beta-1}\left[\lambda\sqrt{(t-\xi)(t-\eta)}\right]}{[(t-\xi)(t-\eta)]^{1-\beta}}dt\right|.$$

Replacing variables by the formula $t = \eta + (\eta-\xi)z$ in the integral and introducing designation $T(x) = e^{|\lambda|x}\tau(x)$, we have

$$L_3(x,y) = |T(x_0)| - \gamma\left|\int\limits_{0}^{(b-\eta)/(\eta-\xi)} T[\eta+(\eta-\xi)z]\right.$$
$$\left.\times\, e^{-|\lambda|(\eta-\xi)(\frac{1}{2}+z)}\overline{J}_{\beta-1}\left[\lambda(\eta-\xi)\sqrt{z(z+1)}\right][z(z+1)]^{\beta-1}dz\right|.$$

Using equality (1.19), from the last equality we obtain

$$L_3(x,y) \geq |T(x_0)|\gamma$$
$$\times \int\limits_{(b-\eta)/(\eta-\xi)}^{+\infty} [z(z+1)]^{\beta-1}e^{-|\lambda|(\eta-\xi)(\frac{1}{2}+z)}\overline{I}_{\beta-1}\left[|\lambda|(\eta-\xi)\sqrt{z(z+1)}\right]dz$$

$$+\gamma \int\limits_0^{(b-\eta)/(\eta-\xi)} [z(z+1)]^{\beta-1} e^{-|\lambda|(\eta-\xi)(\frac{1}{2}+z)} \left\{ |T(x_0)| \overline{I}_{\beta-1}\left[|\lambda|\sqrt{z(z+1)}\right] \right.$$
$$\left. - \left| T[\eta+(\eta-\xi)z] \overline{J}_{\beta-1}\left[\lambda(\eta-\xi)\sqrt{z(z+1)}\right] \right| \right\} dz. \qquad (2.10)$$

It is evident that by virtue of $u(x,y) \not\equiv 0$ in $\overline{\Omega}$, the inequality $\tau(x) \not\equiv 0$, $x \in [a,b]$ is valid. Therefore $\max\limits_{[a,b]} |T(x)| > 0$. Let $\max\limits_{[a,b]} |T(x)| = |T(x_0)|$. Then it is not difficult to verify that, for $\forall (x,y) \in \overline{\Omega} \setminus \overline{AB}$, the inequality

$$|T(x_0)| \overline{I}_{\beta-1}\left[|\lambda|(\eta-\xi)\sqrt{z(z+1)}\right]$$
$$- \left| T[\eta+(\eta-\xi)z] \overline{J}_{\beta-1}\left[\lambda(\eta-\xi)\sqrt{z(z+1)}\right] \right| \geq 0 \qquad (2.11)$$

is true. Moreover, using the remainder of convergent positive integral (1.19) we obtain, $\forall (x,y) \in \overline{\Omega} \setminus \overline{AB}$,

$$\int\limits_{(b-\eta)/(\eta-\xi)}^{+\infty} [z(z+1)]^{\beta-1} e^{-|\lambda|(\eta-\xi)(\frac{1}{2}+z)} \overline{I}_{\beta-1}\left[|\lambda|(\eta-\xi)\sqrt{z(z+1)}\right] dz > 0. \quad (2.12)$$

Taking $|T(x_0)| > 0, \gamma > 0$ into account and inequalities (2.11), (2.12), from (2.10) we obtain that $L_3(x,y) > 0$, $\forall (x,y) \in \overline{\Omega} \setminus \overline{AB}$, hence the statement of Theorem 2.4 follows. □

Theorem 2.4 can be proved similarly in the case $u|_{\overline{AC}} = 0$.

Theorem 2.5. *Let $u(x,y)$ be a non-trivial and continuous in $\overline{\Omega}$ solution of the problem* (1.4) *at $\lambda \in R$, vanishing on $\overline{AC}$ (or on $\overline{BC}$). Then the maximum in $\overline{\Omega}$ of the function $e^{-|\lambda|x}|u(x,y)|$ $[e^{|\lambda|x}|u(x,y)|]$ is achieved on $\overline{AB}$.*

Proof. We consider the case $u(x,y)|_{\overline{AC}} = 0$. Then the function $u(x,y)$ has the form (1.10). We make up the difference $e^{-|\lambda|x_0}|u(x_0,0)| - e^{-|\lambda|x}|u(x,y)| = L_4(x,y)$, where $x_0 \in (a,b]$, $(x,y) \in \overline{\Omega} \setminus \overline{AB}$. Introducing the designation $e^{-|\lambda|x}u(x,0) = T(x)$ and taking (1.10) into account, we obtain

$$L_4(x,y) = |T(x_0)| - \left| T(\xi)e^{|\lambda|y} - \frac{\lambda^2}{4}(\eta-\xi)\int\limits_a^{\xi} T(t)e^{|\lambda|(t-x)} \overline{J}_1\left[\lambda\sqrt{(\xi-t)(\eta-t)}\right] dt \right|.$$

From here we have

$$L_4(x,y) \geq |T(x_0)| - |T(\xi)|e^{|\lambda|y} - \frac{\lambda^2}{2}(-y)\int\limits_a^{\xi} \left| T(t)e^{|\lambda|(t-x)} \overline{J}_1\left[\lambda\sqrt{(\xi-t)(\eta-t)}\right] \right| dt.$$

Transforming the right-hand side of this inequality, we rewrite it in the form

$$L_4(x,y) \geq |T(x_0)|L_5(x,y)e^{|\lambda|y} + [|T(x_0)| - |T(\xi)|]\, e^{|\lambda|y} \tag{2.13}$$
$$+ \frac{\lambda^2}{2}(-y)\int\limits_a^\xi \left\{|T(x_0)| - \left|T(t)\overline{J}_1\left[\lambda\sqrt{(\xi-t)(\eta-t)}\right]\right|\right\} e^{|\lambda|(t-x)}dt,$$

where $L_5(x,y) = e^{-|\lambda|y} - 1 + |\lambda|(-y)[e^{|\lambda|(a-\xi)} - 1]/2$.

Since $u(x,y) \not\equiv 0$ in $\overline{\Omega}$, then $u(x,0) = \tau(x) \not\equiv 0$ on $[a,b]$. Therefore $\max\limits_{[a,b]} e^{-|\lambda|x}|u(x,0)| = \max\limits_{[a,b]} |T(x)| > 0$.

Let $\max\limits_{[a,b]} |T(x)| = |T(x_0)|$. Then for $\forall(x,y) \in \overline{\Omega}\backslash\overline{AB}$ inequalities

$$|T(x_0)| - |T(\xi)| \geq 0, \qquad |T(x_0)| - \left|T(t)\overline{J}_1\left[\lambda\sqrt{(\xi-t)(\eta-t)}\right]\right| \geq 0 \tag{2.14}$$

are true.

Substituting in $L_5(x,y)$ an expansion of the function $e^{-|\lambda|y}$ in series, we have

$$L_5(x,y) = |\lambda|(-y)\left\{\frac{1}{2}e^{|\lambda|(a-\xi)} + \frac{1}{2} + \cdots + \frac{1}{n!}[|\lambda|(-y)]^{n-1} + \cdots\right\}.$$

Hence it follows that

$$L_5(x,y) \geq 0, \qquad \forall(x,y) \in \overline{\Omega}\backslash\overline{AB}, \tag{2.15}$$

moreover at $\lambda \neq 0$ strict inequality is fulfilled.

Taking inequalities (2.14), (2.15) and $|T(x_0)| > 0$, into account from (2.13) we find that $L_4(x,y) \geq 0$, $\forall(x,y) \in \overline{\Omega}\backslash\overline{AB}$. Theorem 2.5 is proved. □

Using formula (1.11), one can prove Theorem 2.5 in the case $u|_{\overline{BC}} = 0$.

Theorem 2.6. *Let $u(x,y)$ be a non-trivial and continuous in $\overline{\Omega}$ solution of the equation* (1.4) *at $i\lambda \in R$, vanishing on $\overline{AC}$. Then if the maximum* (*minimum*) *of the function $e^{-|\lambda|x}u(x,y)$ in $\overline{\Omega}$ is positive* (*negative*), *then it is achieved on $\overline{AB}$.*

Proof. Since $i\lambda \in \mathrm{R}$ and $\overline{J}_1(ix) = \overline{I}_1(|x|)$, then according to formula (1.10), function $u(x,y)$ has the form

$$u(x,y) = \tau(\xi) - \frac{|\lambda|^2}{2}y\int\limits_a^\xi \tau(t)\overline{I}_1[|\lambda|\sqrt{(\xi-t)(\eta-t)}]dt. \tag{2.16}$$

We make up the difference $e^{-|\lambda|x_0}u(x_0,0) - e^{-|\lambda|x}u(x,y) = L_6(x,y)$, where $x_0 \in (a,b]$, $(x,y) \in \overline{\Omega}\backslash\overline{AB}$.

Introducing designation $e^{-|\lambda|x}u(x,0) = T(x)$ and using formula (2.16), we obtain

$$L_6(x,y) = T(x_0) - T(\xi)e^{|\lambda|y} + \frac{|\lambda|^2}{2}y\int\limits_a^\xi T(t)e^{|\lambda|(t-x)}\overline{I}_1[|\lambda|\sqrt{(\xi-t)(\eta-t)}]dt.$$

Replacing the variables $t = \xi - (\eta - \xi)z$, and taking $\overline{I}_1(z) = (2/z)I_1(z)$ into account, from the last equality we have

$$L_6(x,y) = T(x_0) - T(\xi)e^{|\lambda|y} + |\lambda|y \int\limits_0^{(\xi-a)/(\eta-\xi)} T[\xi - (\eta-\xi)z] \times [z(z+1)]^{-\frac{1}{2}} e^{-|\lambda|(\eta-\xi)(\frac{1}{2}+z)} I_1[|\lambda|(\eta-\xi)\sqrt{z(z+1)}]dz.$$

Performing some evaluations in the right side of this equality, we find

$$L_6(x,y) = [T(x_0) - T(\xi)]e^{|\lambda|y} + T(x_0)L_7(x,y) + |\lambda|(-y) \int\limits_0^{(\xi-a)/(\eta-\xi)} \{T(x_0) - T[\xi - (\eta-\xi)z]\} \times [z(z+1)]^{-\frac{1}{2}} e^{-|\lambda|(\eta-\xi)(\frac{1}{2}+z)} I_1[|\lambda|(\eta-\xi)\sqrt{z(z+1)}]dz, \tag{2.17}$$

where

$$L_7(x,y) = 1 - e^{|\lambda|y} + |\lambda|y \int\limits_0^{(\xi-a)/(\eta-\xi)} [z(z+1)]^{-\frac{1}{2}} e^{-|\lambda|(\eta-\xi)(\frac{1}{2}+z)} I_1[|\lambda|(\eta-\xi)\sqrt{z(z+1)}]dz.$$

Assume that $\max\limits_{\overline{\Omega}} e^{-|\lambda|x}u(x,y) > 0$. Then inequality $\max\limits_{[a,b]} T(x) > 0$ is valid.

Let $\max\limits_{[a,b]} T(x) = T(x_0)$. Then inequalities

$$T(x_0) - T(\xi) \geq 0, \qquad T(x_0) - T[\xi - (\eta-\xi)z] \geq 0, \qquad \forall (x,y) \in \overline{\Omega} \setminus \overline{AB} \tag{2.18}$$

are true.

Using equality (1.20), one can easily prove that

$$L_7(x,y) = |\lambda|(-y) \int\limits_{(\xi-a)/(\eta-\xi)}^{+\infty} [z(z+1)]^{-\frac{1}{2}} e^{-|\lambda|(\eta-\xi)(\frac{1}{2}+z)} I_1[|\lambda|(\eta-\xi)\sqrt{z(z+1)}]dz \geq 0,$$

moreover at $\lambda \neq 0$ strict inequality is fulfilled.

Taking this and inequalities (2.18), $T(x_0) > 0$, from (2.17) we obtain that $L_6(x,y) \geq 0, (x,y) \in \overline{\Omega} \setminus \overline{AB}$. Theorem 2.6 is proved. □

The case $\min\limits_{\overline{\Omega}} e^{-|\lambda|x}|u(x,y)| < 0$ can be proved similarly.

The following theorems are also valid.

Theorem 2.7. *Let $u(x,y)$ be a non-trivial and continuous solution of the equation (1.4) in $\overline{\Omega}$ at $i\lambda \in R$, vanishing on $\overline{BC}$. Then if the maximum (minimum) of the function $e^{|\lambda|x}u(x,y)$ in $\overline{\Omega}$ is positive (negative), then it is achieved on $\overline{AB}$.*

Theorem 2.8. *Let $u(x,y)$ be a non-trivial and continuous solution of the equation* (1.4) *in* $\overline{\Omega}$ *at* $\lambda=\lambda_1+i\lambda_2$, $\lambda_1\lambda_2\neq 0$, $\lambda_,\lambda_2\in R$, *vanishing on* $\overline{AC}$ (*or on* $\overline{BC}$). *Then the maximum of function* $e^{-|\lambda|x}|u(x,y)|$ $\left[e^{|\lambda|x}|u(x,y)|\right]$ *in* $\overline{\Omega}$ *is achieved on* $\overline{AB}$.

We omit a proof since it can be done as in Theorems 2.4 and 2.6.

3. Extremum principle for the operator $C_{sx}^{k,\lambda}$

Before establishing of an EP for operators $C_{sx}^{1,\lambda}$, we change it to a more convenient form for further investigation.

We rewrite $C_{sx}^{1,\lambda}$ in the form

$$C_{sx}^{1,\lambda}[\tau(x)]\equiv\frac{1}{\Gamma(2\beta)}\frac{d}{dx}\int\limits_s^x\frac{\overline{J}_{\beta-1}[\lambda(x-t)]}{|x-t|^{1-2\beta}}\tau(t)dt \tag{3.1}$$

$$+\frac{\lambda^2\mathrm{sign}(x-s)}{4\beta(\beta+1)\Gamma(2\beta)}\int\limits_s^x|x-t|^{2\beta}\overline{J}_{\beta+1}[\lambda(x-t)]\tau(t)dt+L_8(x,y),$$

where

$$L_8(x,y)=\frac{1}{\Gamma(2\beta)}\frac{d}{dx}\int\limits_s^x\frac{\overline{J}_{\beta}[\lambda(x-t)]-\overline{J}_{\beta-1}[\lambda(x-t)]}{|x-t|^{1-2\beta}}\tau(t)dt.$$

By virtue of equality (1.7),

$$L_8(x,y)=\frac{\lambda^2}{4\beta(1+\beta)\Gamma(2\beta)}\frac{d}{dx}\int\limits_s^x|x-t|^{1+2\beta}\overline{J}_{\beta+1}[\lambda(x-t)]\tau(t)dt.$$

Differentiating and taking formula (1.18) into account, we have

$$L_8(x,y)=\frac{\lambda^2(1+2\beta)\mathrm{sign}(x-s)}{4\beta(\beta+1)\Gamma(2\beta)}\int\limits_s^x|x-t|^{2\beta}\overline{J}_{\beta+1}[\lambda(x-t)]\tau(t)dt$$

$$-\frac{\lambda^2\mathrm{sign}(x-s)}{2\beta\Gamma(2\beta)}\int\limits_s^x|x-t|^{2\beta}\frac{\lambda^2(x-t)^2}{4(\beta+1)(\beta+2)}\overline{J}_{\beta+2}[\lambda(x-t)]\tau(t)dt.$$

Applying formula (1.7) to the function $\overline{J}_{\beta+2}[\lambda(x-t)]$, we find

$$L_8(x,y)=\frac{\lambda^2(1+2\beta)\mathrm{sign}(x-s)}{4\beta(\beta+1)\Gamma(2\beta)}\int\limits_s^x|x-t|^{2\beta}\overline{J}_{\beta+1}[\lambda(x-t)]\tau(t)dt \tag{3.2}$$

$$+\frac{\lambda^2\mathrm{sign}(x-s)}{2\beta\Gamma(2\beta)}\int\limits_s^x|x-t|^{2\beta}\left\{\overline{J}_{\beta}[\lambda(x-t)]-\overline{J}_{\beta+1}[\lambda(x-t)]\right\}\tau(t)dt.$$

Substituting (3.2) into (3.1), we find

$$C_{sx}^{1,\lambda}[\tau(x)] \equiv \frac{1}{\Gamma(2\beta)} \frac{d}{dx} \int\limits_s^x \frac{\overline{J}_{\beta-1}[\lambda(x-t)]\tau(t)}{|x-t|^{1-2\beta}} dt + \frac{\lambda^2 \mathrm{sign}(x-s)}{\Gamma(1+2\beta)} \int\limits_s^x |x-t|^{2\beta} \overline{J}_\beta[\lambda(x-t)]\tau(t)dt. \quad (3.3)$$

Theorem 3.1. *Let* $\lambda \in R$, $\tau(x) \in C^{(0,\alpha)}[a,b]$, $\alpha > 1-2\beta$ *and* $\max\limits_{[a,b]} |\tau(x)| = |\tau(x_0)| > 0$, $x_0 \in (a,b)$. *Then, if* $\tau(x_0) > 0$ (< 0), *then the inequalities*

$$C_{ax}^{1,\lambda}[\tau(x)]|_{x=x_0} > 0 \ (< 0), \qquad C_{bx}^{1,\lambda}[\tau(x)]|_{x=x_0} > 0 \ (< 0) \quad (3.4)$$

are true.

Proof. Consider the operator $C_{ax}^{1,\lambda}$ in the form (3.3) and rewrite it in the form

$$C_{ax}^{1,\lambda}[\tau(x)] = \lim_{\epsilon\to 0} \frac{1}{\Gamma(2\beta)} \frac{d}{dx} \int\limits_a^{x-\epsilon} \frac{\overline{J}_{\beta-1}[\lambda(x-t)]}{(x-t)^{1-2\beta}} \tau(t)dt + \frac{\lambda^2}{\Gamma(1+2\beta)} \int\limits_a^x (x-t)^{2\beta} \overline{J}_\beta[\lambda(x-t)]\tau(t)dt. \quad (3.5)$$

Differentiating in (3.5) and using formula (1.18), we have

$$\Gamma(2\beta) C_{ax}^{1,\lambda}[\tau(x)] = \lim_{\epsilon\to 0} \left\{ \epsilon^{2\beta-1} \overline{J}_{\beta-1}(\lambda\epsilon)\tau(x-\epsilon) - (1-2\beta) \int\limits_a^{x-\epsilon} (x-t)^{2\beta-2} \overline{J}_{\beta-1}[\lambda(x-t)]\tau(t)dt \right\}.$$

Adding and subtracting expression $(1-2\beta)\tau(x) \int\limits_a^{x-\epsilon} (x-t)^{2\beta-2}dt$, from here we have

$$\Gamma(2\beta) C_{ax}^{1,\lambda}[\tau(x)] = \lim_{\epsilon\to 0} \left\{ \epsilon^{2\beta-1} [\overline{J}_{\beta-1}(\lambda\epsilon)\tau(x-\epsilon) - \tau(x)] + \frac{\tau(x)}{(x-a)^{1-2\beta}} + (1-2\beta) \int\limits_a^{x-\epsilon} \frac{\tau(x) - \tau(t)\overline{J}_{\beta-1}[\lambda(x-t)]}{(x-t)^{2-2\beta}} dt \right\}. \quad (3.6)$$

Since $\tau(x) \in C^{(0,\alpha)}[a,b]$ and $\lambda \in \mathrm{R}$ is fixed, then equalities

$$|\tau(x) - \tau(x-\epsilon)\overline{J}_{\beta-1}(\lambda\epsilon)| = \epsilon^\alpha O(1),$$
$$|\tau(x) - \tau(t)\overline{J}_{\beta-1}[\lambda(x-t)]| = (x-t)^\alpha O(1)$$

are true.

By virtue of these equalities and $\alpha > 1-2\beta$, there exists a limit in (3.6) and the equality

$$\Gamma(2\beta)C_{ax}^{1,\lambda}[\tau(x)] = \tau(x)(x-a)^{2\beta-1} + (1-2\beta)\int\limits_a^x \frac{\tau(x)-\tau(t)\overline{J}_{\beta-1}[\lambda(x-t)]}{(x-t)^{2-2\beta}}dt \tag{3.7}$$

is valid.

In (3.7) we set $x = x_0$. Then, if $\tau(x_0) > 0$ (< 0), then $\tau(x_0)-\tau(t)\overline{J}_{\beta-1}[\lambda(x_0-t)] \geq 0$ (≤ 0) is true. If we consider this and $\tau(x_0) > 0$ (< 0), then from (3.7) there follows the first inequality of (3.4). The second inequality can be proved similarly. □

Remark 3.2. In the work [7] when conditions of Theorem 3.1 and the condition $\delta \geq |\lambda|$ are fulfilled, the validity of inequalities

$$C_{ax}^{1,\lambda}[e^{\delta x}\tau(x)]|_{x=x_0} > 0 \ \ (<0), \qquad C_{bx}^{1,\lambda}[e^{-\delta x}\tau(x)]|_{x=x_0} > 0 \ \ (<0)$$

is proved.

Theorem 3.3. *Let $i\lambda \in R$, $T(x) \in C^{(0,\alpha)}[a,b]$, $\alpha > 1-2\beta$ and $\max\limits_{[a,b]} T(x) = T(x_0) > 0$ $\left[\min\limits_{[a,b]} T(x) = T(x_0) < 0\right]$, $x_0 \in (a,b)$. Then the inequalities*

$$C_{ax}^{1,\lambda}[e^{|\lambda|x}T(x)]|_{x=x_0} > 0 \ \ (<0), \qquad C_{bx}^{1,\lambda}[e^{-|\lambda|x}T(x)]|_{x=x_0} > 0 \ \ (<0) \tag{3.8}$$

are true.

Proof. Considering $i\lambda \in \mathrm{R}$, $\overline{J}_{\beta-1}(ix) = \overline{I}_{\beta-1}(|x|)$ and a form (3.3) of the operator $C_{ax}^{1,\lambda}$, as in (3.5), we have

$$\Gamma(2\beta)C_{ax}^{1,\lambda}[e^{|\lambda|x}T(x)] = \lim_{\epsilon\to 0}\left\{\frac{d}{dx}\int\limits_a^{x-\epsilon}(x-t)^{2\beta-1}\overline{I}_{\beta-1}[|\lambda|(x-t)]e^{|\lambda|t}T(t)dt - \frac{|\lambda|^2}{2\beta}\int\limits_a^{x-\epsilon}(x-t)^{2\beta}\overline{I}_{\beta}[|\lambda|(x-t)]e^{|\lambda|t}T(t)dt\right\}.$$

After differentiating and applying formula (1.18), we get

$$\Gamma(2\beta)C_{ax}^{1,\lambda}[e^{|\lambda|x}T(x)] = \lim_{\epsilon\to 0}\left\{\epsilon^{2\beta-1}\overline{I}_{\beta-1}(|\lambda|\epsilon)e^{|\lambda|(x-\epsilon)}T(x-\epsilon) - (1-2\beta)\int\limits_a^{x-\epsilon}(x-t)^{2\beta-2}\overline{I}_{\beta-1}[|\lambda|(x-t)]e^{|\lambda|t}T(t)dt\right\}. \tag{3.9}$$

Integrating by parts and using equality (1.18), it is not difficult to verify that

$$(1-2\beta)\int\limits_a^{x-\epsilon}(x-t)^{2\beta-2}e^{|\lambda|t}\overline{I}_{\beta-1}[|\lambda|(x-t)]dt \tag{3.10}$$
$$=\epsilon^{2\beta-1}e^{|\lambda|(x-\epsilon)}\overline{I}_{\beta-1}(|\lambda|\epsilon)-(x-a)^{2\beta-1}e^{|\lambda|a}\overline{I}_{\beta-1}[|\lambda|(x-a)]$$
$$+|\lambda|\int\limits_a^{x-\epsilon}(x-t)^{2\beta-1}\left\{\frac{|\lambda|(x-t)}{2\beta}\overline{I}_{\beta}[|\lambda|(x-t)]-\overline{I}_{\beta-1}[|\lambda|(x-t)]\right\}e^{|\lambda|t}dt.$$

Considering (3.10), we can rewrite the expression (3.9) in the form

$$\Gamma(2\beta)C_{ax}^{1,\lambda}[e^{|\lambda|x}T(x)] \tag{3.11}$$
$$=\lim_{\epsilon\to 0}\Bigg\{\epsilon^{2\beta-1}\overline{I}_{\beta-1}(|\lambda|\epsilon)e^{|\lambda|(x-\epsilon)}[T(x-\epsilon)-T(x)]$$
$$+(1-2\beta)\int\limits_a^{x-\epsilon}(x-t)^{2\beta-2}[T(x)-T(t)]\overline{I}_{\beta-1}[|\lambda|(x-t)]e^{|\lambda|t}dt$$
$$+T(x)(x-a)^{2\beta-1}e^{|\lambda|a}\overline{I}_{\beta-1}[|\lambda|(x-a)]++T(x)\int\limits_a^{x-\epsilon}(x-t)^{2\beta-1}$$
$$\times\left[|\lambda|\overline{I}_{\beta-1}[|\lambda|(x-t)]-\frac{1}{2\beta}|\lambda|^2(x-t)\overline{I}_{\beta}[|\lambda|(x-t)]\right]e^{|\lambda|t}dt\Bigg\}.$$

Hence passing to a limit at $\epsilon\to 0$ and taking $T(x)\in C^{(0,\alpha)}[a,b], \alpha>1-2\beta$ into account, we have

$$\Gamma(2\beta)C_{ax}^{1,\lambda}[e^{|\lambda|x}T(x)] \tag{3.12}$$
$$=(1-2\beta)\int\limits_a^{x}(x-t)^{2\beta-2}[T(x)-T(t)]\overline{I}_{\beta-1}[|\lambda|(x-t)]e^{|\lambda|t}dt$$
$$+T(x)\Bigg\{(x-a)^{2\beta-1}e^{|\lambda|a}\overline{I}_{\beta-1}[|\lambda|(x-a)]$$
$$+\int\limits_a^{x}(x-t)^{2\beta-1}\left[|\lambda|\overline{I}_{\beta-1}[|\lambda|(x-t)]-\frac{1}{2\beta}|\lambda|^2(x-t)\overline{I}_{\beta}[|\lambda|(x-t)]\right]e^{|\lambda|t}dt\Bigg\}.$$

Using formula (1.21) one can easily show that

$$\int\limits_a^{x}(x-t)^{2\beta-1}e^{|\lambda|t}\overline{I}_{\beta-1}[|\lambda|(x-t)]dt \tag{3.13}$$
$$=\frac{1}{2\beta}(x-a)^{2\beta}e^{|\lambda|x}{}_2F_2\left[\beta-\frac{1}{2},2\beta;2\beta-1,2\beta+1;-2|\lambda|(x-a)\right],$$

$$\int_a^x (x-t)^{2\beta} e^{|\lambda| t} \overline{I}_\beta[|\lambda|(x-t)]dt \tag{3.14}$$
$$= \frac{1}{1+2\beta}(x-a)^{2\beta+1} e^{|\lambda|x} {}_1F_1\left[\beta+\frac{1}{2}; 2+2\beta; -2|\lambda|(x-a)\right].$$

Considering equality [7]

$$e^{-z}\overline{I}_{\beta-1}(z) = {}_1F_1\left(\beta-\frac{1}{2}; 2\beta-1; -2z\right), \qquad z>0$$

and using expansions of functions ${}_1F_1$ and ${}_2F_2$ into series, by a comparison of coefficients at equal degrees of z, it is not difficult to verify that

$$e^{-z}\overline{I}_{\beta-1}(z) + \frac{z}{2\beta}\, {}_2F_2\left(\beta-\frac{1}{2}, 2\beta; 2\beta-1, 2\beta+1; -2z\right) \tag{3.15}$$
$$-\frac{z^2}{2\beta(1+2\beta)} {}_1F_1\left(\beta+\frac{1}{2}; 2+2\beta; -2z\right) = {}_1F_1\left(\beta-\frac{1}{2}; 2\beta; -2z\right).$$

On the base of equalities (3.13), (3.14), (3.15), from (3.12) it follows that

$$\Gamma(2\beta)e^{-|\lambda|x} C_{ax}^{1,\lambda}[e^{|\lambda|x}T(x)]$$
$$= (1-2\beta)\int_a^x \frac{T(x)-T(t)}{(x-t)^{2-2\beta}} e^{|\lambda|(t-x)} \overline{I}_{\beta-1}[|\lambda|(x-t)]dt$$
$$+ T(x)(x-a)^{2\beta-1} {}_1F_1\left[\beta-\frac{1}{2}; 2\beta; -2|\lambda|(x-a)\right]. \tag{3.16}$$

Let $\max_{[a,b]} T(x) = T(x_0) > 0$ $\left[\min_{[a,b]} T(x) = T(x_0) < 0\right]$, $x_0 \in (a,b)$. Then $T(x_0) - T(t) \geq 0$ (≤ 0) for $\forall t \in [a,b]$. Besides from (1.22) it follows that ${}_1F_1\left[\beta-\frac{1}{2}; 2\beta; -2|\lambda|(x_0-a)\right] > 0$.

By virtue of the fact that $T(x_0) > 0$ (< 0), $1-2\beta > 0$, $\overline{I}_{\beta-1}[|\lambda|(x-t)] > 0$, from (3.16) at $x = x_0$ follows the first inequality of (3.8). The second part of the inequality (3.8) can be proved analogously. □

Remark 3.4. In the work [5], the inequality (3.8) is proved by fulfilling conditions of the Theorem 3.3 and $|\lambda| < 1/(b-a)$.

Remark 3.5. The equality (3.16) can be obtained by formula (2.6). In fact, introducing designation $T(x) = e^{-|\lambda|x}\tau(x)$ in (2.6), we have

$$e^{-|\lambda|x}u(x,y) = \gamma(\eta-\xi)^{1-2\beta}e^{-|\lambda|x}\int_a^\xi \frac{e^{|\lambda|t}T(t)\overline{I}_{\beta-1}[\lambda\sqrt{(\xi-t)(\eta-t)}]}{[(\xi-t)(\eta-t)]^{1-\beta}}dt. \tag{3.17}$$

From here, differentiating by y and passing to a limit at $y \to 0$, and according to (1.12), we have

$$\lim_{y\to 0}\frac{\partial}{\partial y}[e^{-|\lambda|x}u(x,y)] = \gamma_0 e^{-|\lambda|x} C_{ax}^{1,\lambda}[e^{|\lambda|x}T(x)]. \tag{3.18}$$

Further, taking $\overline{J}_{\beta-1}(iz) = \overline{I}_{\beta-1}(|z|)$ into account at $i\lambda \in \mathrm{R}$ and replacing the variables $t = \xi - (\eta - \xi)z$, from (3.17) we get

$$e^{-|\lambda|x}u(x,y) = \gamma \int\limits_{0}^{(\xi-a)/(\eta-\xi)} T[\xi - (\eta-\xi)z][z(z+1)]^{\beta-1}e^{-|\lambda|(\eta-\xi)(\frac{1}{2}+z)} \times \overline{I}_{\beta-1}[|\lambda|(\eta-\xi)\sqrt{z(z+1)}]dz. \tag{3.19}$$

We assume the expression

$$\frac{e^{-|\lambda|x}u(x,0) - e^{-|\lambda|x}u(x,y)}{0-y} = L_9(x,y). \tag{3.20}$$

Using equality (1.19) and the designation $T(x) = e^{-|\lambda|x}u(x,0)$, one can rewrite the expression $L_9(x,y)$ in the form

$$L_9(x,y) = \gamma(-y)^{-1}T(x) \int\limits_{(\xi-a)/(\eta-\xi)}^{+\infty} [z(z+1)]^{\beta-1}e^{-|\lambda|(\eta-\xi)(\frac{1}{2}+z)} \times \overline{I}_{\beta-1}\left[|\lambda|(\eta-\xi)\sqrt{z(z+1)}\right]dz + \gamma(-y)^{-1} \int\limits_{0}^{(\xi-a)/(\eta-\xi)} \{T(x) - T[\xi-(\eta-\xi)z]\}[z(z+1)]^{\beta-1} \times e^{-|\lambda|(\eta-\xi)(\frac{1}{2}+z)}\overline{I}_{\beta-1}\left[|\lambda|(\eta-\xi)\sqrt{z(z+1)}\right]dz. \tag{3.21}$$

Replacing variables by formula $t = \xi - (\eta - \xi)z$ in the integrals of (3.21) and substituting it into (3.20), and also passing to a limit at $y \to 0$, we have

$$\lim_{y\to 0}\frac{\partial}{\partial y}[e^{-|\lambda|x}u(x,y)] = \gamma(2-4\beta)^{1-2\beta}\left\{T(x)L_{10}(x) + \int\limits_{a}^{x}\frac{T(x)-T(t)}{(x-t)^{2-2\beta}}e^{|\lambda|(t-x)}\overline{I}_{\beta-1}[|\lambda|(x-t)]dt\right\}, \tag{3.22}$$

where

$$L_{10}(x) = \int\limits_{-a}^{+\infty}(x+t)^{2\beta-2}e^{-|\lambda|(t+x)}\overline{I}_{\beta-1}[|\lambda|(x+t)]dt.$$

Comparing (3.18) and (3.22), we get

$$\Gamma(2\beta)e^{-|\lambda|x}C_{ax}^{1,\lambda}[e^{|\lambda|x}T(x)] = (1-2\beta)T(x)L_{10}(x) + (1-2\beta)\int\limits_{a}^{x}\frac{T(x)-T(t)}{(x-t)^{2-2\beta}}e^{|\lambda|(t-x)}\overline{I}_{\beta-1}[|\lambda|(x-t)]dt. \tag{3.23}$$

Calculating integral $L_{10}(x)$ by the formula (1.22) and substituting it into (3.23) we obtain the equality (3.16).

Theorem 3.6. *Let* $\lambda = \lambda_1 + i\lambda_2$, $\lambda_1\lambda_2 \neq 0$, $\lambda_1, \lambda_2 \in R$; $T(x) \in C^{(0,\alpha)}[a,b]$, $\alpha > 1-2\beta$, *and the function* $\overline{T}(x)-$ *is conjugate to function* $T(x)$. *Then, inequalities*

$$Re\left\{\frac{\overline{T}(x)}{|T(x)|}C_{ax}^{1,\lambda}\left[e^{|\lambda|x}T(x)\right]\right\}\Bigg|_{x=x_0} > 0, \tag{3.24}$$

$$Re\left\{\frac{\overline{T}(x)}{|T(x)|}C_{bx}^{1,\lambda}\left[e^{-|\lambda|x}T(x)\right]\right\}\Bigg|_{x=x_0} > 0 \tag{3.25}$$

are true if $\max\limits_{[a,b]}|T(x)| = |T(x_0)| > 0$, $x_0 \in (a,b)$.

Proof. Assume in (3.5) that $\tau(x) = e^{|\lambda|x}T(x)$ and differentiating, we have

$$\Gamma(2\beta)C_{ax}^{1,\lambda}[e^{|\lambda|x}T(x)] = \lim_{\epsilon\to 0}\Bigg\{\epsilon^{2\beta-1}\overline{J}_{\beta-1}(\lambda\epsilon)e^{|\lambda|(x-\epsilon)}T(x-\epsilon) \tag{3.26}$$
$$-(1-2\beta)\int\limits_a^{x-\epsilon}(x-t)^{2\beta-2}\overline{J}_{\beta-1}[\lambda(x-t)]e^{|\lambda|t}T(t)dt\Bigg\}.$$

Adding and deducting the expression

$$(1-2\beta)T(x)\lim_{\epsilon\to 0}\int\limits_a^{x-\epsilon}(x-t)^{2\beta-2}\overline{I}_{\beta-1}[|\lambda|(x-t)]e^{|\lambda|t}dt$$

and taking equality (3.10) into account, from equality (3.26), we get

$$\Gamma(2\beta)C_{ax}^{1,\lambda}[e^{|\lambda|x}T(x)] \tag{3.27}$$
$$=\lim_{\epsilon\to 0}\Bigg\{\epsilon^{2\beta-1}e^{|\lambda|(x-\epsilon)}\left[\overline{J}_{\beta-1}(|\lambda|\epsilon)T(x-\epsilon)-\overline{I}_{\beta-1}(|\lambda|\epsilon)T(x)\right]$$
$$+(1-2\beta)\int\limits_a^{x-\epsilon}(x-t)^{2\beta-2}\left[T(x)\overline{I}_{\beta-1}[|\lambda|(x-t)]-T(t)\overline{J}_{\beta-1}[\lambda(x-t)]\right]e^{|\lambda|t}dt$$
$$+T(x)(x-a)^{2\beta-1}e^{|\lambda|a}\overline{I}_{\beta-1}[|\lambda|(x-a)]$$
$$+T(x)\int\limits_a^{x-\epsilon}(x-t)^{2\beta-1}\left[|\lambda|\overline{I}_{\beta-1}[|\lambda|(x-t)]-\frac{1}{2\beta}|\lambda|^2(x-t)\overline{I}_{\beta}[|\lambda|(x-t)]\right]e^{|\lambda|t}dt\Bigg\}.$$

Passing to a limit at $\epsilon \to 0$ and considering $T(x) \in C^{(0,\alpha)}[a,b]$, $\alpha > 1-2\beta$, and equalities (3.13), (3.14), (3.15), from (3.27) we have

$$\Gamma(2\beta)e^{-|\lambda|x}C_{ax}^{1,\lambda}[e^{|\lambda|x}T(x)]$$
$$= (1-2\beta)\int_a^x \left\{T(x)\overline{I}_{\beta-1}[|\lambda|(x-t)] - T(t)\overline{J}_{\beta-1}[\lambda(x-t)]\right\} e^{|\lambda|(t-x)}(x-t)^{2\beta-2}dt$$
$$+ T(x)(x-a)^{2\beta-1}{}_1F_1[\beta-1/2; 2\beta; -2|\lambda|(x-a)].$$

From here it follows that

$$\Gamma(2\beta)e^{-|\lambda|x}\mathrm{Re}\left\{\frac{\overline{T}(x)}{|T(x)|}C_{ax}^{1,\lambda}[e^{|\lambda|x}T(x)]\right\}$$
$$= (1-2\beta)\int_a^x \frac{|T(x)|\overline{I}_{\beta-1}[|\lambda|(x-t)] - \mathrm{Re}\left\{\frac{\overline{T}(x)}{|T(x)|}T(t)\overline{J}_{\beta-1}[\lambda(x-t)]\right\}}{(x-t)^{2-2\beta}}e^{|\lambda|(t-x)}dt$$
$$+ |T(x)|(x-a)^{2\beta-1}{}_1F_1[\beta-1/2; 2\beta; -2|\lambda|(x-a)]. \tag{3.28}$$

Let $\max\limits_{[a,b]}|T(x)| = |T(x_0)| > 0$, $x_0 \in (a,b)$. Then for $\forall t \in [a,b]$ the inequality

$$|T(x_0)|\overline{I}_{\beta-1}[|\lambda|(x_0-t)] - \mathrm{Re}\left\{\frac{\overline{T}(x_0)}{|T(x_0)|}T(t)\overline{J}_{\beta-1}[\lambda(x_0-t)]\right\} \geq 0 \tag{3.29}$$

is true.

If considering inequality (3.29) and

$$|T(x_0)| > 0,\ \Gamma(2\beta) > 0,\ {}_1F_1[\beta-1/2; \quad 2\beta; -2|\lambda|(x_0-a)] > 0,$$

then from equality (3.28) in the point $x = x_0$ follows inequality (3.24). □

Inequality (3.25) can be proved analogously.

For completeness of information we mention an EP for operators $C_{ax}^{0,\lambda}$ and $C_{bx}^{0,\lambda}$, which was established in [5].

Theorem 3.7. *Let $\lambda \in R$ and $T(x) \in C[a,b] \cap C^1(a,b)$. Then, at $T(x_0) > 0$ (< 0) the inequality*

$$\begin{aligned} C_{ax}^{0,\lambda}[e^{|\lambda|x}T(x)]|_{x=x_0} &\geq 0 \quad (\leq 0), \\ C_{bx}^{0,\lambda}[e^{-|\lambda|x}T(x)]|_{x=x_0} &\geq 0 \quad (\leq 0) \end{aligned} \tag{3.30}$$

is true if $\max\limits_{[a,b]}|T(x)| = |T(x_0)|$, $x_0 \in (a,b)$.

Theorem 3.8. *Let $i\lambda \in R$ and $T(x) \in C[a,b] \cap C^1(a,b)$. Then inequality* (3.30) *is true if $\max\limits_{[a,b]}T(x) = T(x_0) > 0 \quad [\min\limits_{[a,b]}T(x) = T(x_0) < 0]$, $x_0 \in (a,b)$.*

Theorem 3.9. *Let* $\lambda = \lambda_1 + i\lambda_2,\ \lambda_1\lambda_2 \neq 0,\ \lambda_1, \lambda_2 \in R;\ T(x) \in C[a,b] \cap C^1(0,1)$. *Then the inequalities*

$$Re\left\{\frac{\overline{T}(x)}{|T(x)|} C_{ax}^{0,\lambda}\left[e^{|\lambda|x}T(x)\right]\right\}\Big|_{x=x_0} \geq 0, \tag{3.31}$$

$$Re\left\{\frac{\overline{T}(x)}{|T(x)|} C_{bx}^{0,\lambda}\left[e^{-|\lambda|x}T(x)\right]\right\}\Big|_{x=x_0} \geq 0 \tag{3.32}$$

are true if $\max\limits_{[a,b]}|T(x)| = |T(x_0)| > 0, x_0 \in (a,b)$.

Remark 3.10. If in Theorems 3.7, 3.8 and 3.9 $\lambda \neq 0$, then in (3.30), (3.31) and (3.32) strict inequalities are valid.

Remark 3.11. The extremum principle for the expression

$$C_{sx}^{k,\lambda}[e^{\text{sign}(x-s)|\lambda|x}p(x)T(x)]$$

can be established by a similar method when $k = 0$ or $k = 1$, and $p(x)$ is a given function.

References

[1] E. Hopf, *Elementare Betrachtungen über die Lösungen partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus.* Sitzungb. Preuss. Akad. Wiss., **19** (1927), pp. 147–152.

[2] L. Nirenberg, *A strop maximum principle for parabolic equations.* Comm. Pure and Appl. Math., **6, 2** (1953), pp. 167–177.

[3] A.V. Bitsadze, *About some mixed type problems.* Dokl. AN SSSR, **70, 4** (1950), pp. 561–565 (in Russian).

[4] S. Agmon, L. Nirenberg and M.H. Protter, *A maximum principle for a class of hyperbolic equations and applications to equations of mixed elliptic-hyperbolic type.* Comm. Pure and Appl. **VI, 4** (1953), pp. 455–470.

[5] O.M. Jokhadze, *Extremum principle for some classes of second order elliptic and parabolic systems.* (Russian) Nonlocal boundary problems and related problems of mathematical biology, informatics and physics (Abstr. intern. conf., 1996), 30-31, Kabard.-Balkarian Center of Russian Academy of Sciences, Nalchik, 1996.

[6] M. Usanetashvili, *On the Dirichlet problem for a second order elliptic system with degeneration on the entire domain boundary.* Georgian Mathematical Journal. **6,4** (1999), pp. 395–400.

[7] M.S. Salakhitinov, A.K. Urinov, *Non-local boundary value problems for the equation mixed type with spectral parameter.* Tashkent: FAN, 1997, p. 168.

[8] H. Bateman, A. Erdelyi, *Higher transcendental functions.* Vol. 2. New York Toronto London, McGraw-Hill book company inc. 1953.

[9] H. Bateman, A. Erdelyi, *Higher transcendental functions.* Vol. 1. New York Toronto London, McGraw-Hill book company inc. 1953.

[10] S.G. Samko, A.A. Kilbas and O.I. Marichev, *Integrals and derivatives fractional order and some of their propositions.* Minsk: Nauka and technics, 1987, p. 688.

[11] S. Zaremba, *Sur un problème toujours possible comprenant, à titre de cas particuliers, le problème de Dirichlet et celui de Neumann,* J. math. pures. appl., **6** (1927), pp. 127–163.

[12] G. Giraud, *Généralisation des problèmes sur les opérations du type elliptique,* Bull. sc. math., **56** (1932), pp. 248–272, pp. 281–312, 316–352.

[13] A.P. Prudnikov, Y.A. Brichkov and O.I. Marichev, *Integrals and series. Special functions*, Moscow: Nauka, 1983, p. 752.

I.U. Khaydarov and M.S. Salakhitdinov
Durmon yuli, 29
P.O. 100125
Tashkent, Uzbekistan
e-mail: ibrohimbek@mail.ru
salakhitdinovms@yahoo.com

A.K. Urinov
Murabbiylar, 19
P.O.
Fergana, Uzbekistan
e-mail: urinovak@mail.ru

Operator Theory:
Advances and Applications, Vol. 216, 233–241

Numerical Investigations of Tangled Flows in a Channel of Constant and Variable Section at Presence of Recirculation Zone

S. Khodjiev

Abstract. In the present work results of numerical investigations of tangled coaxial flows in a channel of constant and variable section are given. For description of the flow full non-stationary two-dimensional equations of Navier-Stokes are used and numerically solved by non-explicit difference scheme, based on linearization finite-difference analogies of original differential equations and next approximate factorization of stabilizing finite-difference operators. Relation of initial parameters of mixed flows, velocity, temperature, pressure and altitude of cross-section in entrance, at which might happen zones of recirculation are showed, moreover optimal tangle of extension of the channel at which zones of revocable flows is defined.

Keywords. Inner flows; recirculation; Navier-Stokes; turbulence.

1. Conditional definitions and abbreviations

a, b are constants in form of channel; $b(x)$ is conditional width of the area of the displacement; C_p, C_v are specific thermal capacity under constant pressure to the volume; E is full specific energy; K_x, K_y are constants for condensation of accounting net on axis x, y; f_0 is half-altitude of the entry section of the channel; L is length of the channel; N_x, N_y are quantity of points of the coordinates x, y; n is derivation by normal; P is pressure; Pr_T is Prandtl's turbulence number; R is universal gas constant; R_1 is half-width of entry of the central active stream; T is temperature; t is time; u, v are components of the velocity along x, y; α is tangle of the extension of channel; η, ξ are transformation of coordinates; C is constant of turbulence; μ is dynamic coefficient of turbulent viscosity; ρ is density; lower indexes: x is partial derivation $\frac{\partial}{\partial x}$; y partial derivation $\frac{\partial}{\partial y}$; i, j are numbers of rated points along the axes x, y; 1 are parameters of wall streams; 2 are parameters of central streams; upper indexes: $'$ is characterized a dimensionless.

2. Introduction

Problems of turbulent displacement of tangled gas flows in channels of constant and variable cutest is interesting by their wide applications to the creation of mixing and heating device, cameras of combustion of the different energy installation. Especially, little-studied area of parameters is interesting, when as a result of mixing and spreading of tangled flows in the channel, zones of recirculation are formed.

Experimental investigations of conditions of the existence and dimensions of recirculation zones at mixing of tangled flows were done in works [1–4]. In these works, experimental investigations is ended by consideration in cases of bigger relation of areas of cross-sections in the entry of channels [1–3], organized attempt of the generalization of the geometric sizes of the zones of recirculation and distribution concentration on axis of the current of tangled coaxial flows in channel of the constant section moreover area of the cross-sections flow at the input comparable [4].

3. Formulation of a problem

Suppose, tangled flows with their gas dynamic parameters enter from coaxial nozzles in channel of constant and variable section, i.e., in the entry of channel there are two flows, characterizing by the velocity u_1, temperature T_1, pressure P_1 (wall stream) and by velocity u_2, temperature T_2, pressure P_2 (central active stream), altitude R_1.

Form of the channel is given as $f(x) = ax + b$ (Fig. 1).

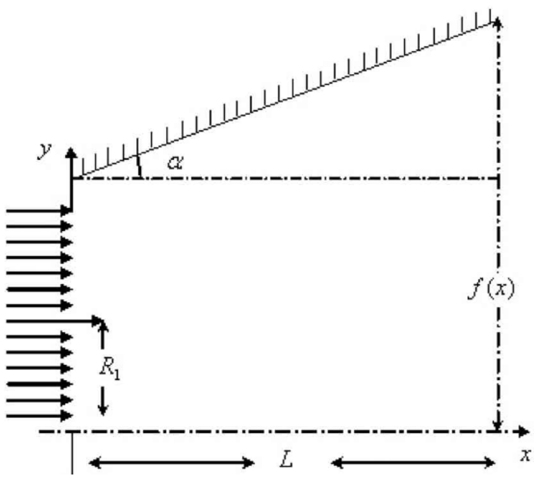


FIGURE 1. The Channel at the input.

For the description of the given flow we use the following main admissions: flow is viscous, two-dimensional, planar; gravity power is absent; heat losses occur due to heat-conduction, i.e., by Fourier's law; Boussinesq's assumption, which says apparent turbulent shift voltages are connected with average deformation through appearing (efficient) scalar turbulent viscosity is used; apparent turbulent heat currents are connected with turbulent numbers of Prandtl, moreover channel of constant and variable section is symmetric. In this case one can restrict by consideration of the current in the field of between axis of the symmetries and one of the wall of the channel. Such current can be described mathematically by means of full system of the equations of the Navier-Stokes [5, 8].

We make some transformations, before giving the system of the Navier-Stokes equations.

We choose as a scalar of length, velocity, time, temperature and pressure f_0, f_0/u_2, u_2^2/R and $\rho_2 u_2^2$ respectively. Choosing as a scalars for physical properties of gas, its coefficients of a transfer as a density, specific heat capacity, viscosity: ρ_2, R, $\rho_2 f_0 u_2$ we get relation between dimensionless with dimensional as follows:

$$\overline{x} = \frac{x}{f_0};\ \overline{y} = \frac{y}{f_0};\ \overline{u} = \frac{u}{u_2};\ \overline{v} = \frac{v}{u_2};\ \overline{E} = \frac{E}{\rho u_2};\ \overline{p} = \frac{p}{\rho_2 u_2};\ \overline{\mu} = \frac{\mu}{\rho_2 f_0 u_2};\ \overline{t} = \frac{t}{\frac{f_0}{u_2}};$$

$$\overline{\rho} = \frac{\rho}{\rho_2};\ \overline{T} = \frac{T}{\frac{u_2}{R_m}};\ \overline{C}_P = \frac{C_P}{R_m};\ \overline{C}_v = \frac{C_v}{R_m};\ \overline{L} = \frac{L}{f_0};\overline{f}(x) = \frac{f(x)}{f_0};\ \overline{R_1} = \frac{R_1}{f_0}. \tag{3.1}$$

Considered domain transform to the quadratic by the following transformation:

$$\xi = \frac{\overline{x}}{\overline{\overline{L}}}, \quad \eta = \frac{y}{f(\overline{x})}. \tag{3.2}$$

Known that stream is characterized by big gradient of gas dynamic parameters in a domain near the wall, therefore it is useful to make transformation of coordinates, which allows condense accounting points near the wall of physical plain saving constant step in accounting plain. As an example for this kind of transformation we use

$$F(y) = \frac{ln[1 + K_y(e-1)\eta]}{ln[1 + K_y(e-1)]}. \tag{3.3}$$

Graduated assignation of input parameters requires condensation of accounting points in the entrance of the channel. For this aim we introduce analytic functions, transforming condensation of accounting set of the entrance of the channel

$$\varphi(x) = 1 - \frac{ln[1 + K_x(e-1)\xi]}{ln[1 + K_x(e-1)]}. \tag{3.4}$$

Using transformations (3.1)–(3.4), the system of the Navier-Stokes equations we can represent in divergent form [5, 6].

Equation of the continuity

$$\frac{\partial}{\partial t} f F_y \varphi_x \rho + \frac{1}{L}\frac{\partial}{\partial x} f F_y \rho u + \frac{\partial}{\partial y}\varphi_x(\rho v + \Omega \rho u) = 0 \tag{3.5}$$

Equation of the motion along the axe x

$$\begin{aligned}&\frac{\partial}{\partial t} fF_y\varphi_x\rho u+\frac{1}{L}\frac{\partial}{\partial x} fF_y(\rho u^2+P)+\frac{\partial}{\partial y}\varphi_x(\rho uv+\Omega(\rho u^2+P))\\&=\frac{1}{L^2}\frac{\partial}{\partial x}\frac{4}{3} fF_y\varphi_x^{-1}\mu u_x+\frac{1}{L}\frac{\partial}{\partial x}\mu\left[\frac{4}{3}\Omega u_y-\frac{2}{3}v_y\right]+\frac{1}{L}\frac{\partial}{\partial y}\mu\left[v_x+\frac{4}{3}u_x\Omega\right]\\&\quad+\frac{\partial}{\partial y} f^{-1}F_y^{-1}\varphi_x\mu\left[\left(\frac{4}{3}\Omega^2+1\right)u_y+\frac{1}{3}\Omega v_y\right].\end{aligned}\tag{3.6}$$

Equation of the motion along the axe y

$$\begin{aligned}&\frac{\partial}{\partial t} fF_y\varphi_x\rho v+\frac{1}{L}\frac{\partial}{\partial x} fF_y\rho uv+\frac{\partial}{\partial y}\varphi_x(\rho v^2+P+\Omega\rho uv)\\&=\frac{1}{L^2}\frac{\partial}{\partial x} fF_y\varphi_x^{-1}\mu v_x+\frac{1}{L}\frac{\partial}{\partial x}\mu(u_y+\Omega v_y)+\frac{1}{L}\frac{\partial}{\partial y}\mu\left[-\frac{2}{3}u_x+\Omega v_x\right]\\&\quad+\frac{\partial}{\partial y} f^{-1}F_y^{-1}\varphi_x\mu\left[\left(\Omega^2+\frac{4}{3}\right)v_y+\frac{1}{3}\Omega u_y\right].\end{aligned}\tag{3.7}$$

Equation of the energy

$$\begin{aligned}&\frac{\partial}{\partial t} fF_y\varphi_x E+\frac{1}{L}\frac{\partial}{\partial x} fF_y(E+\rho)u+\frac{\partial}{\partial y}\varphi_x((E+P)v+\Omega(E+P)u)\\&=\frac{1}{L^2}\frac{\partial}{\partial x} fF_y\varphi_x^{-1}\mu\left(vv_x+\frac{4}{3}uu_x+\frac{C_P}{Pr_T}T_x\right)\\&\quad+\frac{1}{L}\frac{\partial}{\partial x}\mu\left(vu_y-\frac{2}{3}uv_y+\Omega vv_y+\frac{4}{3}\Omega uu_y+\frac{C_P}{Pr_T}\Omega T_y\right)\\&\quad+\frac{1}{L}\frac{\partial}{\partial y}\mu\left[uv_x-\frac{2}{3}vu_x+\Omega vv_x+\frac{4}{3}\Omega uu_x+\frac{C_P}{Pr_T}\Omega T_x\right]\\&\quad+\frac{\partial}{\partial y} f^{-1}F_y^{-1}\varphi_x\mu\Bigg[\left(\Omega^2+\frac{4}{3}\right)vv_y+\left(\frac{4}{3}\Omega^2+1\right)uu_y\\&\quad+(\Omega^2+1)\frac{C_P}{Pr_T}T_y+\frac{1}{3}\Omega(uv_y+vu_y)\Bigg].\end{aligned}\tag{3.8}$$

Equation of condition

$$\begin{aligned}P&=\rho T\\E&=\rho C_vT+\frac{1}{2}\rho(u^2+v^2),\Omega=-\eta\frac{f'}{f}.\end{aligned}\tag{3.9}$$

Effective turbulent viscous is represented via the sum of laminar and turbulent viscous in the form of

$$\mu=\mathrm{const}\,T^{0.6472}+C\rho b^2(x)|\frac{\partial u}{\partial y}|.\tag{3.10}$$

4. Method of solving

System of equations (3.5)–(3.9) with relation (3.10) is numerically solved by inexplicit difference-scheme, based on linearization of finite-difference analogy of initial differential equations and next approximate factorization of stabilizing finite-difference operators [5–6].

Boundary conditions for the system of equations (3.5)–(3.10) are formulated as follows: on the wall of the channel conditions of sticking and non-elapse, distribution of temperatures (or assumption of adiabatic wall) are used, also on an axe of the channel we put condition of symmetry. In the entrance of the channel ($x = 0$) gas dynamic parameters wall and central flows are given and in the "exit" we put weak conditions.

As a initial conditions ($t = 0$) we use homogeneous in transverse direction of gas dynamic parameter's field, moreover transverse component of the velocity we take as a zero. In every variant on the wall condition for P in the form $\partial P/\partial \overrightarrow{n} = 0$ is putted. Constancy of P supposed to be not in transverse to the whole border layer, but only in across of layer with thickness adjoining to the wall. This method gives a possibility to obtain stable numerical solution for the flow in non-isolate border layer and for the flow with isolation of stream [7].

Serial accounting of investigations was done at constant steps of the accounting set $Nx \times Ny = 31 \times 41$ or by refinement 21×31 with coefficients of condensation $K_x = -0.4701$ and $K_y = 1.2644$, corresponding to the uniform step in the entrance of the channel at $N_x = 41$, and in the domain, near of the wall $N_y = 51$.

5. Numerical results

As a base object of the investigation we choose the channel (similarly to the work [3]) with geometric characteristics: $D = 188$mm ($f_0 = 94$mm half-altitude), $L = 1.4m$ In calculations we suppose that all two streams of the air: heat capacity at constant pressure and heat capacity at constant volume are constant, and $Pr_T = Pr = 0.7$ (variant number 24 with conditions of slide on the wall). Variants of calculations are given in Table 1, in last column of which were putted symbol "+" which means that in an initial part of the channel there are recirculation zones. Variants 28–43 for extending channels with angle of extension $\alpha = 6°$ ($L/f_0 = 10$) and $\alpha = 11°$ ($L/f_0 = 5$).

From the analysis of obtained results follow:

At minor relations of initial values of velocities ($u_2/u_1 = 0.022$, $u_1 = 6.9m/s$) and equal initial values of temperature ($T_2/T_1 = 1$), pressure ($P_2/P_1 = 1$), also at large values of u_2/u_1 ($u_2/u_1 = 45.072$) at small length of channel (5.319) in entry part one can see full stop of tangled flow by creation of recirculation zones, and longitude occupies 20 percent of channel cross-cutting.

At minor relations of initial values of velocities ($u_2/u_1 = 0.022$) and high temperature ($T_2/T_1 = 2.333$; $P_2/P_1 = 1$) of passive stream (stream with slow velocity) recirculation zone is not observed in channel with constant cutting, and

№	u_2/u_1	R_1/f_0	L/f_0	T_2/T_1	P_2/P_1	
1	0,0222	0,5	14,8936	1	1	+
2	0,0965	0,5	14,8936	1	1	-
3	45,0725	0,5	5,3191	1	1	+
4	45,0725	0,5	14,8936	1	1	-
5	10,3667	0,26	14,8936	1	1	-
6	1,4706	0,26	14,8936	1	1	-
7	0,0222	0,26	14,8936	2,3333	1	-
8	0,0222	0,5	14,8936	2,3333	1	-
9	0,0222	0,5	5,3191	0,4286	1	+
10	72,4638	0,26	5,3191	2,333	1	+
11	72,4638	0,26	5,3191	0,6	1	+
12	72,4638	0,26	14,8936	0,6	1	-
13	45,0725	0,5	14,8936	2,3333	1	+
14	45,0725	0,5	5,3191	0,4286	1	+
15	45,0725	0,26	14,8936	2,3333	1	+
16	10,3667	0,26	14,8936	0,6000	1	-
17	72,4638	0,5	14,8936	1	2	+
18	72,4638	0,5	14,8936	2,333	2	-
19	72,4638	0,5	14,8936	1	4	-
20	72,4638	0,5	14,8936	2,333	4	+
21	0,02222	0,26	14,8936	1	2	-
22	0,02222	0,5	14,8936	1	2	-
23	0,02222	0,26	14,8936	1,6666	2	-
24	0,02222	0,5	14,8936	1	1	-
25	45,0725	0,26	14,8936	1	2	+
26	45,0725	0,5	14,8936	0,4286	2	+
27	45,0725	0,5	14,8936	1,6666	2	+
28	44,4285	0,5	10	1	1	-
29	44,4285	0,26	10	1	1	-
30	71,428	0,26	10	1	1	-
31	44,4285	0,26	5	1	1	-
32	44,4285	0,5	5	1	1	-
33	1,60775	0,26	10	1	1	-
34	1,60775	0,5	10	1	1	-
35	1,60775	0,5	5	1	1	-
36	1,60775	0,26	5	1	1	-
37	45,072	0,26	5	1,6666	1	-
38	45,072	0,5	5	1,6666	1	-
39	71,4285	0,26	5	1,6666	1	-
40	71,4285	0,5	5	1,0000	1	-
41	45,072	0,5	10	1,0000	2	-
42	45,072	0,26	10	1,0000	2	+
43	45,072	0,5	10	2,3333	2	+

Table 1

at large relation of initial velocities ($u_2/u_1 = 45.072$; variant 13) recirculation zone occupies 25 percent of entry section of channel when active stream (stream with strong velocity) in entry of channel occupies twice less area than entry section (variant 15), and its longitude in entry part occupies about 50 percent of cross-section. In this case in entry part of channel fast decrease of velocity, increase of pressure, temperature were observed. Later these parameters will be equal. Moreover, recirculation zones in wider diapason changing of relations of initial velocities $10.36 \le u_2/u_1 \le 72.46$, but at minor relations of temperature $T_2/T_1 = 0.6$ of tangled flow in long channels $L/f_0 = 14$ are not observed.

Recirculation zone is not observed at uncounted regimes of tangled flows ($P_2/P_1 = 2$) and at minor relations of initial velocity ($u_2/u_1 = 0.022$), temperatures ($1 \le T_2/T_1 \le 1.666$) on axe of entry part of channel of constant cross. At enough large relations of initial velocities of tangled flows ($u_2/u_1 = 72.46$), pressure ($P_2/P_1 = 4$) and temperatures ($1 \le T_2/T_1 \le 2.333$) lead to the faster growth of axial values of lengthwise velocity in entry crossing of channel, and at deviation from the crossing of channel to the faster fall (Fig. 2, where cross distribution of

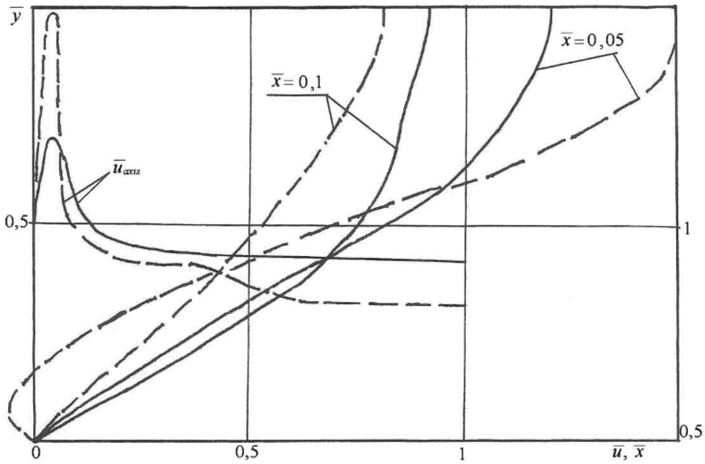


FIGURE 2. Distribution of cross velocity in various section of channel and along the axe of channel (u^*): No 19; No 20.

lengthwise velocity in various distances from the entry section and from the axial changing along the channel: continuous lines belong to variant 19, and dotted lines to 20).

Big recirculation zone observed at unrated ($P_2/P_1 = 2$), mixture of sub and supersonic tangled flows in entry part of domain ($u_1 = 6.9$ m/s; $u_2 = 311$ m/s, variant 25) occupying 60 percent of cross-section and riches up to half of area of the channel when active stream occupies 1/4 part of area of entry part of channel (see Fig. 3).

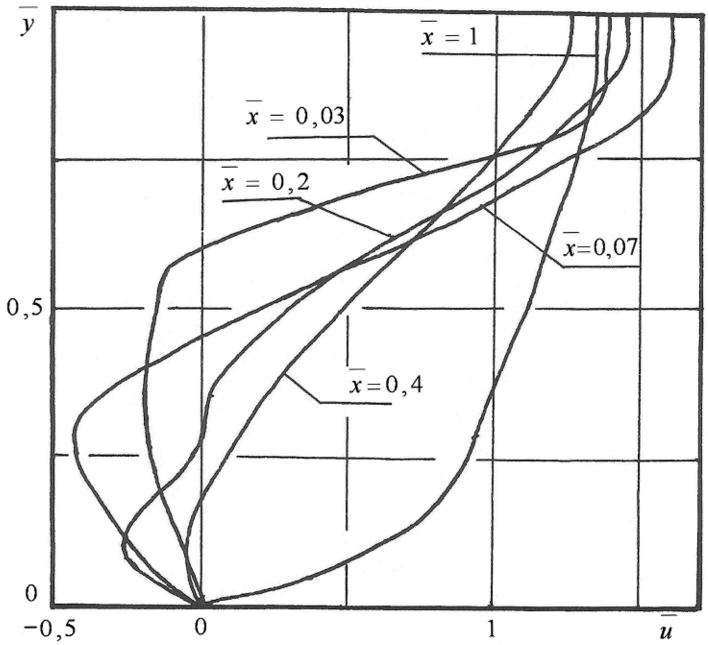


FIGURE 3. Distribution of cross velocity in various sections of channel at $u_1 = 6,9\ m/s$; $u_2 = 311\ m/s$; $P_2/P_1 = 2$ (variant 25).

Some cross distribution of pressure in various distances from the entry section of channel (variant 27), when active stream has high pressure. In first sections of channel cross changing of pressure is harmonic.

Special interest lies on less studied domain where role of index of channel's extension to the processes of mixing tangled flows is considered.

References

[1] A. Craya, R. Curtet. Publications Scientifiques et Technologues ministèr de L'Air, Mars, 1960, N-359.

[2] R. Kurte, M. Barchilon. Teoreticheskije osnovy inzhenernikh raschetov. 1964, 86, D.4.

[3] V.K. Bajev, V.A. Konstantinovsky, I.V. Sidirov. Uslovija sushestvovanije i prodolnije ramery resirkuljarnikh zon pri vzajmodejstvije sverkhzvukovoj struji s ogranichennym sputnym dozvukovym potokom. -Izv. SO AN SSSR, 1970, vyp. 2, N-8, s. 37–45.

[4] V.K. Bajev, V.A. Konstantinovsky, I.V. Sidirov. Smeshenije sputnykh potokov v kanale postojannogo sechenija pri nalichii zony resirkuljatsii. Fizika gorenija i vzriva. 1972, N-1, s. 70–76.

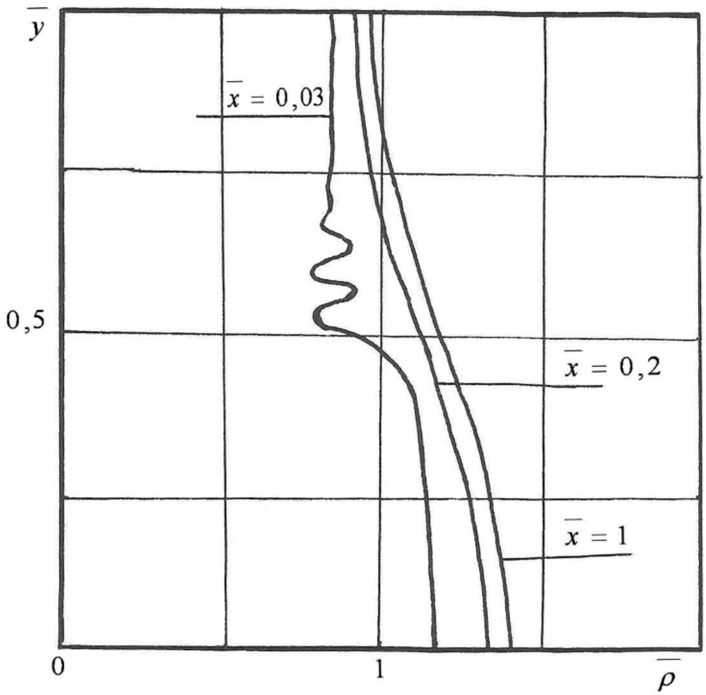


FIGURE 4. Cross distribution of pressure in various distances from the entry section of channel (variant 27).

[5] R.M. Bim, R.F. Worming. Nejavnaja faktorizovannaja raznostnaja schema dlja uravnenija Nav'e-Stoksa techenija szhimajemogo gaza. Raketnaja tekhnika i kosmonavtika, 1978, T. 16, N-4, s. 145–150.
[6] S. Khodjiev, Z.Sh. Jumaev, O.O. Yodgorov. Chislennij raschet vnutrennikh techenij szhimajemogo gaza po nejavnofaktorizovannoj raznostnoj scheme. Izv. AN. UzSSR. Ser. Tekh.Nauk. Tashkent: Fan. 1990, N-2, s. 28–34.
[7] P. Rouch. Vychislitelnaja gidrodinamika. M.: Mir, 1980, 616 s.
[8] L.G. Lojtsjanskij. Mekhanika zhidkosti i gaza. M.: Nauka, 1987, 840 s.

S. Khodjiev
Boboqulov, 12
P.O. 100125
Bukhara, Uzbekistan
e-mail: itatai@mail.ru

Operator Theory:
Advances and Applications, Vol. 216, 243–259

The Optimal Interior Regularity for the Critical Case of a Clamped Thermoelastic System with Point Control Revisited

Catherine Lebiedzik and Roberto Triggiani

Communicated by F. Bucci and I. Lasiecka

Abstract. In the case of clamped thermoelastic systems with interior point control defined on a bounded domain Ω, the critical case is $n = \dim\,\Omega = 2$. Indeed, an optimal interior regularity theory was obtained in [Triggiani, *Discrete Contin. Dyn. Syst.*, 2007] for $n = 1$ and $n = 3$. However, in this reference, an 'ϵ-loss' of interior regularity has occurred due to a peculiar pathology: the incompatibility of the B.C. of the spaces $H_0^{3/2}(\Omega)$ and $H_{00}^{3/2}(\Omega)$. This problem for $n = 2$ was rectified in [Triggiani, *J. Differential Equations*, 2008]: this establishes the sought-after interior regularity of the thermoelastic problem through a technical analysis based on sharp *boundary* (trace) regularity theory of Kirchhoff and wave equations. As an additional bonus, a sharp boundary regularity of the elastic displacement is also obtained. In the present paper, we revisit that problem using a technique developed by these authors to circumvent the pathology of the incompatible boundary conditions. This yields a more direct proof of the optimal interior regularity (but not of the boundary regularity).

Mathematics Subject Classification (2000). 35; 93.

Keywords. Interior regularity, thermoelastic plate, point control.

1. Introduction, orientation, model

1.1. The pathology of the critical case $n = 2$

The present paper is a successor to [32]. This work dealt with a thermoelastic system defined on a bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2$, subject to the action of interior point control exercised in the elastic equation and satisfying clamped/Dirichlet boundary conditions (see system (1.1a–d) below. Paper [32] succeeded in providing an optimal *interior* regularity for this model, after an original 'ε-loss' of regularity which was suffered in [31]. In the process, [32] obtained also a new sharp *boundary* regularity result for the elastic displacement. In the cases $n = 1, 3$, optimal *interior* regularity results – from the control space to the state space – were obtained in the prior work [31]. The case of $n = 2$ is critical, in the sense that its analysis encounters a subtle technical difficulty due to the incompatibility of the boundary conditions, *yet within the same topological level*, between the Sobolev space $H_0^{\frac{3}{2}}(\Omega)$ and the Sobolev space $H_{00}^{\frac{3}{2}}(\Omega)$, so that $H_{00}^{\frac{3}{2}}(\Omega) \subsetneq H_0^{\frac{3}{2}}(\Omega)$, with a finer topology [24, p. 66]. This is the pathology that caused the 'ε-loss' of regularity in [31], in the case $n = 2$. To circumvent this technical obstacle and secure the optimal *interior* regularity also in the case $n = 2$, a radically new approach was pursued in [32]. It replaces the *interior* $\rightarrow$ *interior* strategy of [31] with a technical *boundary* $\rightarrow$ *interior* strategy. The latter consists of obtaining interior regularity results through an analysis that proceeds from the boundary: it involves both the wave equation (at an interpolated level with respect to the theory of [16]) as well as Kirchhoff plate equations [12, 13, 17, 25]. In addition, [32] uses pseudo-differential analysis. It slashes the equations by two pseudo-differential operators: one in the time-derivative, and one in the space variables that are tangential to the boundary $\Gamma = \partial\Omega$. Though technical and extensive, this approach has an additional *advantage*: as a bonus, it provides – besides the sought-after optimal *interior* regularity, the original objective – also a sharp *boundary* regularity of $\Delta w|_\Sigma$, w being the elastic displacement. The latter result does *not* follow from the optimal interior regularity for w via trace theory, see (1.11d) below, and hence is a new, additional sharp boundary regularity result. It should be noted that all these difficulties are tied to the *clamped* boundary conditions which are a pathological case [21]. They do not occur in the case of hinged boundary conditions [30].

A 2-dimensional thermoelastic plate with clamped boundary conditions and subject to interior point control – the model of the present paper – is an ideal wall of a structural acoustic chamber, subject to piezo-ceramic control action, for the purpose of noise reduction [1, 2, 3, 6, 15, 22, 4, 5] to quote a few references.

1.2. Orientation in the new approach

The present paper revisits the optimal regularity theory of [32] for the thermoelastic plate with interior point control and clamped boundary conditions, in the critical case $n = 2$. It then provides a new *interior* $\rightarrow$ *interior* proof to obtain optimal *interior* regularity results in this case, thus matching [32]. The novel proof is inspired by an idea (or trick) that was introduced in [23] to circumvent (in the greater complexity of a structural acoustic model with thermoelastic wall) a technical difficulty akin to that more specifically described in Remarks 2.2 and 2.3 below. To be sure, implementation of this idea encounters, however, some serious technical difficulties of its own, of a different nature: some of which – dealing with the identification of $\mathcal{D}(\mathbb{A}_\gamma^2)$ in (3.16) below and with the subsequent formula (3.17) – have been resolved in [21, Lemma 4.2, p. 466]; and others of which – dealing with the delicate interpolation result (3.27) below – were resolved in [23, Proposition 3.1b, Eq. (3.33)]. Putting all these ingredients together provides the new proof, which re-establishes the optimal *interior* regularity results of [32]. Being, in its own way, an *interior* $\rightarrow$ *interior* proof, it cannot recover the sharp boundary regularity result on $\Delta w|_\Sigma$ obtained in [32].

1.3. The model. A canonical thermoelastic point control problem with clamped/Dirichlet boundary conditions

Let Ω be an open bounded domain in $\mathbb{R}^n$, $n = 1,2,3$, with sufficiently smooth boundary Γ for $n = 2,3$. On Ω, we consider the following thermoelastic problem in the unknown $\{w(t,x), \theta(t,x)\}$:

$$\begin{cases} w_{tt} - \gamma\Delta w_{tt} + \Delta^2 w + \Delta\theta = \delta u & \text{in } (0,T]\times\Omega \equiv Q; & (1.1a)\\ \theta_t - \Delta\theta - \Delta w_t = 0 & \text{in } Q; & (1.1b)\\ w(0,\,\cdot\,) = w_0;\ w_t(0,\,\cdot\,) = w_1;\ \theta(0,\,\cdot\,) = \theta_0 & \text{in } \Omega; & (1.1c)\\ w|_\Sigma \equiv 0;\ \left.\dfrac{\partial w}{\partial\nu}\right|_\Sigma \equiv 0;\ \theta|_\Sigma \equiv 0 & \text{on } (0,T]\times\Gamma \equiv \Sigma, & (1.1d)\end{cases}$$

with homogeneous clamped/Dirichlet B.C., under the influence of the scalar point control term $u \in L_2(0,T)$, which acts through the Dirac distribution δ concentrated at the origin, assumed to be an interior point of Ω. In (1.1a) the constant γ is taken to be positive: $\gamma > 0$ throughout the paper. In this case, the free system ($u \equiv 0$) generates a s.c. thermoelastic contraction semigroup (Proposition 1.3 below). Further information is available in [9, 10, 31]. To express the results below, we need to introduce the following setting [20], [28], [29], [31]: the positive self-adjoint operator B (norm equivalence):

$$\begin{aligned} Bf &= -\Delta f; & \mathcal{D}(B) &\equiv H^2(\Omega)\cap H_0^1(\Omega);\\ B_\gamma &= (I+\gamma B); & \mathcal{D}(B_\gamma^{\frac12}) &= \mathcal{D}(B^{\frac12}) = H_0^1(\Omega), \end{aligned} \tag{1.2}$$

as well as the elastic operator, still positive self-adjoint,

$$Af = \Delta^2 f, \quad \mathcal{D}(A) = \left\{ f \in H^4(\Omega):\ f|_\Gamma = \left.\frac{\partial f}{\partial\nu}\right|_\Gamma = 0 \right\}. \tag{1.3}$$

We recall that, with equivalent norms [29]

$$\begin{cases} \mathcal{D}(A^{\frac34}) \equiv H^3(\Omega)\cap H_0^2(\Omega) \equiv \left\{ f\in H^3(\Omega):\ f|_\Gamma = \left.\dfrac{\partial f}{\partial\nu}\right|_\Gamma = 0\right\}; & (1.4a)\\ \mathcal{D}(A^{\frac12}) \equiv H_0^2(\Omega); \quad \mathcal{D}(A^{\frac14}) \equiv H_0^1(\Omega) = \mathcal{D}(B^{\frac12}); & (1.4b)\end{cases}$$

$$\begin{aligned} \mathcal{D}(A^{\frac38}) &= [\mathcal{D}(A^{\frac12}), \mathcal{D}(A^{\frac14})]_{\frac12} = [H_0^2(\Omega), H_0^1(\Omega)]_{\frac12} \equiv H_{00}^{\frac32}(\Omega)\\ &\subset [\mathcal{D}(B), \mathcal{D}(B^{\frac12})]_{\frac12} = \mathcal{D}(B^{\frac34}) = \mathcal{D}(B_\gamma^{\frac34}) = H_0^{\frac32}(\Omega); \end{aligned} \tag{1.5}$$

$$\begin{aligned} \mathcal{D}(A^{\frac18}) &= [\mathcal{D}(A^{\frac14}), L_2(\Omega)]_{\frac12} = [H_0^1(\Omega), L_2(\Omega)]_{\frac12}\\ &= H_{00}^{\frac12}(\Omega) = \mathcal{D}(B^{\frac14}) = \mathcal{D}(B_\gamma^{\frac14}), \end{aligned} \tag{1.6}$$

see [24] for these Sobolev spaces. We note that by (1.2), (1.4) we have (properly)

$$\begin{gathered}\mathcal{D}(A^{\frac{1}{2}}) \subset \mathcal{D}(B); \text{ hence } BA^{-\frac{1}{2}} \in \mathcal{L}(L_2(\Omega)), \text{ while} \\ A^{\frac{1}{2}}B^{-1} \text{ is an unbounded operator on } L_2(\Omega),\end{gathered} \tag{1.7a}$$

$\mathcal{L}(E)$ being the Banach space of bounded operators on a Banach space E. Similarly, by (1.5) we have

$$\begin{gathered}\mathcal{D}(A^{\frac{3}{8}}) \subset \mathcal{D}(B^{\frac{3}{4}}); \text{ hence } B^{\frac{3}{4}}A^{-\frac{3}{8}} \in \mathcal{L}(L_2(\Omega)), \text{ while} \\ A^{\frac{3}{8}}B^{-\frac{3}{4}} \text{ is an unbounded operator on } L_2(\Omega).\end{gathered} \tag{1.7b}$$

In both cases, (1.7a) and (1.7b), the topological level of $\mathcal{D}(A^{\frac{1}{2}})$ and $\mathcal{D}(B)$, as well as of $\mathcal{D}(A^{\frac{3}{8}})$ and $\mathcal{D}(B^{\frac{3}{4}})$, is the same, but subtle differences in the boundary conditions occur.

Remark 1.1. The fact that under the clamped B.C. (1.3), the operator $BA^{-\frac{1}{2}}$ and $B^{\frac{3}{4}}A^{-\frac{3}{8}}$ are *not* isomorphisms on $L_2(\Omega)$, as noted in (1.7), is a major technical difference over the hinged case of [30], and are responsible for additional technical difficulties. In fact, they are precisely these differences of the B.C. between $\mathcal{D}(A^{\frac{3}{8}}) \equiv H_{00}^{\frac{3}{2}}(\Omega)$ and $\mathcal{D}(B^{\frac{3}{4}}) = H_0^{\frac{3}{2}}(\Omega)$ that are responsible for causing the pathology and the technical difficulties described in the Orientation. Refining the information of (1.7a) by adjointness, we recall that [21, Proposition 2.3, p. 453]:

$$A^{-\frac{1}{2}}B_\gamma g \in L_2(\Omega) \Longleftrightarrow g \in \tilde{L}_2(\Omega),$$

where the space $\tilde{L}_2(\Omega)$ can be characterized in a few ways:

(i) Either as the dual space of $\mathcal{D}(A^{\frac{1}{2}})$ with respect to space $\mathcal{D}(B_\gamma^{\frac{1}{2}})$ as a pivot space, endowed with the norm

$$\|f\|^2_{\mathcal{D}(B_\gamma^{\frac{1}{2}})} = (B_\gamma^{\frac{1}{2}}f, B_\gamma^{\frac{1}{2}}f)_{L_2(\Omega)} = ((I+\gamma B)f, f)_{L_2(\Omega)}$$

[21, Eqn. (2.29), p. 452], as in (1.14) below;

(ii) or else as (isometric to) the factor space $L_2(\Omega)/\mathcal{H}$ where

$$\mathcal{H} \equiv \{h \in L_2(\Omega) : (1-\gamma\Delta)h = 0 \text{ in } H^{-2}(\Omega)\} = \mathcal{N}(1-\gamma\Delta)$$

[21, Section 2.4, p. 456].

An additional property is noted in (3.17) below and is critically used in the arguments of Section 3. This fact permits the refinement in [21, Section 4.4, p. 473] of the interior regularity in [30] of the purely elastic problem (2.1a-b-c) for $n = 3$, to yield, ultimately, $w_{tt} \in L_2(0, T; \tilde{L}_2(\Omega))$, as in (1.9d) below, for the corresponding thermoelastic problem (1.1a-b-c) for $n = 3$. □

1.4. Regularity results

Paper [31] showed the following results.

Theorem 1.1 ([31]). *With reference to problem* (1.1) *with* $\gamma > 0$ *and zero initial conditions:* $w_0 = w_1 = \theta_0 = 0$, *we have the following regularity result. Let*

$$u \in L_2(0,T). \tag{1.8}$$

Then, continuously, where $\epsilon > 0$ *is arbitrary and for any* p, $1 < p < \infty$*:*

(i) *for* $n = \dim \Omega = 3$,

$$w \in C([0,T]; \mathcal{D}(A^{\frac{1}{2}}) = H_0^2(\Omega)); \tag{1.9a}$$

$$w_t \in C([0,T]; \mathcal{D}(A^{\frac{1}{4}}) = \mathcal{D}(B^{\frac{1}{2}}) = H_0^1(\Omega)); \tag{1.9b}$$

$$\theta \in L_p(0,T; \mathcal{D}(B^{\frac{1}{2}}) = H_0^1(\Omega)) \cap C([0,T]; \mathcal{D}(B^{\frac{1}{2}-\epsilon}) = H_0^{1-2\epsilon}(\Omega)); \tag{1.9c}$$

$$w_{tt} \in L_2(0,T; \tilde{L}_2(\Omega)); \tag{1.9d}$$

(ii) *for* $n = \dim \Omega = 1$,

$$w \in C([0,T]; \mathcal{D}(A^{\frac{3}{4}}) = H^3(\Omega) \cap H_0^2(\Omega)); \tag{1.10a}$$

$$w_t \in C([0,T]; \mathcal{D}(A^{\frac{1}{2}}) = H_0^2(\Omega)); \tag{1.10b}$$

$$\theta \in C([0,T]; \mathcal{D}(B) \quad = H^2(\Omega) \cap H_0^1(\Omega)); \tag{1.10c}$$

$$w_{tt} \in L_2(0,T; \mathcal{D}(B^{\frac{1}{2}}) \ = H_0^1(\Omega)). \tag{1.10d}$$

As already stated in the Orientation, the results of Theorem 1.1 for $n = 3$ and $n = 1$ are optimal. The optimal result for $n = 2$ was proven in [32]:

Theorem 1.2 ([32]). *Let* $n = \dim \Omega = 2$ *and assume* (1.8) *for the corresponding problem* (1.1). *Then, continuously, the following interior regularity holds true:*

$$\begin{cases} w \in C([0,T]; \mathcal{D}(A^{\frac{5}{8}}) \equiv H^{\frac{5}{2}}(\Omega) \cap H_0^2(\Omega)); & (1.11a) \\ w_t \in C([0,T]; \mathcal{D}(A^{\frac{3}{8}}) = H_{00}^{\frac{3}{2}}(\Omega)); & (1.11b) \\ \theta \in L_p(0,T; \mathcal{D}(B^{\frac{3}{4}}) \equiv H_0^{\frac{3}{2}}(\Omega)) \cap C([0,T]; \mathcal{D}(B^{\frac{3}{4}-\frac{\epsilon}{2}}) & \\ \qquad \equiv H_0^{\frac{3}{2}-\epsilon}(\Omega)), \quad 1 < p < \infty. & (1.11c) \end{cases}$$

Theorem 1.3 ([32]). *Assume the hypotheses of Theorem* 1.2. *Then, still continuously in* $u \in L_2(0,T)$, *the following boundary regularity of the elastic component holds true:*

$$\Delta w|_{\Sigma} \in L_2(0,T; L_2(\Gamma)) \equiv L_2(\Sigma). \tag{1.11d}$$

The boundary regularity (1.11d) does not follow from (1.11a) via trace theory. It is a new, additional regularity result.

Remark 1.2. Theorem 1.1 for $n = 3$, $n = 1$, as well as Theorem 1.2 for $n = 2$, show consistency in the following sense. The position variable w gains in regularity "$\frac{1}{8}$ in terms of fractional power of A," while decreasing the dimension from $n =$

3 to $n = 2$ to $n = 1$. The same occurs for the velocity variable w_t, which – moreover – is consistently "$\frac{1}{4}$ less regular in terms of fractional power of A" than the corresponding regularity of w for $n = 3, 2, 1$. □

The goal of the present paper is to give a more direct proof of the interior regularity (1.11a-c) of Theorem 1.2. This proof, however, will not cover Theorem 1.3.

1.5. Further preliminaries

For future discussion, we need further preliminary background from [31], [28], [29]. By (1.2), (1.3), we may rewrite (1.1) abstractly first as

$$\begin{cases} (I + \gamma B)w_{tt} + Aw - B\theta = \delta u; & (1.12a) \\ \theta_t + B\theta + Bw_t = 0; & (1.12b) \end{cases}$$

next, as the first-order equation

$$\dot{y} = -\mathbb{A}_\gamma y + \mathcal{B}u, \ y(0) = [w_0, w_1, \theta_0] \in Y_\gamma; \ y(t) = [w(t), w_t(t), \theta(t)]; \tag{1.13a}$$

$$-\mathbb{A}_\gamma = \begin{bmatrix} 0 & I & 0 \\ -B_\gamma^{-1}A & 0 & B_\gamma^{-1}B \\ 0 & -B & -B \end{bmatrix}; \qquad \mathcal{B}u = \begin{bmatrix} 0 \\ B_\gamma^{-1}\delta u \\ 0 \end{bmatrix}; \tag{1.13b}$$

$$\mathcal{D}(\mathbb{A}_\gamma) = \mathcal{D}(A^{\frac{3}{4}}) \times \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(B); \tag{1.13c}$$

$$\begin{aligned} Y_\gamma &= \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(B_\gamma^{\frac{1}{2}}) \times L_2(\Omega); \\ B_\gamma &= I + \gamma B; \ (x_1, x_2)_{\mathcal{D}(B_\gamma^{\frac{1}{2}})} = ((I + \gamma B)x_1, x_2)_{L_2(\Omega)}. \end{aligned} \tag{1.14}$$

The space Y_γ is the natural energy space for problem (1.1a–d).

Below, in Section 3, equation (3.5), we shall also need the following domains of fractional power of $\mathbb{A}_\gamma$:

$$\mathcal{D}(\mathbb{A}_\gamma^s) = [\mathcal{D}(\mathbb{A}_\gamma), Y_\gamma]_{1-s} = \mathcal{D}(A^{\frac{1}{2}+\frac{s}{4}}) \times \mathcal{D}(A^{\frac{1}{4}+\frac{s}{4}}) \times \mathcal{D}(B^s), \quad 0 \le s \le 1, \tag{1.15}$$

obtained from (1.14) for $\mathcal{D}(\mathbb{A}_\gamma)$ and (1.15) for Y_γ via

$$\begin{cases} [\mathcal{D}(A^{\frac{3}{4}}), \mathcal{D}(A^{\frac{1}{2}})]_{1-s} = \mathcal{D}(A^{\frac{1}{2}+\frac{s}{4}}); \ [\mathcal{D}(A^{\frac{1}{2}}), \mathcal{D}(A^{\frac{1}{4}})]_{1-s} = \mathcal{D}(A^{\frac{1}{4}+\frac{s}{4}}); \\ [\mathcal{D}(B), L_2(\Omega)]_{1-s} = \mathcal{D}(B^s). \end{cases} \tag{1.16}$$

and [19, p. 5], via the Lumer-Phillips theorem, or a corollary thereof [26, pp. 14–15], one may readily show the following known well-posedness result [20].

Proposition 1.4. *The operator* $-\mathbb{A}_\gamma$ *in* (1.13) *is the infinitesimal generator of a s.c. semigroup of contractions* $e^{-\mathbb{A}_\gamma t}$ *on the space* Y_γ *defined by* (1.14)*, as well as on* $\mathcal{D}(\mathbb{A}_\gamma^s)$*,* $0 < s \le 1$.

We conclude this section with a standard regularity result for the self-adjoint, analytic semigroup e^{-Bt}, to be invoked repeatedly in the sequel [17, Prop. 0.1, p. 4]:

the map

$$f \to \int_0^t e^{-B(t-\tau)} f(\tau)d\tau : \text{ continuous}$$

$$L_2(0,T;L_2(\Omega)) \to L_2(0,T;\mathcal{D}(B)) \cap C([0,T];\mathcal{D}(B^{\frac{1}{2}})). \tag{1.17}$$

$$L_p(0,T;L_2(\Omega)) \to L_p(0,T;\mathcal{D}(B)) \text{ for all } 1 < p < \infty; \tag{1.18}$$

$$L_\infty(0,T;L_2(\Omega)) \to C([0,T];\mathcal{D}(B^{1-\epsilon})), \text{ for all } 0 < \epsilon \le \frac{1}{2}. \tag{1.19}$$

In (1.17), the case $p = 2$ is shown by Laplace transform [14, Appendix]; the case $1 < p < \infty$ in (1.18) is much harder [8]; see also [11, p. 112]. Finally, (1.19) follows by convolution of an L_1-function $B^{1-\epsilon}e^{-Bt}$ with an L_∞-function f [27, p. 26, p. 29].

2. Proof of Theorem 1.2: Preliminaries

Henceforth, we shall focus on the case $n = 2$ only. Thus, when invoking results from [31], we shall confine only to the case $n = 2$.

2.1. The auxiliary ψ- and h-problems

First, as in [31], following [7], we introduce the *uncoupled* Kirchhoff problem corresponding to (1.1) with zero I.C.:

$$\psi_{tt} - \gamma\Delta\psi_{tt} + \Delta^2\psi = \delta u \quad \text{in } (0,T] \times \Omega \equiv Q; \tag{2.1a}$$

$$\psi(0,\cdot) = 0, \ \psi_t(0,\cdot) = 0 \quad \text{in } \Omega; \tag{2.1b}$$

$$\psi|_\Sigma \equiv 0; \ \left.\frac{\partial\psi}{\partial\nu}\right|_\Sigma \equiv 0 \quad \text{in } (0,T] \times \Gamma \equiv \Sigma. \tag{2.1c}$$

Regarding the sharp (optimal) regularity of problem (2.1), we then invoke [28, Theorem 3.1, p. 410] and obtain that:

for $n = \dim \Omega = 2$, and for $u \in L_2(0,T)$ as in (1.8), then, continuously:

$$\psi \in C([0,T];\mathcal{D}(A^{\frac{5}{8}}) \equiv H^{\frac{5}{2}}(\Omega) \cap H_0^2(\Omega)); \tag{2.2a}$$

$$\psi_t \in C([0,T];\mathcal{D}(A^{\frac{3}{8}}) \equiv H_{00}^{\frac{3}{2}}(\Omega)) \subset C([0,T];\mathcal{D}(B^{\frac{3}{4}}) = H_0^{\frac{3}{2}}(\Omega)); \tag{2.2b}$$

$$B\psi_{tt} \in L_2(0,T;[\mathcal{D}(A^{\frac{3}{8}})]'); \tag{2.2c}$$

Next, with ψ_t provided by problem (2.1), and hence satisfying (2.2b) ($n = 2$), continuously in $u \in L_2(0,T)$, we next consider the *uncoupled* heat problem corresponding to (1.1) with zero I.C.:

$$\begin{cases} h_t - \Delta h - \Delta\psi_t \equiv 0 & \text{in } Q; & (2.3a)\\ h(0,\cdot) = 0 & \text{in } \Omega; \quad \text{or } h_t = -Bh - B\psi_t; & (2.3b)\\ h|_\Sigma \equiv 0 & \text{in } \Sigma. & (2.3c)\end{cases}$$

where $\Delta\psi_t$ is rewritten as $-B\psi_t$, since $\psi_t|_\Sigma = 0$ by (2.1c). Its solution is

$$h(t) = -\int_0^t e^{-B(t-\tau)}B\psi_t(\tau)d\tau \tag{2.4}$$

$$= -\int_0^t B^{\frac{1}{4}}e^{-B(t-\tau)}B^{\frac{3}{4}}\psi_t(\tau)d\tau \in L_p(0,T;\mathcal{D}(B^{\frac{3}{4}}) = H_0^{\frac{3}{2}}(\Omega))$$

$$\cap\, C([0,T];\mathcal{D}(B^{\frac{3}{4}-\epsilon})), \quad n=2 \tag{2.5a}$$

$$h_t(t) \in L_p(0,T;[\mathcal{D}(B^{\frac{1}{4}})]' \equiv [H_{00}^{\frac{1}{2}}(\Omega)]'), \tag{2.5b}$$

for any $\epsilon > 0$, and for all $1 < p < \infty$. The regularity in (2.5a) follows from the general regularity result (1.18) and (1.19) via $B^{\frac{3}{4}}\psi_t = (B^{\frac{3}{4}}A^{-\frac{3}{8}})A^{\frac{3}{8}}\psi_t \in C([0,T];L_2(\Omega)) \subset L_p(0,T;L_2(\Omega))$ $(n=2)$, see (2.2b) and (1.7b). The case for $n=2$ is not using ψ_t in an optimal way. See Remark 2.1 below.

Remark 2.1. For $n=2$, the regularity in (2.5a) appears to be optimal, even though we only used $\psi_t \in C([0,T];\mathcal{D}(B^{\frac{3}{4}}) \equiv H_0^{\frac{3}{2}}(\Omega))$ rather than the slightly sharper $\psi_t \in C([0,T];\mathcal{D}(A^{\frac{3}{8}}) \equiv H_{00}^{\frac{3}{2}}(\Omega))$, given by (2.2b). For $n=2$, the desirable regularity $h \in L_p(0,T;\mathcal{D}(A^{\frac{3}{8}}) \equiv H_{00}^{\frac{3}{2}}(\Omega))$ appears to be false. This subtle difference on the boundary conditions between $H_0^{\frac{3}{2}}(\Omega)$ and $H_{00}^{\frac{3}{2}}(\Omega)$ had the negative impact in the semigroup approach of [31] by forcing the use of (1.16) for $s = \frac{1}{2} - \epsilon$ – that is, (1.18) – and a consequent loss of "ϵ," in the regularity of $\{z, z_t, q\}$ below in (2.10), hence of $\{w, w_t\}$ for $n=2$. Instead, if it were true that

$$h \in L_p(0,T;\mathcal{D}(A^{\frac{3}{8}}) \equiv H_{00}^{\frac{3}{2}}(\Omega)),\ 1<p<\infty,\ n=2, \tag{2.5c}$$

we would be allowed to use (1.15) for $s = \frac{1}{2}$, in (2.10). But (2.5c) is not true.

2.2. The reduced $\{z, q\}$-problem

Setting new variables as in [30, Eqn. (2.12)], [31, Eqn. (2.9)]

$$z = w - \psi; \quad q = \theta - h, \tag{2.6}$$

we likewise readily find from (1.1), (2.1), (2.3) that $\{z, q\}$ solves the following thermoelastic problem

$$\begin{cases} z_{tt} - \gamma\Delta z_{tt} + \Delta^2 z + \Delta q = -\Delta h & \text{in } Q; & (2.7a)\\ q_t - \Delta q - \Delta z_t = 0 & \text{in } Q; & (2.7b)\\ z(0,\cdot) = 0;\ z_t(0,\cdot) = 0;\ q(0,\cdot) = 0 & \text{in } \Omega; & (2.7c)\\ z|_\Sigma \equiv 0;\ \left.\dfrac{\partial z}{\partial \nu}\right|_\Sigma \equiv 0;\ q|_\Sigma = 0 & \text{in } \Sigma, & (2.7d) \end{cases}$$

with the term $-\Delta h = Bh$ known via problem (2.3). Recall the operator $\mathbb{A}_\gamma$ in (1.14): the abstract version of problem (2.7) is (with $B_\gamma = (I + \gamma B)$):

$$\begin{cases} (I+\gamma B)z_{tt} + Az - Bq &= Bh; \\ q_t + Bq + Bz_t &= 0, \end{cases} \quad \text{or} \quad \frac{d}{dt}\begin{bmatrix} z \\ z_t \\ q \end{bmatrix} = -\mathbb{A}_\gamma \begin{bmatrix} z \\ z_t \\ q \end{bmatrix} + \begin{bmatrix} 0 \\ B_\gamma^{-1}Bh \\ 0 \end{bmatrix}. \tag{2.8}$$

The solution $\{z, z_t, q\}$ of problem (2.8) with zero I.C. is

$$\begin{bmatrix} z(t) \\ z_t(t) \\ q(t) \end{bmatrix} = \int_0^t e^{-\mathbb{A}_\gamma(t-\tau)} \begin{bmatrix} 0 \\ B_\gamma^{-1}Bh(\tau) \\ 0 \end{bmatrix} d\tau, \tag{2.9}$$

where we seek to show well-posedness and regularity of (2.9).

Remark 2.2. (the '$\varepsilon-$' loss in regularity.) For the case $n = 2$, [31, Eqn. (2.14)] obtained, with $\epsilon > 0$ arbitrary, and recalling (1.18) and (1.15) for $s = \frac{1}{2} - \varepsilon$:

$$\begin{bmatrix} z(t) \\ z_t(t) \\ q(t) \end{bmatrix} \in C([0,T]; \mathcal{D}(\mathbb{A}_\gamma^{\frac{1}{2}-\epsilon}))$$

$$= C\left([0,T]; \begin{bmatrix} \mathcal{D}(A^{\frac{5}{8}-\frac{\epsilon}{4}}) \equiv H^{\frac{5}{2}-\epsilon}(\Omega) \cap H_0^2(\Omega) \\ \mathcal{D}(A^{\frac{3}{8}-\frac{\epsilon}{4}}) \equiv H_0^{\frac{3}{2}-\epsilon}(\Omega) = \mathcal{D}(B^{\frac{3}{4}-\frac{\epsilon}{2}}) \\ \mathcal{D}(B^{\frac{1}{2}-\epsilon}) = H_0^{1-2\epsilon}(\Omega) \end{bmatrix}\right), \quad n = 2. \qquad \begin{matrix} (2.10\text{a}) \\ (2.10\text{b}) \\ (2.10\text{c}) \end{matrix}$$

Indeed, (2.10) follows from (2.9), and Proposition 1.4, via the (critical) fact that by (2.5a) ($n = 2$), we have *a fortiori*

$$B_\gamma^{-1}Bh \in L_p(0,T; \mathcal{D}(B^{\frac{3}{4}-\frac{\epsilon}{2}}) = \mathcal{D}(A^{\frac{3}{8}-\frac{\epsilon}{4}})), \; n = 2, \tag{2.10d}$$

and thus, by (1.18), with $1 < p < \infty$, $B_\gamma^{-1}B = \frac{1}{\gamma}I - B_\gamma^{-1}$:

$$\begin{bmatrix} 0 \\ B_\gamma^{-1}Bh \\ 0 \end{bmatrix} \in L_p(0,T; \mathcal{D}(\mathbb{A}_\gamma^{\frac{1}{2}-\epsilon})), \; n = 2. \tag{2.11}$$

Moreover, $e^{\mathbb{A}_\gamma t}$ restricts to a s.c. semigroup on $\mathcal{D}(\mathbb{A}_\gamma^s)$. Then, (2.11) yields (2.10), via Proposition 1.2.

Remark 2.3. We note that, in the above argument after [31], in the case $n = 2$, the loss of $\epsilon > 0$ suffered in (2.11) was incurred in order to force $B_\gamma^{-1}Bh$ into the second component space of the domain of the 'largest' fractional power of $\mathbb{A}_\gamma$, see (1.16). Since $\mathcal{D}(A^{\frac{3}{8}}) = H_{00}^{\frac{3}{2}}(\Omega) \subsetneq \mathcal{D}(B^{\frac{3}{4}}) = H_0^{\frac{3}{2}}(\Omega) = H^{\frac{3}{2}}(\Omega) \cap H_0^1(\Omega)$, see (1.5), with a strictly finer topology [24, Thm. 11.7, p. 66], then the vector $[0, B_\gamma^{-1}Bh, 0] \notin$

$\mathcal{D}(\mathbb{A}_\gamma^{\frac{1}{2}})$, see (1.16), or (1.18); while, instead, $[0, B_\gamma^{-1}Bh, 0] \in \mathcal{D}(\mathbb{A}_\gamma^{\frac{1}{2}-\epsilon})$, $\forall\, \epsilon > 0$, see (1.18).

This obstacle was resolved in [32] by using a radically different, technical *boundary* → *interior* approach. In the present paper, we overcome this difficulty by a new strategy which is introduced in the next section. It is a novel *interior* → *interior* strategy.

3. New strategy for the proof of the main theorem 1.2

From now on, the proof is quite different from that of [31], or [32]: the first led to optimal results for $n = 1$, $n = 3$, but incurred in a loss of $\epsilon > 0$ for $n = 2$ (Remark 2.3 and Remark 2.1). The second obtained the required optimal regularity of the $\{z, q\}$-problem by using a radically different, technical *boundary* → *interior* approach. In the present paper, we use the trick displayed in (3.11), (3.12) which requires the new regularity result (3.13). Establishing (3.13) is a delicate issue which, in turn, requires the peculiar interpolation result (3.27), whose proof is given in the Appendix.

Step 1. We start with equation (2.9) above:

$$\begin{bmatrix} z(t) \\ z_t(t) \\ q(t) \end{bmatrix} = \int_0^t e^{-\mathbb{A}_\gamma(t-\tau)} \begin{bmatrix} 0 \\ B_\gamma^{-1}Bh(\tau) \\ 0 \end{bmatrix} d\tau, \tag{3.1}$$

where $B_\gamma^{-1}B$ acts 'like' the identity. By the regularity in (2.5a),

$$B_\gamma^{-1}Bh, \quad h \in L_p(0,T; \mathcal{D}(B^{\frac{3}{4}}) = H_0^{\frac{3}{2}}(\Omega)) \tag{3.2}$$

and as such

$$h \notin L_p(0,T; \mathcal{D}(A^{\frac{3}{8}}) = H_{00}^{\frac{3}{2}}(\Omega)) \tag{3.3}$$

and, as explained in Remark 2.1, herein lies the obstacle. We *want* to show that

$$\begin{bmatrix} z(t) \\ z_t(t) \\ q(t) \end{bmatrix} \in C([0,T]; \mathcal{D}(\mathbb{A}_\gamma^{\frac{1}{2}})) \tag{3.4}$$

where by (1.15) with $s = \frac{1}{2}$

$$\mathcal{D}(\mathbb{A}_\gamma^{\frac{1}{2}}) = \mathcal{D}(A^{\frac{5}{8}}) \times \mathcal{D}(A^{\frac{3}{8}}) \times \mathcal{D}(B^{\frac{1}{2}}) \tag{3.5}$$

but (3.3) does not allow us to claim (3.4) directly from (3.1) and (3.5).

Step 2. Instead, we will use a trick developed in [23] in which we will return to $-\mathbb{A}_\gamma$ given by (1.13):

$$-\mathbb{A}_\gamma = \begin{bmatrix} 0 & I & 0 \\ -B_\gamma^{-1}A & 0 & B_\gamma^{-1}B \\ 0 & -B & -B \end{bmatrix} \tag{3.6}$$

and verify that

$$\begin{bmatrix} 0 \\ B_\gamma^{-1}Bh(\cdot) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ -B_\gamma^{-1}A & 0 & B_\gamma^{-1}B \\ 0 & -B & -B \end{bmatrix} \begin{bmatrix} -A^{-1}Bh(\cdot) \\ 0 \\ 0 \end{bmatrix} \tag{3.7}$$

or

$$\mathbb{A}_\gamma^{-1} \begin{bmatrix} 0 \\ B_\gamma^{-1}Bh(\cdot) \\ 0 \end{bmatrix} = \begin{bmatrix} A^{-1}Bh(\cdot) \\ 0 \\ 0 \end{bmatrix} \tag{3.8}$$

note that

$$A^{-1}Bh = A^{-\frac{7}{8}}(A^{-\frac{1}{8}}B^{\frac{1}{4}})B^{\frac{3}{4}}h \tag{3.9}$$

Since $\mathcal{D}(A^{\frac{1}{8}}) = \mathcal{D}(B^{\frac{1}{4}})$ by (1.6), $(A^{-\frac{1}{8}}B^{\frac{1}{4}})$ is an isomorphism, and $B^{\frac{3}{4}}h \in L_p(0, T; L_2(\Omega))$ by (3.2). So, clearly, we have

$$A^{-1}Bh \in L_p(0, T; \mathcal{D}(A^{\frac{7}{8}})) \tag{3.10}$$

We can rewrite (3.1) as

$$\begin{bmatrix} z(t) \\ z_t(t) \\ q(t) \end{bmatrix} = \int_0^t \mathbb{A}_\gamma e^{-\mathbb{A}_\gamma(t-\tau)} \mathbb{A}_\gamma^{-1} \begin{bmatrix} 0 \\ B_\gamma^{-1}Bh(\tau) \\ 0 \end{bmatrix} d\tau \tag{3.11}$$

$$= \int_0^t \mathbb{A}_\gamma e^{-\mathbb{A}_\gamma(t-\tau)} \begin{bmatrix} A^{-1}Bh(\tau) \\ 0 \\ 0 \end{bmatrix} d\tau \tag{3.12}$$

where application of (3.8) allows us to go from (3.11) to (3.12) and the term $A^{-1}Bh \in L_p(0, T; \mathcal{D}(A^{\frac{7}{8}}))$ by (3.10). The question is: to which domain of fractional power $\mathcal{D}(\mathbb{A}^r)$ does the vector on the right-hand side of (3.12) belong? We shall show below that

$$\begin{bmatrix} A^{-1}Bh(\tau) \\ 0 \\ 0 \end{bmatrix} \in L_p(0, T; \mathcal{D}(\mathbb{A}_\gamma^{\frac{3}{2}})), \quad \text{or} \quad \mathbb{A}_\gamma^{\frac{3}{2}} \begin{bmatrix} A^{-1}Bh(\tau) \\ 0 \\ 0 \end{bmatrix} \in L_p(0, T; Y_\gamma) \tag{3.13}$$

so that using (3.13) in (3.12) we obtain

$$\mathbb{A}_\gamma^{\frac{1}{2}} \begin{bmatrix} z(t) \\ z_t(t) \\ q(t) \end{bmatrix} = \int_0^t e^{-\mathbb{A}_\gamma (t-\tau)} \mathbb{A}_\gamma^{\frac{3}{2}} \begin{bmatrix} A^{-1}Bh(\tau) \\ 0 \\ 0 \end{bmatrix} d\tau \tag{3.14}$$

$$= C([0,T]; Y_\gamma) \tag{3.15}$$

as desired. Thus, (3.15) proves (3.4), as soon as we establish (3.13).

Step 3 (Proof of (3.13)). To do so, we need first to show that

$$\mathcal{D}(\mathbb{A}_\gamma^2) = A^{-1}\mathcal{H}^\perp \times \mathcal{Y}_2 \times \mathcal{Y}_3 \tag{3.16}$$

where $\mathcal{Y}_2$, $\mathcal{Y}_3$ are some (not specified) spaces and with [21, eq. (4.11)]

$$A^{-1}\mathcal{H}^\perp = \mathcal{D}(A^{\frac{1}{2}} B_\gamma^{-1} A) \cong A^{-1} \tilde{L}_2(\Omega) \tag{3.17}$$

where, as in [21, eqns. (2.5), (2.6), p. 448] and also at the end of Section 1.2:

$$\mathcal{H} \equiv \left\{ h \in L_2(\Omega) : (1-\gamma\Delta)h = 0 \text{ in } H^{-2}(\Omega) \right\} = \mathcal{N}\left\{(1-\gamma\Delta)\right\} \tag{3.18}$$

$$\mathcal{H}^\perp \equiv \left\{ f \in L_2(\Omega) : (f,h)_{L_2(\Omega)} = 0, \quad \forall h \in \mathcal{H} \right\} \tag{3.19}$$

$$L_2(\Omega) = \mathcal{H} + \mathcal{H}^\perp \qquad \text{orthogonal sum;} \quad \tilde{L}_2(\Omega) \cong L_2(\Omega)/\mathcal{H}\,. \tag{3.20}$$

Step 3(i) (Proof of (3.16)): We calculate via (3.6)

$$\mathbb{A}_\gamma \mathbb{A}_\gamma \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -\mathbb{A}_\gamma \begin{bmatrix} 0 & I & 0 \\ -B_\gamma^{-1}A & 0 & B_\gamma^{-1}B \\ 0 & -B & -B \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{3.21}$$

$$= \begin{bmatrix} 0 & I & 0 \\ -B_\gamma^{-1}A & 0 & B_\gamma^{-1}B \\ 0 & -B & -B \end{bmatrix} \begin{bmatrix} x_2 \\ -B_\gamma^{-1}Ax_1 + B_\gamma^{-1}Bx_3 \\ -Bx_2 - Bx_3 \end{bmatrix} \tag{3.22}$$

We display only the term of interest, that is, x_1. We require that

$$\mathbb{A}_\gamma^2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -B_\gamma^{-1}Ax_1 + B_\gamma^{-1}Bx_3 \\ \cdots \\ BB_\gamma^{-1}Ax_1 + \cdots \end{bmatrix} \in \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(B_\gamma^{\frac{1}{2}}) \times L_2(\Omega) = Y_\gamma \tag{3.23}$$

or

$$A^{\frac{1}{2}} B_\gamma^{-1} A x_1 \in L_2(\Omega)\,. \tag{3.24}$$

The (3.24) implies via (3.17) that

$$x_1 \in \mathcal{D}(A^{\frac{1}{2}} B_\gamma^{-1} A) = A^{-1}\mathcal{H}^\perp, \tag{3.25}$$

and (3.16) is proved.

Step 3(ii): Equation (1.15) with $s = 1$ gives that

$$\mathcal{D}(\mathbb{A}_\gamma) = \mathcal{D}(A^{\frac{3}{4}}) \times \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(B)\,. \tag{3.26}$$

Step 3(iii) (Interpolation between (3.16) and (3.26)): First, Proposition 3.1b of [23, eqn. (3.33)] gives the following interpolation result:

$$\left[A^{-1}\mathcal{H}^\perp, \mathcal{D}(A^{\frac{3}{4}})\right]_{\frac{1}{2}} = \mathcal{D}(A^{\frac{7}{8}}) \tag{3.27}$$

whose proof is provided in the Appendix. By interpolation between (3.16) and (3.26), we then obtain via (3.27):

$$[\mathcal{D}(\mathbb{A}_\gamma^2), \mathcal{D}(\mathbb{A}_\gamma)]_{\frac{1}{2}} = \mathcal{D}(\mathbb{A}_\gamma^{\frac{3}{2}}) = \mathcal{D}(A^{\frac{7}{8}}) \times \tilde{\mathcal{Y}}_2 \times \tilde{\mathcal{Y}}_3. \tag{3.28}$$

Finally, (3.10) then implies that

$$\begin{bmatrix} A^{-1}Bh(\tau) \\ 0 \\ 0 \end{bmatrix} \in L_p(0, T; \mathcal{D}(\mathbb{A}_\gamma^{\frac{3}{2}})) \tag{3.29}$$

and (3.13) is proved. This completes the proof of (3.14), hence of Theorem 1.2, recalling (1.15) for $s = \frac{1}{2}$.

Appendix A: Proof of (3.27)

Proposition A.1. [Proposition 3.1b of [23]] *The following interpolation result holds true:*

$$\left[A^{-1}\mathcal{H}^\perp, \mathcal{D}(A^{\frac{3}{4}})\right]_{\frac{1}{2}} = \mathcal{D}(A^{\frac{7}{8}}). \tag{A.1}$$

To prove the interpolation result (A.1), we shall seek to fall into the setting of [24, Section 14.3, p. 96–98]. This is an interpolation result between subspaces; that is, between spaces subject to additional constraints. To this end, let (we use the notation of [24, Section 14.3]):

$$X = \mathcal{D}(A) \subset \Phi, \quad \mathcal{X} = \mathcal{H}^\perp = \bar{\mathcal{X}} \subset \Psi, \quad \delta = A, \tag{A.2}$$

so that we may equivalently rewrite $A^{-1}\mathcal{H}^\perp$ as

$$\begin{aligned} A^{-1}\mathcal{H}^\perp = (X)_{\delta,\mathcal{X}} &= \{x \in X : \delta x \in \mathcal{X}\} \\ &= \left\{x \in \mathcal{D}(A) : Ax \in \mathcal{H}^\perp\right\}. \end{aligned} \tag{A.3}$$

Similarly, we set

$$Y = \mathcal{D}(A^{\frac{3}{4}}) = \Phi, \quad \mathcal{Y} = \left[\mathcal{D}(A^{\frac{1}{4}})\right]' = \bar{\mathcal{Y}} \equiv \Psi, \quad \delta = A, \quad \delta \in \mathcal{L}(\Phi; \Psi), \tag{A.4}$$

so that $\delta \in \mathcal{L}(X;\bar{\mathcal{X}}) \cap \mathcal{L}(Y;\bar{\mathcal{Y}})$ as well and we may equivalently rewrite $\mathcal{D}(A^{\frac{3}{4}})$ as

$$\mathcal{D}(A^{\frac{3}{4}}) = (Y)_{\delta,\mathcal{Y}} = \{y \in Y : \quad \delta y \in \mathcal{Y}\} = \left\{ y \in \mathcal{D}(A^{\frac{3}{4}}) : \quad Ay \in \left[\mathcal{D}(A^{\frac{1}{4}})\right]' \right\}. \tag{A.5}$$

In (A.4), (A.5), $[\,\cdot\,]'$ denotes duality with respect to $L_2(\Omega)$ as a pivot space. Then, our original object $\left[A^{-\frac{1}{2}}\mathcal{H}^{\perp}, \mathcal{D}(A^{\frac{3}{4}})\right]_{\frac{1}{2}}$ is accordingly equivalently rewritten via (A.3) and (A.4) as

$$\left[A^{-\frac{1}{2}}\mathcal{H}^{\perp}, \mathcal{D}(A^{\frac{3}{4}})\right]_{\frac{1}{2}} = [(X)_{\delta,\mathcal{X}}, (Y)_{\delta,\mathcal{Y}}]_{\frac{1}{2}}\,. \tag{A.6}$$

Finally, to verify the remaining assumption in [24, Eqn. (14.23)(iii)], we take $\mathcal{G} = A^{-1}$, $\chi \in \bar{\mathcal{X}} + \bar{\mathcal{Y}} = [\mathcal{D}(A^{\frac{1}{4}})]'$, and $r = 0$. We can now appeal to [24, Theorem 14.3 p. 97] to get

$$\left[A^{-\frac{1}{2}}\mathcal{H}^{\perp}, \mathcal{D}(A^{\frac{3}{4}})\right]_{\frac{1}{2}} = [(X)_{\delta,\mathcal{X}}, (Y)_{\delta,\mathcal{Y}}]_{\frac{1}{2}} = \left([X,Y]_{\frac{1}{2}}\right)_{\delta,[\mathcal{X},\mathcal{Y}]_{\frac{1}{2}}}. \tag{A.7}$$

But from (A.2) and (A.4), we compute

$$[X,Y]_{\frac{1}{2}} = \left[\mathcal{D}(A), \mathcal{D}(A^{\frac{3}{4}})\right]_{\frac{1}{2}} = \mathcal{D}(A^{\frac{7}{8}}). \tag{A.8}$$

as desired. Via (A.7) and (A.8), our sought after conclusion (A.1) will be established, as soon as we verify that the required constraint

$$\delta\left([X,Y]_{\frac{1}{2}}\right) \in [\mathcal{X},\mathcal{Y}]_{\frac{1}{2}} \tag{A.9}$$

is automatically satisfied. Via (A.8) and $\delta = A$, $\mathcal{X} = \mathcal{H}^{\perp}$ and $\mathcal{Y} = [\mathcal{D}(A^{\frac{1}{4}})]'$ in (A.2),(A.4), we re-write the terms in (A.9) explicitly as

$$\delta\left([X,Y]_{\frac{1}{2}}\right) = A\mathcal{D}(A^{\frac{7}{8}}) = [\mathcal{D}(A^{\frac{1}{8}})]'; \tag{A.10}$$

$$[\mathcal{X},\mathcal{Y}]_{\frac{1}{2}} = \left[\mathcal{H}^{\perp}, [\mathcal{D}(A^{\frac{1}{4}})]'\right]_{\frac{1}{2}} = \left[\mathcal{D}(A^{\frac{1}{4}}), (\mathcal{H}^{\perp})'\right]'_{\frac{1}{2}}. \tag{A.11}$$

where in the last step we have invoked the duality result [24, Theorem 6.2, p. 29]. In conclusion, via (A.10) and (A.11), verifying the validity of statement (A.9) means establishing that

$$\left[\mathcal{D}(A^{\frac{1}{8}})\right]' \subset \left[\mathcal{D}(A^{\frac{1}{4}}), (\mathcal{H}^{\perp})'\right]'_{\frac{1}{2}}, \tag{A.12}$$

where $(\mathcal{H}^{\perp})'$ denotes duality with respect to the $L_2(\Omega)$-topology. In turn (A.12) is equivalent to

$$\left[\mathcal{D}(A^{\frac{1}{4}}), (\mathcal{H}^{\perp})'\right]_{\frac{1}{2}} \subset \mathcal{D}(A^{\frac{1}{8}}) = \left[\mathcal{D}(A^{\frac{1}{4}}), L_2(\Omega)\right]_{\frac{1}{2}}, \tag{A.13}$$

which is plainly true. Thus, the required condition (A.9) has been verified. In conclusion, (A.7), (A.8), (A.9) cumulatively establish the validity of (A.1).

Acknowledgment

The research of Catherine Lebiedzik was partially supported by the National Science Foundation under grant DMS-0708076. The research of Roberto Triggiani was partially supported by the National Science Foundation under grant DMS-0104305 and the AFOSR under grant FA9550-09-1-0459.

References

[1] G. Avalos, Wellposedness of a structural acoustic model with point control, *Differential geometric methods in the control of partial differential equations* (Boulder, CO, 1999), 1–22, Contemp. Math. 268, Amer. Math. Soc., Providence, RI, 2000.

[2] G. Avalos and I. Lasiecka, Exponential stability of a simply supported thermoelastic system without mechanical dissipation, *SIAM J. Math. Anal.* 29(1) (1998), 155–182.

[3] G. Avalos and I. Lasiecka, Differential Riccati equations for the active control of a problem in structural acoustics, *J. Optim. Theory Appl.* 91(3) (1996), 695–728.

[4] F. Bucci, Control theoretic properties of structural acoustic models with thermal effects, I. Singular estimates, *J. Evol. Equ.* 7 (2007), 387–414.

[5] F. Bucci, I. Lasiecka and R. Triggiani, Singular estimates and uniform stability of coupled systems of hyperbolic/parabolic PDE's, *Abstr. Appl. Anal.* 7(4) (2002), 169–236.

[6] J. Cagnol, I. Lasiecka, C. Lebiedzik and J.-P. Zolésio, Uniform stability in structural acoustic models with flexible curved walls, *J. Differential Equations* 186(1) (2003), 88–121.

[7] M. Camurdan and R. Triggiani, Sharp regularity of a coupled system of a wave and a Kirchhoff equation with point control arising in noise reduction, *Differential Integral Equations* 12(1) (1999), 101–118.

[8] L. De Simon, Una applicazione della teoria degli integrali singolari allo studio delle equazioni differenziali lineari astratte del primo ordine, *Rend. Sem. Mat. Univ. Padova* 34 (1964), 205–223.

[9] M. Eller, I. Lasiecka and R. Triggiani, Simultaneous exact/approximate boundary controllability of thermoelastic plates with variable coefficient, Marcel Dekker Lecture Notes Pure and Applied Mathematics 216 (February 2001), 109–230, invited paper for the special volume entitled *Shape Optimization and Optimal Designs*, J.P. Zolesio (Editor). [Preliminary version, an invited paper in *Semigroup of Operators and Applications*, Birkhäuser (2000), 335–351, A.V. Balakrishnan (editor).]

[10] M. Eller, I. Lasiecka and R. Triggiani, Simultaneous exact/approximate boundary controllability of thermoelastic plates with variable thermal coefficient and clamped controls, *Discrete Contin. Dynam. Systems* 7(2) (2001), 283–301.

[11] H.O. Fattorini, *The Cauchy Problem*, Encyclopedia of Mathematics and its Applications, Addison-Wesley, 1983.

[12] J. Lagnese, *Boundary Stabilization of Thin Plates*, SIAM, Philadelphia, 1989.

[13] J. Lagnese and J. L. Lions, *Modelling Analysis and Control of Thin Plates*, Research in Applied Mathematics, Vol. 6, Masson, Paris, 1988.

[14] I. Lasiecka, Unified theory for abstract parabolic boundary problems – A semigroup approach, *Appl. Math. Optim.* 6 (1980), 287–333.

[15] I. Lasiecka and C. Lebiedzik, Decay rates in interactive hyperbolic-parabolic PDE models with thermal effects on the interface, *Appl. Math. Optim.* 42(2) (2000), 127–167.

[16] I. Lasiecka, J. L. Lions and R. Triggiani, Nonhomogeneous boundary value problems for second-order operators, *J. Math. Pures Appl.* 65 (1986), 149–152.

[17] I. Lasiecka and R. Triggiani, Exact controllability and uniform stabilization of Euler-Bernoulli equations with boundary control only in $\Delta w|_\Sigma$, *Boll. Un. Mat. Ital.* (7)5-B (1991), 665–702.

[18] I. Lasiecka and R. Triggiani, Sharp regularity theory for elastic and thermoelastic Kirchhoff equations with free boundary conditions, *Rocky Mountain J. Math.* 30(3) (2000), 981–1024.

[19] I. Lasiecka and R. Triggiani, *Control theory for partial differential equations: Continuous and approximation theories*, Cambridge University Press, Encyclopedia of Mathematics and its Applications, vols. 74 and 75, January 2000.

[20] I. Lasiecka and R. Triggiani, Structural decomposition of thermo-elastic semigroups with rotational forces, *Semigroup Forum* 60 (2000), 16–66.

[21] I. Lasiecka and R. Triggiani, Factor spaces and implications on Kirchhoff elastic and thermoelastic systems with clamped boundary conditions, *Abst. Appl. Anal.* 6(8) (2001), 1–48. Preliminary announcement in Oberwolfach Proceedings: Control of Complex Systems, Kluwer.

[22] C. Lebiedzik, Uniform stability of a coupled structural acoustic system with thermoelastic effects and weak structural damping, *Differential geometric methods in the control of partial differential equations* (Boulder, CO, 1999), 327–340, *Contemp. Math.* 268, Amer. Math. Soc., Providence, RI, 2000.

[23] C. Lebiedzik and R. Triggiani, Optimal Regularity and Optimal Control of a Thermoelastic Structural Acoustic Model with Point Control and Clamped Boundary Conditions, Invited paper, *Control and Cybernetics*. Special issue on "50 Years of Optimal Control", Bedlewo, Poland, September 2008, Volume 28, 4 (2009), 1461–1500.

[24] J. L. Lions and E. Magenes, *Non-homogeneous Boundary Value Problems and Applications*, vol. 1, Springer-Verlag, New York, 1972.

[25] N. Ourada and R. Triggiani, Uniform stabilization of the Euler-Bernoulli equation with feedback operator only in the Neumann boundary condition, *Differential Integral Equations* 4 (1991), 277–291.

[26] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, 1983.

[27] C. Sadosky, *Interpolation of Operators and Singular Integrals*, Marcel Dekker, 1979.

[28] R. Triggiani, Regularity with interior point control, Part II: Kirchhoff equations, *J. Differential Equations* 103 (1993), 394–421.

[29] R. Triggiani, Sharp regularity for thermo-elastic mixed problems, *Applicable Analysis* 77 (2001), 419–433. [Reprint of an electronically faulty version in *Applicable Anal.*, 73(3-4) (1999), 557–572.

[30] R. Triggiani, Sharp regularity of hyperbolic-dominated thermoelastic systems with point control: the hinged case, *J. Math. Anal. Appl.* 313 (2007), 530–542.

[31] R. Triggiani, Sharp regularity of hyperbolic-dominated thermoelastic systems with point control: the clamped case, *Discrete Contin. Dyn. Syst.*, Supplement 2007, 993–1004.

[32] R. Triggiani, The Critical Case of Clamped Thermoelastic Systems with Interior Point Control: Optimal Interior and Boundary Regularity Results, *J. Differential Equations* 245 (2008) 3764–3805.

Catherine Lebiedzik
Department of Mathematics
Wayne State University
656 W Kirby
Detroit
MI 48202, USA
e-mail: kate@math.wayne.edu

Roberto Triggiani
Department of Mathematics
University of Virginia / KFUPM, Dhahran, 31261, S.A.
Charlottesville
VA 22903, USA
e-mail: rt7u@virginia.edu

Operator Theory:
Advances and Applications, Vol. 216, 261–274

Multidimensional Controllability Problems with Memory

Paola Loreti and Daniela Sforza

Communicated by F. Bucci and I. Lasiecka

Abstract. We study reachability problems for a class of partial integro-differential equations arising in viscoelasticity theory. Our approach is based on the Hilbert Uniqueness Method and nonharmonic analysis techniques.

Mathematics Subject Classification (2000). 42A38; 45K05; 93C20.

Keywords. Reachability, Hilbert Uniqueness Method, Ingham estimates.

1. Introduction

In this work we study reachability problems for the following partial integro-differential equations

$$u_{tt}(t,x) - \triangle u(t,x) + \beta \int_0^t e^{-\eta(t-s)} \triangle u(s,x)ds = 0\,, \quad t \in (0,T)\,,\ x \in \Omega, \tag{1.1}$$

where $\triangle$ denotes the Laplace operator in an open ball Ω of radius R in $\mathbb{R}^N$ $(N \geq 2)$ and $0 < \beta < \eta$. The solution u of (1.1) is subject to null initial data

$$u(0,x) = u_t(0,x) = 0, \qquad x \in \Omega\,, \tag{1.2}$$

and boundary conditions

$$u(t,x) = g(t,x) \qquad t \in [0,T]\,,\ x \in \partial\Omega\,, \tag{1.3}$$

where $\partial\Omega$ denotes the boundary of Ω.

If we consider g as a control function, our reachability problem consists in proving the existence of $g \in L^2((0,T) \times \partial\Omega)$ such that a weak solution of equation (1.1), subject to boundary conditions (1.3), moves from the null state to a given one in finite control time. To be more precise, we adopt the same definition of reachability problems for systems with memory given by several authors in the literature, see for example [17, 6, 7, 11, 13, 14, 20, 21]. Indeed, we mean the following: given $T > 0$, $u_0 \in L^2(\Omega)$ and $u_1 \in H^{-1}(\Omega)$, find $g \in L^2((0,T) \times \partial\Omega)$ such that the weak solution $u \in C([0,T], L^2(\Omega)) \cap C^1([0,T], H^{-1}(\Omega))$ of problem (1.1)–(1.3) verifies the final conditions

$$u(T,x) = u_0(x)\,, \qquad u_t(T,x) = u_1(x)\,, \quad x \in \Omega\,. \tag{1.4}$$

Our goal is to achieve such result without any smallness assumption on the convolution kernel, as suggested by J.-L. Lions in [17, p. 258]. Moreover, due to the finite speed of propagation, we expect that the controllability time T will be sufficiently large. Indeed, we will find that $T > 2$, see Theorem 7.1. The one-dimensional case $N = 1$ has been studied in [18, 19].

As it is well known, a common way for studying exact controllability problems is the so-called Hilbert Uniqueness Method, see [10, 15, 16, 17]. We will apply this method to equation (1.1). The HUM method is based on a "uniqueness theorem" for the adjoint problem. To prove such uniqueness theorem we employ some typical techniques of nonharmonic analysis, see [26]. This approach relies on Fourier series development for the solution of the adjoint problem, that exhibits an expansion of type (6.4) below. In this framework Ingham type estimates (see [5]) play an important role. Indeed, if we apply to functions (6.4) inverse and direct inequalities obtained in [18, 19] (see Theorems 3.1 and 3.2) then we are able to prove our reachability result.

To sum up, our approach is based on nonharmonic analysis, in particular on Ingham type estimates, which could be of interest in themselves. However, our methodology brings about some restrictions on the convolution kernel. We refer to [22] for exact boundary controllability for the Gurtin-Pipkin heat equation with more general kernels. That first-order equation leads to a second-order problem similar to ours, but with only one reachability condition on the final data. In our approach, we are able to handle both conditions: on state and on speed. In addition, we also obtain a sharp estimate for the controllability time, which is not present in [22] due to the more general setting.

Exponential kernels arise in linear viscoelasticity theory, such as in the analysis of Maxwell fluids or Poynting-Thomson solids, see, e.g., [23, 25]. For other references in viscoelasticity theory see the seminal papers of Dafermos [1, 2] and [24, 12].

Other papers related to our problem are [3, 14, 27, 28], where the approach is not of Ingham type.

The plan of our paper is the following. In Section 2 we give some preliminary results. In Section 3 we recall Ingham type theorems. In Section 4 we describe the HUM method. In Section 5 we recall some known facts concerning the eigenfunctions of the Laplace operator in a ball. In Section 6 we show that the solution of the adjoint problem can be written as a Fourier series. Finally, in Section 7 we give our reachability result.

2. Preliminaries

For any $T \in (0, \infty)$, we denote by $L^1(0,T)$ the usual spaces of measurable functions $u : (0,T) \to \mathbb{R}$ such that one has

$$\|u\|_1 := \int_0^T |u(t)|\, dt < \infty .$$

We denote by $L^1_{\mathrm{loc}}(0,\infty)$ the space of functions belonging to $L^1(0,T)$ for any $T \in (0,\infty)$.

Classical results for integral equations (see, e.g., [4, Theorem 2.3.5]) ensure that, for any kernel $k \in L^1_{\mathrm{loc}}(0,\infty)$ and any $v \in L^1_{\mathrm{loc}}(0,\infty)$, the problem

$$u(t) - k * u(t) = v(t), \qquad t \geq 0 , \tag{2.1}$$

admits a unique solution $u \in L^1_{\mathrm{loc}}(0,\infty)$. In particular, there is a unique solution $\varrho_k \in L^1_{\mathrm{loc}}(0,\infty)$ of

$$\varrho_k(t) - k * \varrho_k(t) = k(t), \qquad t \geq 0 . \tag{2.2}$$

Such a solution is called the *resolvent kernel* of k. Furthermore, the solution u of (2.1) is given by the variation of constants formula

$$u(t) = v(t) + \varrho_k * v(t), \qquad t \geq 0 , \tag{2.3}$$

where ϱ_k is the resolvent kernel of k. We recall the following result, see, e.g., [19, Lemma 2.1].

Lemma 2.1. *Given $k \in L^1_{\mathrm{loc}}(0,\infty)$ and $v \in L^1(0,T)$, $T > 0$, a function $u \in L^1(0,T)$ is a solution of*

$$u(t) - \int_t^T k(s-t)u(s)ds = v(t) \qquad a.e. \quad t \in (0,T) ,$$

if and only if

$$u(t) = v(t) + \int_t^T \varrho_k(s-t)v(s)\ ds \qquad a.e. \quad t \in (0,T) ,$$

where ϱ_k is the resolvent kernel of k.

Let Ω be an open ball Ω of radius R in $\mathbb{R}^N$ $(N \geq 2)$. In the following we consider $L^2(\Omega)$ and $H^1_0(\Omega)$ endowed with the standard norms

$$\|u\|^2 = \int_\Omega |u(x)|^2\ dx , \qquad \|u\|^2_{H^1_0(\Omega)} = \int_\Omega |\nabla u(x)|^2\ dx ,$$

and $H^{-1}(\Omega)$ is endowed with the dual norm of $\|\cdot\|_{H^1_0(\Omega)}$.

3. Ingham type theorems

In [19, Theorems 1.1 and 1.2] Ingham type inverse and direct inequalities have been proved. In this section we recall those results, presented them in a slight different formulation.

In the following two theorems we consider functions of the type

$$t \mapsto \sum_{n=1}^{\infty} \left(R_n e^{r_n t} + C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t}\right) \qquad t \geq 0$$

with $r_n, R_n \in \mathbb{R}$ and $\omega_n\,, C_n \in \mathbb{C}$ such that the sequences $\{r_n\}$, $\{\Im\omega_n\}$ are bounded and

$$\sum_{n=1}^{\infty} |R_n|^2 < +\infty\,, \qquad \sum_{n=1}^{\infty} |C_n|^2 < +\infty\,.$$

Theorem 3.1. *Let $\{\omega_n\}_{n\in\mathbb{N}}$ and $\{r_n\}_{n\in\mathbb{N}}$ be sequences of pairwise distinct numbers such that $r_n \neq i\omega_m$ for any $n\,, m \in \mathbb{N}$. Assume*

$$\liminf_{n\to\infty}(\Re\omega_{n+1} - \Re\omega_n) = \gamma > 0\,,$$
$$\lim_{n\to\infty} \Im\omega_n = \alpha\,, \qquad r_n \leq -\Im\omega_n \qquad \forall\ n \geq n'\,,$$
$$|R_n| \leq \frac{\mu}{n^\nu}|C_n| \quad \forall\ n \geq n'\,, \qquad |R_n| \leq \mu|C_n| \quad \forall\ n \leq n'\,,$$

for some $n' \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $\mu > 0$ and $\nu > 1/2$. Then, for any $T > 2\pi/\gamma$ we have

$$\int_0^T \Big|\sum_{n=1}^{\infty} \left(R_n e^{r_n t} + C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t}\right)\Big|^2 dt \geq c_1(T) \sum_{n=1}^{\infty} |C_n|^2\,, \tag{3.1}$$

where $c_1(T)$ is a positive constant.

Theorem 3.2. *Assume*

$$\liminf_{n\to\infty}(\Re\omega_{n+1} - \Re\omega_n) = \gamma > 0\,,$$
$$\lim_{n\to\infty} \Im\omega_n = \alpha\,,$$
$$|R_n| \leq \frac{\mu}{|n|^\nu}|C_n| \quad \forall\ n \geq n'\,, \qquad |R_n| \leq \mu|C_n| \quad \forall\ n \leq n'\,,$$

for some $n' \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $\mu > 0$ and $\nu > 1/2$. Then, for any $T > \pi/\gamma$ we have

$$\int_{-T}^T \Big|\sum_{n=1}^{\infty} \left(R_n e^{r_n t} + C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t}\right)\Big|^2 dt \leq c_2(T) \sum_{n=1}^{\infty} |C_n|^2\,, \tag{3.2}$$

where $c_2(T)$ is a positive constant.

4. Hilbert Uniqueness Method

To render the paper self-contained, in this section we describe the Hilbert Uniqueness Method.

To begin, we consider the integro-differential equation

$$u_{tt}(t,x) - \triangle u(t) + \int_0^t k(t-s)\triangle u(s,x)ds = 0 \qquad t \in (0,T)\,, \quad x \in \Omega\,, \tag{4.1}$$

where $k \in L^1_{\rm loc}(0,\infty)$, with null initial conditions

$$u(0,x) = u_t(0,x) = 0 \qquad x \in \Omega\,, \tag{4.2}$$

and boundary conditions

$$u(t,x) = g(t,x) \qquad t \in (0,T)\,, \quad x \in \partial\Omega\,. \tag{4.3}$$

For a reachability problem we mean the following: given $T > 0$, $u_0 \in L^2(\Omega)$ and $u_1 \in H^{-1}(\Omega)$, find $g \in L^2((0,T)\times\partial\Omega)$ such that the weak solution u of problem (4.1)-(4.3) verifies the final conditions

$$u(T,x) = u_0(x)\,, \qquad u_t(T,x) = u_1(x)\,. \tag{4.4}$$

To explain how we can solve a reachability problem by the HUM method, we proceed as follows.

Given $z_0 \in C_c^\infty(\Omega)$ and $z_1 \in C_c^\infty(\Omega)$, we introduce the *adjoint* equation of (4.1), that is

$$z_{tt}(t,x) - \triangle z(t,x) + \int_t^T k(s-t)\triangle z(s,x)ds = 0 \qquad t \in [0,T]\,, \quad x \in \Omega\,, \tag{4.5}$$

$$z(t,x) = 0 \qquad t \in [0,T] \quad x \in \partial\Omega\,, \tag{4.6}$$

with final data

$$z(T,\cdot) = z_0\,, \qquad z_t(T,\cdot) = z_1\,. \tag{4.7}$$

The above problem is well posed, see, e.g., [23].

If we denote by ν the outward unit normal vector to $\partial\Omega$ and $\partial_\nu z(t,x)$ the normal derivative of z, we consider the problem

$$\begin{cases} \varphi_{tt}(t,x) - \triangle\varphi(t,x) + \displaystyle\int_0^t k(t-s)\triangle\varphi(s,x)ds = 0 \qquad t \in [0,T]\,, \quad x \in \Omega\,, \\ \varphi(t,x) = \partial_\nu z(t,x) - \displaystyle\int_t^T k(s-t)\partial_\nu z(s,x)ds \quad t \in [0,T] \quad x \in \partial\Omega\,, \\ \varphi(0,\cdot) = \varphi_t(0,\cdot) = 0\,. \end{cases} \tag{4.8}$$

It can be proved as in the non integral case that non homogeneous problem (4.8) admits a unique solution φ. Then, we can define the linear operator

$$\Psi(z_0,z_1) = (-\varphi_t(T,\cdot), \varphi(T,\cdot))\,, \qquad (z_0,z_1) \in C_c^\infty(\Omega) \times C_c^\infty(\Omega)\,. \tag{4.9}$$

Let $(\xi_0,\xi_1)\in C_c^\infty(\Omega)\times C_c^\infty(\Omega)$ and ξ the solution of

$$\begin{cases} \xi_{tt}(t,x)-\triangle\xi(t,x)+\int_t^T k(s-t)\triangle\xi(s,x)ds=0 \qquad t\in[0,T]\,,\quad x\in\Omega\,,\\ \xi(t,x)=0 \qquad t\in[0,T] \quad x\in\partial\Omega\,,\\ \xi(T,\cdot)=\xi_0\,, \qquad \xi_t(T,\cdot)=\xi_1\,. \end{cases} \tag{4.10}$$

We prove that

$$\langle\Psi(z_0,z_1),(\xi_0,\xi_1)\rangle = \int_0^T\int_{\partial\Omega}\varphi(t,x)\Big(\partial_\nu\xi(t,x)-\int_t^T k(s-t)\partial_\nu\xi(s,x)ds\Big)\,dxdt\,. \tag{4.11}$$

Indeed, multiplying the equation in (4.8) by $\xi(t,x)$ and integrating on $[0,T]\times\Omega$ we have

$$\int_0^T\int_\Omega\varphi_{tt}(t,x)\xi(t,x)\ dx\ dt-\int_0^T\int_\Omega\triangle\varphi(t,x)\xi(t,x)\ dx\ dt + \int_0^T\int_\Omega\int_0^t k(t-s)\triangle\varphi(s,x)\ ds\ \xi(t,x)\ dx\ dt=0\,.$$

If we take into account that

$$\int_0^T\int_0^t k(t-s)\triangle\varphi(s,x)\ ds\ \xi(t,x)\ dt=\int_0^T\triangle\varphi(s,x)\int_s^T k(t-s)\ \xi(t,x)\ dt\ ds$$

and integrate by parts twice both respect to t and respect to x, then we have

$$\begin{aligned}&\int_\Omega\varphi_t(T,x)\xi(T,x)\ dx-\int_\Omega\varphi(T,x)\xi_t(T,x)\ dx\\ &\quad+\int_0^T\int_\Omega\varphi(t,x)\Big(\xi_{tt}(t,x)-\triangle\xi(t,x)+\int_t^T k(s-t)\triangle\xi(s,x)ds\Big)\ dx\ dt\\ &\quad+\int_0^T\int_{\partial\Omega}\varphi(t,x)\partial_\nu\xi(t,x)\ dx\ dt\\ &\quad-\int_0^T\int_{\partial\Omega}\varphi(t,x)\int_t^T k(s-t)\partial_\nu\xi(s,x)ds\ dxdt=0\,.\end{aligned}$$

Since ξ is the solution of (4.10), we have that (4.11) holds.

Now, taking $(\xi_0,\xi_1)=(z_0,z_1)$ in (4.11), we have

$$\langle\Psi(z_0,z_1),(z_0,z_1)\rangle=\int_0^T\int_{\partial\Omega}\Big|\partial_\nu z(t,x)-\int_t^T k(s-t)\partial_\nu z(s,x)ds\Big|^2\ dxdt\,. \tag{4.12}$$

So, we can introduce the semi-norm

$$\|(z_0,z_1)\|_F:=\Big(\int_0^T\int_{\partial\Omega}\Big|\partial_\nu z(t,x)-\int_t^T k(s-t)\partial_\nu z(s,x)ds\Big|^2\ dxdt\Big)^{1/2} \tag{4.13}$$

for any $(z_0,z_1)\in C_c^\infty(\Omega)\times C_c^\infty(\Omega)$.

In view of Lemma 2.1 $\|\cdot\|_F$ is a norm if and only if the following uniqueness theorem holds.

Theorem 4.1. *If z is the solution of problem* (4.5)–(4.7) *such that*

$$\partial_\nu z(t,x) = 0\,, \qquad \forall (t,x) \in [0,T] \times \partial\Omega\,,$$

then

$$z(t,x) = 0 \qquad \forall (t,x) \in [0,T] \times \Omega\,.$$

If Theorem 4.1 holds true, then we can define the Hilbert space F as the completion of $C_c^\infty(\Omega) \times C_c^\infty(\Omega)$ for the norm (4.13). Moreover, the operator Ψ extends uniquely to a continuous operator, denoted again by Ψ, from F to the dual space F' in such a way that $\Psi : F \to F'$ is an isomorphism.

In conclusion, if we prove a uniqueness result as Theorem 4.1 and

$$F = H_0^1(\Omega) \times L^2(\Omega),$$

then we can solve the reachability problem (4.1)–(4.4).

5. The eigenfunctions of the Laplace operator in a ball

In this section we first recall some basic facts regarding Bessel type functions (see, e.g., [9]), which will be useful to treat the eigenfunctions of the Laplace operator in a ball.

Let us introduce the Bessel functions of any real order p by the formula

$$J_p(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(p+j+1)} \left(\frac{x}{2}\right)^{p+2j} \qquad x \geq 0\,, \tag{5.1}$$

where Γ is the gamma function.

Lemma 5.1. *Let p be a nonnegative real number. The following equality holds for every positive real number c:*

$$2c^2 \int_0^1 r|J_p(cr)|^2 \; dr = c^2|J_p'(c)|^2 + (c^2 - p^2)|J_p(c)|^2\,. \tag{5.2}$$

As for the location of the zeros of the Bessel functions, the following result holds.

Proposition 5.2.

(a) *For any given real number p, the positive zeros of $J_p(x)$ are simple and they form an infinite strictly increasing sequence $\{\lambda_n\}$ tending to infinity.*
(b) *The difference sequence $\{\lambda_{n+1} - \lambda_n\}$ converges to π.*
(c) *The sequence $\{\lambda_{n+1} - \lambda_n\}$ is strictly decreasing if $|p| > 1/2$, strictly increasing if $|p| < 1/2$ and constant if $p = \pm 1/2$.*

We may assume without loss of generality that Ω is the unit ball of $\mathbb{R}^N$: the general case then follows easily by a linear change of variables. We shall consider the case $N \geq 2$. Let us also recall that the spherical harmonics of order $m \in \mathbb{N}$ are the restrictions to the unit sphere $\partial\Omega$ of the homogeneous polynomials of order m.

Lemma 5.3. *The spherical harmonics of order $m \in \mathbb{N}$ form a finite-dimensional subspace S_m in $L^2(\partial\Omega)$. These subspaces are mutually orthogonal and their linear hull is dense in $L^2(\partial\Omega)$.*

By using hyperspherical coordinates (ρ, θ) with $0 \leq \rho \leq 1$ and $\theta \in \partial\Omega$, we can describe the eigenfunctions of the Laplace operator.

Proposition 5.4. *The eigenfunctions of $-\triangle$ with the homogeneous Dirichlet boundary condition are the functions*

$$E_{mk}(\rho, \theta) := \rho^{1-\frac{N}{2}} J_{m-1+\frac{N}{2}}(\lambda_{mk}\rho) H_m(\theta)\,, \tag{5.3}$$

where $m \in \mathbb{N} \cup \{0\}$, $k \in \mathbb{N}$, $H_m \in S_m$ and for each m we denote by $\{\lambda_{mk}\}_{k\in\mathbb{N}}$ the strictly increasing sequence of positive zeros of the Bessel function $J_{m-1+\frac{N}{2}}(x)$. The corresponding eigenvalue of the eigenfunction $E_{mk}(\rho, \theta)$ is λ_{mk}^2.

6. Fourier series of the solution

Let Ω be the unit ball of $\mathbb{R}^N$, $N \geq 2$, and $T > 0$. For any $v_0 \in H_0^1(\Omega)$ and $v_1 \in L^2(\Omega)$ there exists a unique weak solution v belonging to $C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ of equation

$$v_{tt}(t,x) - \triangle v(t,x) + \beta \int_0^t e^{-\eta(t-s)} \triangle v(s,x)\,ds = 0\,, \qquad t \in (0,T), \quad x \in \Omega\,, \tag{6.1}$$

verifying the Dirichlet boundary condition

$$v(t,x) = 0 \qquad t \in (0,T), \quad x \in \partial\Omega\,, \tag{6.2}$$

and the initial conditions

$$v(0,\cdot) = v_0\,, \qquad v_t(0,\cdot) = v_1\,. \tag{6.3}$$

If we expand the initial data v_0 and v_1 according to the eigenfunctions E_{mk} of $-\triangle$ with the homogeneous Dirichlet boundary condition, see (5.3), then we obtain the expressions

$$v_0(\rho, \theta) = \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} \alpha_{mk} E_{mk}(\rho, \theta)\,, \quad \|v_0\|^2_{H_0^1(\Omega)} = \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} \alpha_{mk}^2 \lambda_{mk}^2 \|E_{mk}\|^2\,,$$

$$v_1(\rho, \theta) = \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} \gamma_{mk} E_{mk}(\rho, \theta)\,, \qquad \|v_1\|^2 = \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} \gamma_{mk}^2 \|E_{mk}\|^2\,.$$

Repeating an analogous procedure to that followed in [19, Section 6], we can write the solution of problem (6.1)–(6.3) as the Fourier series,

$$v(t,\rho,\theta)=\sum_{m=0}^{\infty}\sum_{k=1}^{\infty}(R_{mk}e^{r_{mk}t}+C_{mk}e^{i\omega_{mk}t}+\overline{C_{mk}}e^{-i\overline{\omega_{mk}}t})E_{mk}(\rho,\theta)\,, \tag{6.4}$$

for any $t\in[0,T]$, $0\le\rho\le 1$ and $\theta\in\partial\Omega$. In the above formula $r_{mk}\,,R_{mk}\in\mathbb{R}$ and $\omega_{mk}\,,C_{mk}\in\mathbb{C}$ are defined by

$$r_{mk}=\beta-\eta-\frac{\beta(\beta-\eta)^2}{\lambda_{mk}^2}+O\Big(\frac{1}{\lambda_{mk}^3}\Big)\,, \tag{6.5}$$

$$R_{mk}=\frac{\alpha_{mk}\beta^2-\alpha_{mk}\eta\beta+\gamma_{mk}\beta+O\Big(\frac{1}{\lambda_{mk}^2}\Big)}{1+(\eta^2+3\beta^2-4\eta\beta)\frac{1}{\lambda_{mk}^2}+O\Big(\frac{1}{\lambda_{mk}^4}\Big)}\;\frac{1}{\lambda_{mk}^2}\,, \tag{6.6}$$

$$\omega_{mk}=\lambda_{mk}+\frac{\beta}{2}\Big(\frac{3}{4}\beta-\eta\Big)\frac{1}{\lambda_{mk}}+O\Big(\frac{1}{\lambda_{mk}^3}\Big)+i\Big[\frac{\beta}{2}-\frac{\beta(\beta-\eta)^2}{2\lambda_{mk}^2}+O\Big(\frac{1}{\lambda_{mk}^3}\Big)\Big]\,, \tag{6.7}$$

$$\begin{aligned}C_{mk}=&\frac{\alpha_{mk}\lambda_{mk}^2+\gamma_{mk}(\frac{\beta}{2}-\eta)+\alpha_{mk}(\beta-\eta)\frac{\beta}{2}+\alpha_{mk}O\Big(\frac{1}{\lambda_{mk}^2}\Big)+\gamma_{mk}O\Big(\frac{1}{\lambda_{mk}^2}\Big)}{2\lambda_{mk}^2+\frac{3}{2}\beta^2-2\beta\eta+O\Big(\frac{1}{\lambda_{mk}^2}\Big)-i\Big[(2\eta-3\beta)\lambda_{mk}+O\Big(\frac{1}{\lambda_{mk}}\Big)\Big]}\\ &-\frac{i\Big[(\gamma_{mk}-\alpha_{mk}\beta+\alpha_{mk}\eta)\lambda_{mk}+\alpha_{mk}O\Big(\frac{1}{\lambda_{mk}}\Big)+\gamma_{mk}O\Big(\frac{1}{\lambda_{mk}}\Big)\Big]}{2\lambda_{mk}^2+\frac{3}{2}\beta^2-2\beta\eta+O\Big(\frac{1}{\lambda_{mk}^2}\Big)-i\Big[(2\eta-3\beta)\lambda_{mk}+O\Big(\frac{1}{\lambda_{mk}}\Big)\Big]}\,.\end{aligned} \tag{6.8}$$

Moreover, there exist some constants $c_1\,,c_2>0$ such that, for any $m\in\mathbb{N}\cup\{0\}$ and sufficiently large k, one has

$$c_1\Big(\alpha_{mk}^2\lambda_{mk}^2+\gamma_{mk}^2\Big)\le\lambda_{mk}^2|C_{mk}|^2\le c_2\Big(\alpha_{mk}^2\lambda_{mk}^2+\gamma_{mk}^2\Big)\,. \tag{6.9}$$

7. A reachability result

In this section we will show our reachability result.

Theorem 7.1. *Let $\eta>3\beta/2$. For any $T>2$, $u_0\in L^2(\Omega)$ and $u_1\in H^{-1}(\Omega)$ there exists $g\in L^2((0,T)\times\partial\Omega)$ such that the weak solution $u\in C([0,T],L^2(\Omega))\cap C^1([0,T],H^{-1}(\Omega))$ of problem*

$$\begin{cases}u_{tt}(t,x)-\triangle u(t,x)+\beta\displaystyle\int_0^t e^{-\eta(t-s)}\triangle u(s,x)ds=0\,, & t\in(0,T),\ x\in\Omega,\\ u(0,x)=u_t(0,x)=0, & x\in\Omega,\\ u(t,x)=g(t,x)\,, \quad t\in(0,T), & x\in\partial\Omega,\end{cases} \tag{7.1}$$

verifies the final conditions

$$u(T,x)=u_0(x)\,,\qquad u_t(T,x)=u_1(x)\,,\quad x\in\Omega\,. \tag{7.2}$$

Proof. To prove our claim, we apply the HUM method described in Section 4. First, we consider the adjoint equation of (7.1), that is

$$z_{tt}(t,x)-\triangle z(t,x)+\beta\int_t^T e^{-\eta(s-t)}\triangle z(s,x)ds=0,\quad t\in(0,T),\quad x\in\Omega\,, \tag{7.3}$$

with the Dirichlet boundary condition

$$z(t,x)=0\qquad t\in(0,T),\quad x\in\partial\Omega\,, \tag{7.4}$$

and final data

$$z(T,\cdot)=z_0\,,\qquad z_t(T,\cdot)=z_1\,, \tag{7.5}$$

where $z_0\in H_0^1(\Omega)$ and $z_1\in L^2(\Omega)$. It is easy to verify that the backward problem (7.3)–(7.5) is equivalent to a Cauchy problem of the type (6.1)–(6.3) with $v(t,x)=z(T-t,x)$. Therefore, we can apply the conclusions of the previous section to write the solution $z(t,x)$ of the adjoint problem as a Fourier series. Indeed, we expand the final data z_0 and z_1 according to the eigenfunctions E_{mk} of $-\triangle$ with the homogeneous Dirichlet boundary condition: by using hyperspherical coordinates (ρ,θ) with $0\le\rho\le1$ and $\theta\in\partial\Omega$, we have

$$z_0(\rho,\theta)=\sum_{m=0}^\infty\sum_{k=1}^\infty\alpha_{mk}E_{mk}(\rho,\theta)\,,\quad \|z_0\|^2_{H_0^1(\Omega)}=\sum_{m=0}^\infty\sum_{k=1}^\infty\alpha_{mk}^2\lambda_{mk}^2\|E_{mk}\|^2\,, \tag{7.6}$$

$$z_1(\rho,\theta)=\sum_{m=0}^\infty\sum_{k=1}^\infty\gamma_{mk}E_{mk}(\rho,\theta)\,,\qquad \|z_1\|^2=\sum_{m=0}^\infty\sum_{k=1}^\infty\gamma_{mk}^2\|E_{mk}\|^2\,, \tag{7.7}$$

with

$$\|E_{mk}\|^2=\int_0^1\rho|J_{m-1+\frac{N}{2}}(\lambda_{mk}\rho)|^2\,d\rho\int_{\partial\Omega}|H_m(\theta)|^2\,d\theta\,. \tag{7.8}$$

Therefore z can be written as in formula (6.4), that is

$$z(t,\rho,\theta)=\sum_{m=0}^\infty\sum_{k=1}^\infty(R_{mk}e^{r_{mk}(T-t)}+C_{mk}e^{i\omega_{mk}(T-t)}+\overline{C_{mk}}e^{-i\overline{\omega_{mk}}(T-t)})E_{mk}(\rho,\theta)\,,$$
$$t\in[0,T],\quad 0\le\rho\le1\,,\quad \theta\in\partial\Omega\,, \tag{7.9}$$

where $r_{mk}\,,R_{mk}\,,\omega_{mk}\,,C_{mk}$ are given by formulas (6.5)–(6.8) respectively. Keeping in mind that, for any $m\in\mathbb{N}\cup\{0\}$, λ_{mk} are zeros of the Bessel function $J_{m-1+\frac{N}{2}}$, it follows that

$$\partial_\nu z(t,1,\theta)=\sum_{m=0}^\infty\sum_{k=1}^\infty\lambda_{mk}J'_{m-1+\frac{N}{2}}(\lambda_{mk})R_{mk}e^{r_{mk}(T-t)}H_m(\theta) \tag{7.10}$$
$$+\sum_{m=0}^\infty\sum_{k=1}^\infty\lambda_{mk}J'_{m-1+\frac{N}{2}}(\lambda_{mk})(C_{mk}e^{i\omega_{mk}(T-t)}+\overline{C_{mk}}e^{-i\overline{\omega_{mk}}(T-t)})H_m(\theta)\,.$$

Since the spherical harmonics of different order are orthogonal in $L^2(\partial\Omega)$ (see Lemma 5.3) we have

$$\begin{aligned}&\int_{\partial\Omega}|\partial_\nu z(t,1,\theta)|^2 d\theta\\&=\sum_{m=0}^{\infty}\Big|\sum_{k=1}^{\infty}\lambda_{mk}J'_{m-1+\frac{N}{2}}(\lambda_{mk})(R_{mk}e^{r_{mk}(T-t)}+C_{mk}e^{i\omega_{mk}(T-t)}+\overline{C_{mk}}e^{-i\overline{\omega_{mk}}(T-t)})\Big|^2\\&\qquad\cdot\int_{\partial\Omega}|H_m(\theta)|^2 d\theta,\end{aligned}$$

whence, integrating from 0 to T, we get

$$\begin{aligned}&\int_0^T\int_{\partial\Omega}|\partial_\nu z(t,1,\theta)|^2 d\theta\ dt\\&=\sum_{m=0}^{\infty}\int_0^T\Big|\sum_{k=1}^{\infty}\lambda_{mk}J'_{m-1+\frac{N}{2}}(\lambda_{mk})(R_{mk}e^{r_{mk}t}+C_{mk}e^{i\omega_{mk}t}+\overline{C_{mk}}e^{-i\overline{\omega_{mk}}t})\Big|^2\ dt\\&\qquad\cdot\int_{\partial\Omega}|H_m(\theta)|^2 d\theta\,.\end{aligned}\tag{7.11}$$

Now, we observe that in view of (6.7) one gets

$$\Re\omega_{m,k+1}-\Re\omega_{mk}=\lambda_{m,k+1}-\lambda_{mk}+\frac{\beta}{2}\Big(\eta-\frac{3}{4}\beta\Big)\Big(\frac{1}{\lambda_{mk}}-\frac{1}{\lambda_{m,k+1}}\Big)+O\Big(\frac{1}{\lambda_{mk}^3}\Big)\,.$$

Taking into account the behavior of zeros of Bessel functions (see Proposition 5.2) and the assumption $\eta>3\beta/2$, the numbers ω_{mk}, r_{mk}, $\lambda_{mk}J'_{m-1+\frac{N}{2}}(\lambda_{mk})R_{mk}$, $\lambda_{mk}J'_{m-1+\frac{N}{2}}(\lambda_{mk})C_{mk}$ verify the conditions of Theorems 3.1 and 3.2. So for any $m\in\mathbb{N}\cup\{0\}$ we can apply those theorems to the function

$$\sum_{k=1}^{\infty}\lambda_{mk}J'_{m-1+\frac{N}{2}}(\lambda_{mk})(R_{mk}e^{r_{mk}t}+C_{mk}e^{i\omega_{mk}t}+\overline{C_{mk}}e^{-i\overline{\omega_{mk}}t})\,.$$

Indeed, thanks to inequalities (3.1) and (3.2), for any $T>2$ we have

$$\begin{aligned}&c_1(T)\sum_{k=1}^{\infty}\lambda_{mk}^2|C_{mk}|^2|J'_{m-1+\frac{N}{2}}(\lambda_{mk})|^2\\&\quad\le\int_0^T\Big|\sum_{k=1}^{\infty}\lambda_{mk}J'_{m-1+\frac{N}{2}}(\lambda_{mk})(R_{mk}e^{r_{mk}t}+C_{mk}e^{i\omega_{mk}t}+\overline{C_{mk}}e^{-i\overline{\omega_{mk}}t})\Big|^2\ dt\\&\quad\le c_2(T)\sum_{k=1}^{\infty}\lambda_{mk}^2|C_{mk}|^2|J'_{m-1+\frac{N}{2}}(\lambda_{mk})|^2\,.\end{aligned}$$

By (7.11), in view of the above inequalities we get

$$c_1(T)\sum_{m=0}^{\infty}\sum_{k=1}^{\infty}\lambda_{mk}^2|C_{mk}|^2|J'_{m-1+\frac{N}{2}}(\lambda_{mk})|^2\int_{\partial\Omega}|H_m(\theta)|^2d\theta$$
$$\le\int_0^T\int_{\partial\Omega}|\partial_\nu z(t,1,\theta)|^2d\theta\ dt$$
$$\le c_2(T)\sum_{m=0}^{\infty}\sum_{k=1}^{\infty}\lambda_{mk}^2|C_{mk}|^2|J'_{m-1+\frac{N}{2}}(\lambda_{mk})|^2\int_{\partial\Omega}|H_m(\theta)|^2d\theta\,, \tag{7.12}$$

and hence Theorem 4.1 holds true. In addition, by (5.2), we have

$$|J'_{m-1+\frac{N}{2}}(\lambda_{mk})|^2=2\int_0^1\rho|J_{m-1+\frac{N}{2}}(\lambda_{mk}\rho)|^2\ d\rho\,,$$

so in view of (7.12) and (7.8), we obtain

$$c_1(T)\sum_{m=0}^{\infty}\sum_{k=1}^{\infty}\lambda_{mk}^2|C_{mk}|^2\|E_{mk}\|^2\le\int_0^T\int_{\partial\Omega}|\partial_\nu z(t,1,\theta)|^2d\theta\ dt \tag{7.13}$$
$$\le c_2(T)\sum_{m=0}^{\infty}\sum_{k=1}^{\infty}\lambda_{mk}^2|C_{mk}|^2\|E_{mk}\|^2\,.$$

Eventually, by (7.6), (7.7) and estimates (6.9) we have that

$$c_1(T)(\|z_0\|^2_{H_0^1(\Omega)}+\|z_1\|^2)\le\int_0^T\int_{\partial\Omega}|\partial_\nu z(t,1,\theta)|^2d\theta\ dt\le c_2(T)(\|z_0\|^2_{H_0^1(\Omega)}+\|z_1\|^2)\,,$$

whence it follows that the space F introduced at the end of Section 4 is

$$H_0^1(\Omega)\times L^2(\Omega)\,.$$

Since the operator Ψ defined in (4.9) is an isomorphism from $H_0^1(\Omega)\times L^2(\Omega)$ to $H^{-1}(\Omega)\times L^2(\Omega)$, if we take $u_0\in L^2(\Omega)$ and $u_1\in H^{-1}(\Omega)$, then there exists a unique $(z_0,z_1)\in H_0^1(\Omega)\times L^2(\Omega)$ such that $\Psi(z_0,z_1)=(-u_1,u_0)$. Denoted by z the weak solution of problem (7.3)–(7.5) with data z_0 and z_1 and by u the weak solution of

$$\begin{cases}u_{tt}(t,x)-\triangle u(t,x)+\beta\displaystyle\int_0^t e^{-\eta(t-s)}\triangle u(s,x)ds=0\,, & t\in(0,T),\ x\in\Omega,\\ u(0,x)=u_t(0,x)=0, & x\in\Omega,\\ u(t,x)=\partial_\nu z(t,x)-\beta\displaystyle\int_t^T e^{-\eta(s-t)}\partial_\nu z(s,x)ds\,, & t\in(0,T),\ x\in\partial\Omega,\end{cases}$$

thanks to the definition of Ψ we have that u verifies the final conditions

$$u(T,x)=u_0(x)\,,\qquad u_t(T,x)=u_1(x)\,,\quad x\in\Omega\,.$$

So, our proof is complete. □

References

[1] C.M. Dafermos, *Asymptotic stability in viscoelasticity.* Arch. Rational Mech. Anal. **37** (1970), 297–308.

[2] C.M. Dafermos, *An abstract Volterra equation with applications to linear viscoelasticity.* J. Differential Equations **7** (1970), 554–569.

[3] X. Fu, J. Yong, X. Zhang, *Controllability and observability of a heat equation with hyperbolic memory kernel.* J. Differential Equations **247** (2009), 2395–2439.

[4] G. Gripenberg, S.O. Londen, O.J. Staffans, *Volterra Integral and Functional Equations.* Encyclopedia Math. Applications, **34** Cambridge Univ. Press, 1990.

[5] A.E. Ingham, *Some trigonometrical inequalities with applications to the theory of series.* Math. Z. **41** (1936), 367–379.

[6] J.U. Kim, *Control of a plate equation with large memory.* Differential Integral Equations **5** (1992), 261–279.

[7] J.U. Kim, *Control of a second-order integro-differential equation.* SIAM J. Control Optim. **31** (1993), 101–110.

[8] V. Komornik, P. Loreti, *Ingham type theorems for vector-valued functions and observability of coupled linear system.* SIAM J. Control Optim. **37** (1998), 461–485.

[9] V. Komornik, P. Loreti, *Fourier series in control theory.* Springer Monographs in Mathematics, Springer-Verlag, 2005.

[10] J.E. Lagnese, J.-L. Lions, *Modelling analysis and control of thin plates.* Recherches en Mathématiques Appliquées (1988), Masson, Paris.

[11] I. Lasiecka, *Controllability of a viscoelastic Kirchhoff plate.* Control and estimation of distributed parameter systems (Vorau, 1988), Internat. Ser. Numer. Math., 91, Birkhäuser, Basel, 1989, 237–247.

[12] G. Lebon, C. Perez-Garcia, J. Casas-Vazquez, *On the thermodynamic foundations of viscoelasticity.* J. Chem. Phys. **88** (1988), 5068–5075.

[13] G. Leugering, *Exact boundary controllability of an integro-differential equation.* Appl. Math. Optim. (1987), 223–250.

[14] G. Leugering, *Boundary controllability of a viscoelastic string.* in G. Da Prato and M. Iannelli editors, Volterra integrodifferential equations in Banach spaces and applications, Harlow, Essex, Longman Sci. Tech., (1989), 258–270.

[15] J.-L. Lions, *Exact controllability, stabilization and perturbations for distributed systems.* SIAM Rev. **30** (1988), 1–68.

[16] J.-L. Lions, *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 1. Contrôlabilité exacte.* with appendices by E. Zuazua, C. Bardos, G. Lebeau and J. Rauch. Recherches en Mathématiques Appliquées, **8** (1988), Masson, Paris.

[17] J.-L. Lions, *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome* 2. *Perturbations.* Recherches en Mathématiques Appliquées, **9** (1988), Masson, Paris.

[18] P. Loreti, D. Sforza, *Exact reachability for second order integro-differential equations.* Comptes Rendus Mathématique **347** (2009), 1153–1158.

[19] P. Loreti, D. Sforza, *Reachability problems for a class of integro-differential equations.* J. Differential Equations **248** (2010), 1711–1755.

[20] P. Loreti, V. Valente, *Partial exact controllability for spherical membranes.* SIAM J. Control Optim. **35** (1997), 641-653.

[21] J.E. Muñoz Rivera, M.G. Naso, *Exact controllability for hyperbolic thermoelastic systems with large memory.* Adv. Differential Equations **9** (2004), 1369–1394.

[22] L. Pandolfi, *The controllability of the Gurtin-Pipkin equation: a cosine operator approach.* Appl. Math. Optim. **52** (2005), 143–165.

[23] J. Prüss, *Evolutionary integral equations and applications.* Monographs in Mathematics, **87** Birkhäuser, 1993.

[24] M. Renardy, W.J. Hrusa, J.A. Nohel, *Mathematical problems in viscoelasticity.* Pitman Monographs Pure Appl.Math., **35** Longman Sci, 1988.

[25] M. Renardy, *Are viscoelastic flows under control or out of control?* Systems Control Lett. **54** (2005), 1183–1193.

[26] D.L. Russell, *Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions.* SIAM Rev. **20** (1978), 639–739.

[27] X. Zhang, E. Zuazua, *Polynomial decay and control of a* 1*-d model for fluid-structure interaction.* C. R. Math. Acad. Sci. Paris **336** (2003), 745–750.

[28] X. Zhang, E. Zuazua, *Polynomial decay and control of a* 1*-d hyperbolic-parabolic coupled system.* J. Differential Equations **204** (2004), 380–438.

Paola Loreti and Daniela Sforza
Dipartimento di Scienze di Base e Applicate per l'Ingegneria
Sezione di Matematica
Sapienza Università di Roma
Via Antonio Scarpa 16
I-00161 Roma, Italy
e-mail: loreti@dmmm.uniroma1.it
sforza@dmmm.uniroma1.it

Operator Theory:
Advances and Applications, Vol. 216, 275–289

The Schrödinger Flow in a Compact Manifold: High-frequency Dynamics and Dispersion

Fabricio Macià

Communicated by V. Smyshlyaev

Abstract. We discuss various aspects of the dynamics of the Schrödinger flow on a compact Riemannian manifold that are related to the behavior of high-frequency solutions. In particular we show that dispersive (Strichartz) estimates fail on manifolds whose geodesic flow is periodic (thus generalizing a well-known result for spheres proved via zonal spherical harmonics). We also address the issue of the validity of observability estimates. We show that the geometric control condition is necessary in manifolds with periodic geodesic flow and we give a new, geometric, proof of a result of Jaffard on the observability for the Schrödinger flow on the two-torus. All our proofs are based on the study of the structure of semiclassical (Wigner) measures corresponding to solutions to the Schrödinger equation.

Mathematics Subject Classification (2000). Primary 35Q40; Secondary 58J40.

Keywords. Schrödinger equation, Zoll manifolds, semiclassical measures, dispersive (Strichartz) estimates, observability estimates.

1. Introduction

Let (M,g) be a compact, smooth Riemannian manifold. The Schrödinger flow on (M,g) associates to an initial datum $u_0 \in L^2(M)$ the solution $u(t,\cdot)$ to the Schrödinger equation:

$$\begin{cases} i\partial_t u(t,x) + \Delta_x u(t,x) = 0, & (t,x) \in \mathbb{R} \times M, \\ u|_{t=0} = u_0. \end{cases} \tag{1.1}$$

Above, Δ_x denotes the Laplace-Beltrami operator corresponding to (M,g). Since M is compact, the spectrum of $-\Delta_x$ consists of eigenvalues $0 = \lambda_0 < \lambda_1 \le \lambda_2 \dots$

This research has been supported by grants MTM2007-61755 (MEC) and Santander-Complutense 34/07-15844.

that tend to infinity. We shall denote by $(\psi_{\lambda_n})_{n\in\mathbb{N}}$ an orthonormal basis of $L^2(M)$ consisting of eigenfunctions $-\Delta_x\psi_{\lambda_n} = \lambda_n\psi_{\lambda_n}$. One has $u(t,\cdot) = e^{it\Delta_x}u_0$ and the following spectral representation holds:

$$e^{it\Delta_x}u_0 = \sum_{n\in\mathbb{N}} e^{-i\lambda_n t}\widehat{u_0}(\lambda_n)\,\psi_{\lambda_n}, \qquad \text{provided } u_0 = \sum_{n\in\mathbb{N}} \widehat{u_0}(\lambda_n)\,\psi_{\lambda_n}. \tag{1.2}$$

Two direct consequences may be extracted from this formula. First, that the dynamics of the Schrödinger flow is almost-periodic; second, that the $L^2(M)$-norm is conserved by $e^{it\Delta_x}$. Note that both these properties hold regardless of the specific geometry of (M,g).

Another dynamical feature of $e^{it\Delta_x}$ that is not so easily interpreted from (1.2) is its *dispersive* character. The high-frequency modes of a solution to the Schrödinger equation travel at a higher speed than their low-frequency counterparts.[1] This results in a regularizing effect on the singularities of the initial datum, which is usually quantified through dispersive estimates (also known as Strichartz estimates) of the type:

$$\left\|e^{it\Delta_x}u_0\right\|_{L^p([0,1]\times M)} \le C\,\|u_0\|_{H^s(M)}. \tag{1.3}$$

Such an estimate is known to hold for $M = \mathbb{R}^d$ when $p = p_0(d) := 2(2+d)/d$ and $s = 0$. For a general d-dimensional compact manifold M, Burq, Gérard and Tzvetkov [3] have shown that (1.3) also holds for $p = p_0(d)$ but with $s = 1/p$ (which is half the exponent given by the Sobolev embedding theorem).

This value of s is not optimal in general; in fact, the infimum $s(p,M)$ of the values s for which (1.3) holds is a quantity that depends heavily on the specific geometry of the manifold M considered. For instance, when M is the flat torus $\mathbb{T}^d$, Bourgain has shown [2] that $s\left(p_0(d),\mathbb{T}^d\right) = 0$ for $d = 1,2$ (although the estimate is actually false for $s = 0$), and it holds for $d = 1$, $p = 4$, $s = 0$ as shown by Zygmund [32]. When (M,g) has periodic geodesic flow, (1.3) holds for $p = 4$, $s > d/4 - 1/2$ and $d \ge 3$ ($s > 1/8$ if $d = 2$), which is again smaller than $1/p$. Moreover, these values are optimal on standard spheres $\mathbb{S}^d$; these results are proved in [3]. These considerations can be interpreted as the fact that the dispersive effect for the Schrödinger flow is stronger on tori than on spheres.

The validity of dispersive estimates is closely related to the *high-frequency behavior* of the solutions to (1.1). This behavior is tested on highly oscillatory sequences of initial data, *i.e.*, sequences $\left(u_0^h\right)$ whose $L^2(M)$-norm is concentrated on frequencies localized towards infinity as $h \to 0^+$. Typical examples of such initial data are (strictly) *h-oscillating sequences* $\left(u_0^h\right)$, which are of the form:

$$u_0^h = \sum_{a/h \le \sqrt{\lambda_n} \le b/h} \widehat{u_0}(\lambda_n)\,\psi_{\lambda_n}, \qquad \text{for some } b > a > 0, \tag{1.4}$$

[1] However, this is readily seen when M is the Euclidean space equipped with the standard metric. The solution issued from a plane-wave initial datum $e^{i\xi\cdot x}$ is precisely $e^{i\xi\cdot(x-t\xi)}$, which travels at velocity ξ.

or those of W.K.B. type, $u_0^h(x) := e^{iS_0(x)/h}$ for some $S_0 \in C^\infty(M)$. For small h, the behavior of $e^{it\Delta_x}u_0^h$ turns out to be related to the dynamics of the geodesic flow of (M,g). In particular, up to times t of the order of h, the classical W.K.B. method gives a very precise description of the structure of these solutions in terms of propagation along geodesics of M; however, it fails to describe the global in time evolution. A simpler, although more general, approach consists in understanding the limiting behavior as $h \to 0^+$ of the position densities:

$$n_h(t) := \left|e^{it\Delta_x}u_0^h\right|^2.$$

This object is physically relevant, in the context of the quantum-classical correspondence principle, as it describes the asymptotic behavior of the position probability density of a free quantum particle propagating in M. If $\left(u_0^h\right)$ is bounded in $L^2(M)$, the measures n_h are bounded in $L^\infty(\mathbb{R};\mathcal{M}_+(M))$, where $\mathcal{M}_+(M)$ stands for the set of positive Radon measures on M. Therefore it has at least a weak-$*$ accumulation point $\nu \in L^\infty(\mathbb{R};\mathcal{M}_+(M))$; these are sometimes called *quantum limits* or *defect measures*.

It can be shown (see for instance [24]) that the support of ν is a union of geodesics of M. The precise structure of the set of such accumulation points depends heavily on the particular dynamical properties of the geodesic flow of M. When it is completely integrable, some results have been obtained in [24, 25] by identifying the structure of the set of *semiclassical* (*or Wigner*) *measures* corresponding to $\left(e^{it\Delta_x}u_0^h\right)$. These are obtained as limits of some microlocal lifts to T^*M of the densities $n_h(t)$, known as Wigner distributions (a systematic presentation is given in [15, 23, 16, 17, 4], see also Section 2 for precise definitions).[2] In Section 3 we shall present a new approach to the structure result of [25] for semiclassical measures on the flat torus $\mathbb{T}^d$.

The knowledge of the structure of the set of quantum limits in M can be used to show the failure of dispersive estimates (1.3) in the case $s = 0$. This is due to the fact that whenever (1.3) holds one has $n_h \in L^{p/2}([0,1]\times M)$, and the same holds for any quantum limit ν. In particular, since $p/2 \geq 1$, (1.3) implies that any quantum limit must be absolutely continuous with respect to the Riemannian measure in M. If one is able to produce a sequence of initial data $\left(u_0^h\right)$ that gives a quantum limit which has a nontrivial singular component then no dispersive estimate may hold for $e^{it\Delta_x}$ in M. We shall apply this strategy to prove, in Section 4, the following result.

Theorem 1.1. *Let (M,g) be a manifold with periodic geodesic flow. Then the dispersive estimate*

$$\left\|e^{it\Delta_x}u_0\right\|_{L^p([0,1]\times M)} \leq C\left\|u_0\right\|_{L^2(M)} \tag{1.5}$$

fails for every $p > 2$.

[2] We refer the reader to [7, 31] for a comparison between the semiclassical measure and the W.K.B. approaches.

As was pointed out by the referee, the failure of the dispersive estimate in this setting can also be obtained combining the optimality of the analogue of the Strichartz estimates for spectral projectors proved by Sogge (see [30], Corollary 5.1.2) together with the precise spectral results for the Laplacian on manifolds with periodic geodesic flow by Duistermaat-Guillemin [10] and Colin de Verdière [8]. This strategy allows to show that estimate (1.5) fails even if the L^2-norm is replaced by a Sobolev norm H^s with $s < \delta(p)$, where

$$\delta(p) := \begin{cases} \frac{d-1}{2}\left(\frac{1}{2} - \frac{1}{p}\right) & \text{if } 2 < p \leq \frac{2(d+1)}{d-1}, \\ d\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{2} & \text{if } p \geq \frac{2(d+1)}{d-1}, \end{cases}$$

denotes Sogge's exponent.

Note that the approach we used to prove Theorem 1.1 cannot be used to disprove the dispersive estimate in the case of the flat torus $\mathbb{T}^2$ and $2 < p < 4 = p_0(2)$ and $s = 0$, since every quantum limit is absolutely continuous with respect to the Lebesgue measure in that case (see [25] for a proof). This suggests that an eventual failure of the dispersive estimate in this case must be realised by a more subtle mechanism.

The third and final aspect of the dynamics of the Schrödinger flow we want to discuss here is related to a quantitative version of the unique continuation property known as *observability*. Take $T > 0$ and an open set $U \subset M$; the Schrödinger flow $e^{it\Delta_x}$ is said to satisfy the observability property for T and U whenever a constant $C = C_{T,U} > 0$ exists such that

$$\|u_0\|^2_{L^2(M)} \leq C \int_0^T \int_U \left|e^{it\Delta_x} u_0(x)\right|^2 dxdt \tag{1.6}$$

for every initial datum $u_0 \in L^2(M)$. Note that the fact that an estimate like (1.6) holds implies that whenever two solutions to the Schrödinger equation are close to each other in $L^2((0,T) \times U)$-norm they must be globally close. In particular, two solutions that coincide in $(0,T) \times U$ must be identical. The observability property is relevant in Control Theory [22], and Inverse Problems [18].

A sufficient condition for (1.6) to hold was found by Lebeau [20] (see also [9]). It is the following.

$$\begin{array}{c}\text{There exists } L_0 > 0 \text{ such that every geodesic} \\ \text{of } (M,g) \text{ of length smaller than } L_0 \text{ intersects } \overline{U}.\end{array} \tag{1.7}$$

However, this condition is not necessary in general, as follows from the works of Jaffard [19] or Burq and Zworski [6]. Nevertheless, we shall show in Section 4 that (1.7) is equivalent to (1.6) when (M,g) has periodic geodesic flow.

Theorem 1.2. *Let* (M,g) *be a compact manifold with periodic geodesic flow. If the observability estimate* (1.6) *holds for some* $T > 0$ *and some open set* $U \subset M$ *then* U *must satisfy* (1.7)*. As a consequence,* (1.6) *and* (1.7) *are equivalent.*

The proof of this result will again be based on the high-frequency properties of the Schrödinger flow; and in particular on the analysis of the set of semiclassical measures on M. Note that the role of semiclassical measures in the context of observability estimates was first noticed by Lebeau [21]. As mentioned before, condition (1.7) is not in general necessary for (1.6) to hold. For instance, when $M = \mathbb{T}^2$, the two-dimensional standard torus equipped with the flat metric, Jaffard [19] proved the following result (see also [5, 27] for related results for eigenfunctions of the Laplacian).

Theorem 1.3. *Let* $(M, g) = (\mathbb{T}^2, \text{flat})$*. Given any* $T > 0$ *and any open set* $U \subset \mathbb{T}^2$ *there exist a constant* $C > 0$ *such that the observability estimate* (1.6) *holds.*

The original proof of this result is based on results on pseudo-periodic functions due to Kahane. In Section 4 we shall give a new proof of this result which is completely microlocal and relies on the structure result for semiclassical measures for the Schrödinger flow on the torus presented in [25].

2. Semiclassical measures and the Schrödinger flow

Semiclassical measures are a very convenient tool in the high-frequency analysis of a sequence (u^h) bounded in $L^2(M)$. These objects are a microlocal version of the well-known defect measures, that describe the local concentration of the $L^2(M)$-norm of (u^h). Assume that (u^h) is bounded in $L^2(M)$; then the sequence of densities

$$n_h := \left|u^h\right|^2 dm$$

is bounded in $L^1(M)$ (here dm stands for the measure on M induced by the Riemannian metric g). Helly's theorem then ensures that, up to extraction of a subsequence, (n_h) weakly converges, as $h \to 0^+$, to a finite, positive Radon measure $\nu \in \mathcal{M}_+(M)$ which is usually called a *defect measure* for (u^h). The support of ν describes the regions on which the "energy" of (u^h) concentrates. For instance, if u^h is supported in some local chart and given by a concentration profile:

$$\frac{1}{h^{d/2}}\rho\left(\frac{x - x_0}{h}\right) \tag{2.1}$$

then one has $\nu(x) = \|\rho\|^2_{L^2(M)}\,\delta(x - x_0)$. On the other hand, if u^h is oscillating, written in a coordinate chart as:

$$\rho(x)\, e^{i\xi_0/h\cdot x}, \tag{2.2}$$

then $\nu(x) = |\rho(x)|^2\, dm$, whatever the value of ξ_0. The inability of defect measures to distinguish between different directions of oscillation turns out to be a serious difficulty when dealing with solutions to wave-type equations. For instance, suppose $M = \mathbb{R}^d$ equipped with the standard metric, and take u_0^h to be of the form

(2.2). A direct computation gives that the solution $e^{iht\Delta_x}u_0^h$ of the *semiclassical* Schrödinger equation issued from u_0^h satisfies:

$$n_h(ht)(x) := \left|e^{iht\Delta_x}u_0^h(x)\right|^2 = \left|e^{iht\Delta_x}\rho(x-t2\xi_0)\right|^2.$$

Therefore the densities $n_h(ht)$ weakly converge, as $h \to 0^+$, to the defect measure $\nu_t(x) := |\rho(x-t2\xi_0)|^2\,dx$ which does depend on ξ_0. In particular, the defect measure of the initial data $\nu_0 = |\rho|^2\,dx$ does not determine uniquely that corresponding to the evolution, since the latter depends explicitly on ξ_0.

This motivates the introduction of an object that takes into account the nature of the oscillations. The *Wigner distribution* w_h of the function u^h achieves this. Given a test function $a \in C_c^\infty(T^*M)$ on the cotangent bundle of M, we defined the action of w_h against a as:

$$\langle w_h, a\rangle := (\operatorname{op}_h(a)\,u^h|u^h)_{L^2(M)},$$

where $\operatorname{op}_h(a)$ denotes the semiclassical pseudodifferential operator of symbol a obtained by Weyl's quantization rule.[3] When M is the Euclidean space equipped with the standard metric, $\operatorname{op}_h(a)$ is defined by the formula:

$$\operatorname{op}_h(a)\,u(x) := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a\left(\frac{x+y}{2}, h\xi\right) u(y)\,e^{i(x-y)\cdot\xi}dy\frac{d\xi}{(2\pi)^d}.$$

This definition extends to a manifold by applying it locally, in a coordinate chart, and then assembling it by means of a partition of unity. This expression for w_h defines it as an element of $\mathcal{D}'(T^*M)$, the set of distributions on T^*M. The Wigner distribution is actually a lift of the densities n_h to phase-space T^*M for, if $\varphi \in C^\infty(M)$ one has $\operatorname{op}_h(\varphi) = \varphi$, the operator defined by multiplication by φ, and therefore,

$$\langle w_h, \varphi\rangle = (\varphi u^h|u^h)_{L^2(M)} = \int_M \varphi n_h.$$

When $M = \mathbb{R}^d$, we may identify $T^*M \equiv \mathbb{R}_x^d \times \mathbb{R}_\xi^d$. If $\varphi \in C_c^\infty(\mathbb{R}^d)$ only depends of ξ then $\operatorname{op}_h(\varphi) = \varphi(hD_x)$ is the Fourier multiplier of symbol φ. Hence,

$$\langle w_h, \varphi\rangle = \int_{\mathbb{R}^d}\varphi(\xi)\left|\widehat{u^h}\left(\frac{\xi}{h}\right)\right|^2\frac{d\xi}{(2\pi h)^d};$$

this shows that the projection of w_h on the variable ξ measures the concentration of the $L^2(\mathbb{R}^d)$-norm of the h-rescaled Fourier transform of u^h. The fact that the limits of Wigner distributions are positive measures is non-trivial, and was proved by Gérard [15] and Lions and Paul [23].

Theorem 2.1. *Let (u^h) be a bounded sequence in $L^2(M)$. Then there exists a subsequence (which we do not relabel) and a finite positive Radon measure $\mu \in \mathcal{M}_+(T^*M)$ such that*

$$w_h \rightharpoonup \mu, \text{ as } h \to 0^+ \text{ in } \mathcal{D}'(T^*M).$$

[3] The books [11, 26] are clear and recent introductions to semiclassical microlocal analysis, we refer the reader to them for background and precise definitions on pseudodifferential operators.

In this situation we say that μ is the *semiclassical measure* of the sequence $\left(u^h\right)$. If in addition, $\left(u^h\right)$ is *h-oscillating*, that is:

$$\limsup_{h\to 0^+} \sum_{\sqrt{\lambda_n}\geq R/h} \left|\widehat{u^h}\left(\lambda_n\right)\right|^2 \to 0, \text{ as } R\to\infty,$$

then the defect measure ν of $\left(u^h\right)$ is obtained by projecting its semiclassical measure μ on the ξ-component:

$$\int_{T_x^*M} \mu\left(x, d\xi\right) = \nu\left(x\right).$$

The additional variable allows to keep track of the directions of oscillation.

A direct computation gives that the semiclassical measure of the oscillating sequence (2.2) is $\left|\rho\left(x\right)\right|^2 dx\delta\left(\xi-\xi_0\right)$, therefore keeping track of the direction of oscillation ξ_0. A particularly interesting example is that of a *wave-packet* or *coherent state*. It is defined as a sequence $\left(u^h\right)$ in $L^2\left(M\right)$, supported in local chart that is written in coordinates as:

$$\frac{1}{h^{d/4}}\rho\left(\frac{x-x_0}{\sqrt{h}}\right)e^{i\xi_0/h\cdot x} \tag{2.3}$$

for some $\rho\in C^\infty\left(M\right)$. The semiclassical measure of this sequence is

$$\|\rho\|^2_{L^2(M)}\,\delta\left(x-x_0\right)\delta\left(\xi-\xi_0\right).$$

For a more detailed account on these issues, we refer the reader to the survey articles [4, 17], and the concise presentation of [16].

Let us now turn to the analysis of semiclassical measures for sequences of solutions to the Schrödinger equation. Let $\left(u_0^h\right)$ be a bounded, h-oscillating sequence in $L^2\left(M\right)$. We define the time-dependent Wigner distributions:

$$\langle w_h\left(t\right), a\rangle := (\mathrm{op}_h\left(a\right)e^{it\Delta_x}u_0^h|e^{it\Delta_x}u_0^h)_{L^2(M)}, \qquad a\in C_c^\infty\left(T^*M\right). \tag{2.4}$$

The following result was proved in [24].

Theorem 2.2. *With the above notations and hypotheses, the following holds. There exists a subsequence, which we do not relabel, and a positive measure* $\mu\in L^\infty\left(\mathbb{R};\mathcal{M}_+\left(T^*M\right)\right)$ *such that:*

$$\lim_{h\to 0^+}\int_{\mathbb{R}}\phi\left(t\right)\langle w_h\left(t\right), a\rangle\, dt = \int_{\mathbb{R}\times T^*M}\phi\left(t\right)a\left(x,\xi\right)\mu_t\left(dx,d\xi\right)dt, \tag{2.5}$$

for every $\phi\in L^1\left(\mathbb{R}\right)$, $a\in C_c^\infty\left(T^*M\right)$. *Moreover, for all* $\varphi\in C^\infty\left(M\right)$,

$$\lim_{h\to 0^+}\int_{\mathbb{R}\times M}\phi\left(t\right)\varphi\left(x\right)\left|e^{it\Delta_x}u_0^h\left(x\right)\right|^2 dm\,dt = \int_{\mathbb{R}\times T^*M}\phi\left(t\right)\varphi\left(x\right)\mu_t\left(dx,d\xi\right)dt, \tag{2.6}$$

and for almost every $t\in\mathbb{R}$, *the measure* μ_t *is invariant by the geodesic flow* ϕ_s^g *of* $\left(M,g\right)$:

$$\int_{T^*M} a\left(\phi_s^g\left(x,\xi\right)\right)\mu_t\left(dx,d\xi\right) = \int_{T^*M} a\left(x,\xi\right)\mu_t\left(dx,d\xi\right), \quad \textit{for every } s\in\mathbb{R}. \tag{2.7}$$

Note that the convergence in (2.5) is precisely the convergence in the weak-$*$ topology in $L^\infty(\mathbb{R}; \mathcal{D}'(T^*M))$. One cannot expect pointwise convergence of the distributions $w_h(t)$ for every $t \in \mathbb{R}$, since as shown in (2.7) the limit measure becomes instantaneously invariant by the geodesic flow. However, if one considers instead the solutions to the semiclassical Schrödinger equation, which corresponds to taking limits of $(w_h(ht))$, the convergence is locally uniform in t, and the limiting measure μ_t is computed through μ_0 by transport along the geodesic flow ϕ_t^g, see [15, 23, 17].

3. Manifolds with completely integrable geodesic flow

In order to gain further insight on the structure of the set of semiclassical measures obtained as a limit (2.5) we must make additional hypotheses on the dynamics of the geodesic flow ϕ_t^g of the manifold under consideration. Here we shall deal with manifolds with *completely integrable* geodesic flow; in particular, we shall focus on two particular classes of geometries: manifolds with periodic geodesic flow (also known as Zoll manifolds, see the book [1] for a comprehensive discussion on this dynamical hypothesis) and the flat torus (which is a model case for completely integrable geodesic flows).

In the first case we have an explicit formula for the semiclassical measure μ_t in terms of that of the initial data μ_0. In [24], the following is proved.

Theorem 3.1. *Let (M, g) be a manifold with periodic geodesic flow. Let $\left(u_0^h\right)$ be as in Theorem 2.2; suppose that (2.5) holds and that $w_h(0)$ converges to a semiclassical measure μ_0. If $\mu_0(\{\xi = 0\}) = 0$ then, for a.e. $t \in \mathbb{R}$ and $a \in C_c^\infty(T^*M)$ we have:*

$$\int_{T^*M} a(x,\xi)\,\mu_t(dx, d\xi) = \int_{T^*M} \langle a \rangle (x,\xi)\,\mu_0(dx, d\xi), \tag{3.1}$$

where $\langle a \rangle$ denotes the average of a along the geodesic flow:

$$\langle a \rangle (x,\xi) := \lim_{T\to\infty} \frac{1}{T} \int_0^T a\left(\phi_t^g(x,\xi)\right) dt.$$

Note that, in particular, μ_t is constant for a.e. $t \in \mathbb{R}$. When $(M, g) = \left(\mathbb{T}^d, \text{flat}\right)$ the situation is rather different, and the structure of μ_t is considerably more involved. In order to get some insight on the form of the limits of $w_h(t)$ start with noticing that Egorov's theorem (see [11, 26]) is an identity when dealing with the Weyl quantization rule on the torus:

$$e^{-it\Delta_x} \operatorname{op}_h(a)\, e^{it\Delta_x} = \operatorname{op}_h\left(a \circ \phi_{t/h}^{\text{flat}}\right).$$

Hence, in view of (2.4), for $\varphi \in L^1(\mathbb{R})$ and $a \in C_c^\infty\left(T^*\mathbb{T}^d\right)$ one has:

$$\int_{\mathbb{R}} \varphi(t) \langle w_h(t), a \rangle\, dt = \left\langle w_h(0), \langle a \rangle_\varphi^h \right\rangle,$$

where

$$\langle a\rangle_{\varphi}^{h}(x,\xi) := \int_{\mathbb{R}} \varphi(t)\, a\left(x+\frac{t}{h}\xi,\xi\right) dt. \tag{3.2}$$

Let us introduce some notation. Denote by $\mathbb{W}$ the set whose elements are straight lines in $\mathbb{Z}^d \setminus \{0\}$ passing through the origin. We have a disjoint union

$$\mathbb{Z}^d = \bigsqcup_{\omega\in\mathbb{W}} \omega \sqcup \{0\}.$$

Given $a \in C_c^{\infty}\left(T^*\mathbb{T}^d\right)$ we have a Fourier series decomposition:

$$a(x,\xi) = \sum_{k\in\mathbb{Z}^d} \widehat{a}(k,\xi)\,\psi_k(x), \qquad \psi_k(x) := \frac{e^{ik\cdot x}}{(2\pi)^{d/2}}.$$

Now, denote by a_ω the orthogonal projection of a into the set of functions in $L^2\left(\mathbb{T}^d\right)$ whose Fourier modes lie in ω, *i.e.*,

$$a_\omega := \sum_{k\in\omega} \widehat{a}(k,\cdot)\,\psi_k.$$

Taking now (3.2) into account we find that:

$$\langle a_\omega\rangle_{\varphi}^{h}(x,\xi) = b_{a,\varphi}^{\omega}\left(x,\xi,\frac{\xi\cdot\nu_\omega}{h}\right),$$

with

$$b_{a,\varphi}^{\omega}(x,\xi,\sigma) := \int_{\mathbb{R}} \varphi(t)\, a_\omega(x+t\sigma\nu_\omega,\xi)\, dt,$$

where ν_ω denotes a unit vector in the direction ω. Therefore, testing $w_h(0)$ against $\langle a_\omega\rangle_{\varphi}^{h}$ amounts to performing a blow-up of $w_h(0)$ in the direction ν_ω. This type of object has been already studied in the literature (in the context of Euclidean space) under the name of two-microlocal semiclassical measures. We refer the reader to the works of Fermanian-Kammerer [13, 12], Fermanian-Kammerer and Gérard [14], Miller [28], and Nier [29]. Following [25] one shows that, given $\omega \in \mathbb{W}$ there exists a positive measure $\mu_{\mathcal{R}}^{0}(\omega,\cdot)$ on

$$I_\omega := \left\{\xi \in \mathbb{R}^d : k\cdot\xi = 0 \text{ for } k \in \omega\right\}$$

taking values in the set of trace-class operators $\mathcal{L}^1\left(L^2(\gamma_\omega)\right)$ on the space of square-summable functions defined on any geodesic γ_ω in the direction ω such that:

$$\lim_{h\to 0^+} \left\langle w_h(0), b\left(x,\xi,\frac{\xi\cdot\nu_\omega}{h}\right)\right\rangle = \operatorname{tr}\int_{I_\omega} \tilde{b}(s,\xi,D_s)\,\mu_{\mathcal{R}}^{0}(\omega,d\xi)$$

where $b \in C_c^{\infty}\left(T^*\mathbb{T}^d \times \mathbb{R}\right)$ is a functions whose non-zero Fourier modes in x corresponds to frequencies in ω. Note that in this case,

$$b(x,\xi,\sigma) = \tilde{b}(x\cdot\nu_\omega,\xi,\sigma)$$

where $\tilde{b} \in C_c^\infty\left(\gamma_\omega \times \mathbb{R}^d \times \mathbb{R}\right)$ is the restriction of $b(\cdot,\xi,\sigma)$ to γ_ω. For every $\xi \in I_\omega$, the pseudodifferential operator $\tilde{b}(s,\xi,D_s)$ is a compact operator in $L^2(\gamma_\omega)$. A straightforward computation then gives:

$$\lim_{h\to 0^+}\left\langle w_h(0), \langle a_\omega\rangle_\varphi^h\right\rangle = \int_{\mathbb{R}} \varphi(t)\,\mathrm{tr}\int_{I_\omega} \tilde{a}_\omega(\cdot,\xi)\,\mu_{\mathcal{R}}^t(\omega,d\xi)\,dt, \tag{3.3}$$

where $\tilde{a}_\omega(\cdot,\xi)$ denotes the operator of multiplication in $L^2(\gamma_\omega)$ by the restriction of $a_\omega(\cdot,\xi)$ to γ_ω, and the trace-class operator-valued measures $\mu_{\mathcal{R}}^t(\omega,\cdot)$ are defined as the solutions to the initial-value problem for a density-matrix Schrödinger equation on $L^2(\gamma_\omega)$:

$$\begin{cases} li\partial_t\mu_{\mathcal{R}}^t(\omega,\xi) = \left[-\partial_s^2, \mu_{\mathcal{R}}^t(\omega,\xi)\right], \\ \mu_{\mathcal{R}}^t(\omega,\xi)\,|_{t=0} = \mu_{\mathcal{R}}^0(\omega,\xi). \end{cases} \tag{3.4}$$

The right-hand side of (3.3) can be written as

$$\int_{\mathbb{R}} \varphi(t)\int_{\mathbb{T}^d\times I_\omega} a_\omega(x,\xi)\,\rho_\omega^t(dx,d\xi),$$

where ρ_ω is a signed measure on I_ω whose projection on x is absolutely continuous with respect to the Lebesgue measure and whose non-zero Fourier modes lie in ω. The measure ρ_ω^t is obtained as the extension to $\mathbb{T}^d \times I_\omega$ of the density defined on $\gamma_\omega \times I_\omega$ by formula (3.3), see [25] (the sum in ω of these two-microlocal measures was called there the *resonant semiclassical measure* of $\left(u_0^h\right)$). Therefore, we recover the main result [25].

Theorem 3.2. *Let* $(M,g) = \left(\mathbb{T}^d, \text{flat}\right)$*, suppose* $\left(u_0^h\right)$ *satisfies the hypotheses of Theorem* 2.2 *and that* $w_h(0) \rightharpoonup \mu_0$ *as* $h \to 0^+$*. Then, for a.e.* $t \in \mathbb{R}$ *we have:*

$$\mu_t = \sum_{\omega\in\mathbb{W}} \rho_\omega^t + dx \otimes \overline{\mu_0},$$

where

$$\overline{\mu_0}(\xi) := (2\pi)^{-d}\int_{\mathbb{T}^d} \mu_0(dy,\xi),$$

and the ρ_ω^t *are defined by the above construction. In particular, they are signed measures concentrated on* $\mathbb{T}^d \times I_\omega$*, their non-zero Fourier modes in* x *are in the line* ω *and its projection on the* x*-component is absolutely continuous with respect to the Lebesgue measure. Moreover, each of the measures*

$$\rho_\omega^t + dx \otimes \overline{\mu_0}\rceil_{I_\omega}$$

is non-negative.

Let us stress that the measures ρ_ω^t are *not determined* by the semiclassical measures of the initial data μ_0. In [24, 25] examples of sequences $\left(u_0^h\right)$ and $\left(v_0^h\right)$ are given having the same semiclassical measure μ_0 but such that their respective time-dependent measures ρ_ω^t differ. In fact, a sufficient condition to have $\rho_\omega^t = 0$ is that

$$\lim_{h\to 0^+} \left\| \chi\left(\nu_\omega \cdot D_x\right) u_0^h \right\|_{L^2(\mathbb{T}^d)} = 0,$$

for every $\chi \in C_c^\infty(\mathbb{R})$ (see [25]).

Note also that the term $\sum_{\omega\in\mathbb{W}} \rho_\omega^t$ is concentrated on the set

$$\Omega := \left\{\xi \in \mathbb{R}^d : \xi \cdot k = 0 \text{ for some } k \in \mathbb{Z}^d \setminus \{0\}\right\}$$

of *resonant frequencies*. When the measure μ_0 of the initial data does not charge this set then the measure μ_t equals $dx \otimes \overline{\mu_0}$. This is an analogue in this context of the averaging formula (3.1).

When $\mu_0(\{\xi = 0\}) = 0$ and $d = 2$, it is proved in [25] that in fact the whole measure $\int_{\mathbb{R}^d} \mu_t(\cdot, d\xi)$ is absolutely continuous with respect to the Lebesgue measure. This is in stark contrast with the situation on Zoll manifolds, where the semiclassical measures μ_t may be singular with respect to the Riemannian measure. This can again be interpreted as the fact that the dispersive effect is much stronger on the torus than on manifolds with periodic geodesic flow.

4. Dispersion and observability for the Schrödinger flow

Let us now turn to the proof of the main results of this article.

Proof of Theorem 1.1. Take $(x_0, \xi_0) \in T^*M$ with $\xi_0 \neq 0$ and let $\left(u_0^h\right)$ be a wave-packet type sequence of initial data, as defined in (2.3) with $\left\|u_0^h\right\|_{L^2(M)} = 1$. Then we have that $w_h(0) \rightharpoonup \delta(x - x_0)\,\delta(\xi, -\xi_0)$ as $h \to 0^+$. The averaging formula (3.1) in Theorem 3.1 then gives, for every $a \in C_c^\infty(T^*M)$:

$$\lim_{h\to 0^+} \int_0^1 \langle w_h(t), a\rangle\, dt = \int_{T^*M} a(x,\xi)\, \delta_\gamma(dx, d\xi),$$

where δ_γ is the Dirac mass supported on γ, the geodesic in T^*M issued from (x_0, ξ_0). Identity (2.6) then gives:

$$\lim_{h\to 0^+} \int_0^1 \int_M \varphi(x) \left| e^{it\Delta_x} u_0^h \right|^2 dt dx = \int_M \varphi(x)\, \delta_{\gamma_M}(dx), \tag{4.1}$$

where γ_M stands for the projection of γ onto M. Since δ_{γ_M} is singular with respect to the Riemannian measure, we conclude that no dispersive estimate may hold for $p > 2$. □

Proof of Theorem 1.2. Suppose that the open set $U \subset M$ does not satisfy the geometric condition (1.7). Therefore, there exists a geodesic γ_M in M that does not intersect $\overline{U}$. Let γ denote the lift of γ_M to T^*M. Let $(x_0, \xi_0) \in \gamma$ and consider the wave-packet sequence $\left(u_0^h\right)$ centered at that point and satisfying $\left\|u_0^h\right\|_{L^2(M)} = 1$.

Reasoning as in the preceding proof, we find that (4.1) holds. In particular, if $\varphi \in C^{\infty}(M)$ is supported in a neighborhood of $\overline{U}$ that does not intersect γ_M we have:

$$\lim_{h\to 0^+} \int_0^1 \int_M \varphi(x) \left| e^{it\Delta_x} u_0^h \right|^2 dt dx = 0.$$

Since $\left\| u_0^h \right\|_{L^2(M)} = 1$ we conclude that no constant $C > 0$ exists such that estimate (1.6) holds. □

Proof of Theorem 1.3. Before proving Jaffard's result Theorem 1.3, we recall that the semiclassical reduction argument in [20] (which combines a Littlewood-Payley decomposition with a unique continuation results for eigenfunctions of the Laplacian) reduces the proof of an observability estimate (1.6) for any function in $L^2(M)$ to establishing it for strictly oscillating sequences of initial data. This, in turn, is equivalent to establishing the following fact.

Let $\left(u_0^h\right)$ *be a strictly h-oscillating sequence* (i.e., *verifying* (1.4)) *such that*

$$\lim_{h\to 0^+} \int_0^T \int_U \left| e^{it\Delta_x} u_0^h(x) \right|^2 dx dt = 0. \tag{4.2}$$

Then

$$\lim_{h\to 0^+} \left\| u_0^h \right\|_{L^2(\mathbb{T}^2)} = 0.$$

This equivalence is a straightforward consequence of the closed graph theorem. Let $\mu \in L^{\infty}\left(\mathbb{R}; \mathcal{M}_+\left(T^*\mathbb{T}^2\right)\right)$ denote the semiclassical measure (in the sense of (2.5)) associated to (possibly a subsequence of) $\left(e^{it\Delta_x} u_0^h\right)$. Suppose moreover that $\left(u_0^h\right)$ has a semiclassical measure μ_0. Our goal is to show that, assuming (1.4)), we can conclude that (4.2) implies that $\mu_0 = 0$. Start with noticing that (4.2) implies that for every $\varphi \in C\left(\mathbb{T}^2\right)$ supported in U we have:

$$\int_0^T \int_U \varphi(x) \mu_t(dx, d\xi) dt = 0.$$

As shown in Theorem 3.2 the measure μ can be written as:

$$\mu_t = \sum_{\omega\in\mathbb{W}} \rho_\omega^t + dx \otimes \overline{\mu_0}$$

and $\rho_\omega^t + dx \otimes \overline{\mu_0}\rceil_{I_\omega} \geq 0$. Moreover, the Fourier coefficients of ρ_ω^t lie in ω.

Since $\left(u_0^h\right)$ is strictly oscillating we have $\mu_0(\{\xi = 0\}) = 0$. Therefore, setting $\Omega := \bigcup_{\omega\in\mathbb{W}} I_\omega$ we have

$$\overline{\mu_0} := \sum_{\omega\in\mathbb{W}} \overline{\mu_0}\rceil_{I_\omega} + \overline{\mu_0}\rceil_{\Omega^c}.$$

Since all the measures $\mu_\omega^t := \rho_\omega^t + dx \otimes \overline{\mu_0}\rceil_{I_\omega}$ are positive, we can write, for a.e. $t \in \mathbb{R}$,

$$\mu_t = \sum_{\omega\in\mathbb{W}} \mu_\omega^t + dx \otimes \overline{\mu_0}\rceil_{\Omega^c},$$

in the sense of weak convergence of measures. Now, if $\varphi \in C\left(\mathbb{T}^2\right)$ is supported in U, the above remarks imply:

$$0 = \sum_{\omega \in \mathbb{W}} \int_0^T \int_{U \times I_\omega} \varphi(x)\, \mu_\omega^t(dx, d\xi)\, dt + T\overline{\mu_0}(\Omega^c) \int_U \varphi(x)\, dx.$$

Since φ is arbitrary we conclude, since μ is positive:

$$\overline{\mu_0}(\Omega^c) = \frac{1}{(2\pi)^2} \mu_0\left(\mathbb{T}^2 \times \Omega^c\right) = 0, \tag{4.3}$$

and, for every $t \in [0, T]$,

$$\mu_\omega^t(U \times I_\omega) = 0.$$

To conclude that $\mu_0 = 0$ it remains to show that μ_0 does not charge the set Ω of resonant frequencies. By construction, $\int_{I_\omega} \mu_\omega^t$ is invariant by translations along directions in I_ω. Therefore, $\mu_\omega^t(U_\omega \times I_\omega) = 0$, where $U_\omega := \{x + s\xi : x \in U, \xi \in I_\omega\}$. Let $\mu_{\mathcal{R}}^0$ denote a resonant Wigner measure corresponding to $\left(u_0^h\right)$ as defined by (3.3). Let γ_ω be the geodesic in $\mathbb{T}^2$ through the origin in the direction ω. Define $m_\omega^t \in \mathcal{L}^1\left(L^2(\gamma_\omega)\right)$ as the Hermitian, positive operators that solve the density-matrix Schrödinger equation:

$$i\partial_t m_\omega^t = \left[-\partial_s^2, m_\omega^t\right], \qquad m_\omega^t|_{t=0} = \mu_{\mathcal{R}}^0(\omega, I_\omega). \tag{4.4}$$

With our preceding notations, we have $m_\omega^t = \mu_{\mathcal{R}}^t(\omega, I_\omega)$. Let $J_\omega := U_\omega \cap \gamma_\omega$, denote by $\mathbf{1}_{J_\omega}$ the characteristic function of J_ω in γ_ω; note that $\mathbf{1}_{U_\omega}(x) = \mathbf{1}_{J_\omega}(x \cdot \nu_\omega)$, where ν_ω is a unit vector in ω. Let $\lambda_{\mathbf{1}_{J_\omega}}$ denote the operator on $L^2(\gamma_\omega)$ acting by multiplication by $\mathbf{1}_{J_\omega}$. Then, Theorem 3.2 shows that

$$\begin{aligned} \operatorname{tr}\left(\lambda_{\mathbf{1}_{J_\omega}} m_\omega^t\right) &= \int_{U_\omega \times I_\omega} \rho_\omega^t(dx, d\xi) + \frac{|U_\omega|}{(2\pi)^2} \operatorname{tr} \mu_{\mathcal{R}}^0(I_\omega) \\ &= \int_{U_\omega \times I_\omega} \mu_\omega^t(dx, d\xi) - |U_\omega| \left[\overline{\mu_0}(I_\omega) - (2\pi)^{-2} \operatorname{tr} \mu_{\mathcal{R}}^0(I_\omega)\right]. \end{aligned}$$

Therefore $\operatorname{tr}\left(\lambda_{\mathbf{1}_{J_\omega}} m_\omega^t\right) + |U_\omega| \left[\overline{\mu_0}(I_\omega) - (2\pi)^{-2} \operatorname{tr} \mu_{\mathcal{R}}^0(I_\omega)\right] = 0$ for $t \in [0, T]$; unique continuation for (4.4) then implies

$$\operatorname{tr} m_\omega^t + |U_\omega| \left[\overline{\mu_0}(I_\omega) - (2\pi)^{-2} \operatorname{tr} \mu_{\mathcal{R}}^0(I_\omega)\right] = 0,$$

for every $t \in \mathbb{R}$. Finally, notice that $\overline{\mu_0}(I_\omega) \geq (2\pi)^{-2} \operatorname{tr} \mu_{\mathcal{R}}^0(I_\omega)$ ([25], Proposition 8). We conclude that $\operatorname{tr} m_\omega^t = 0$ and, consequently, $\mu_\omega^0\left(\mathbb{T}^2 \times I_\omega\right) = \operatorname{tr} \mu_{\mathcal{R}}^0(I_\omega) = \operatorname{tr} m_\omega^0 = 0$ as well. Therefore, we have shown that $\mu_0(\mathbb{T}^2 \times \Omega) = 0$, combining this with (4.3) we conclude that $\mu_0 = 0$ as we wanted to prove. □

References

[1] Arthur L. Besse. *Manifolds all of whose geodesics are closed*, volume 93 of *Ergebnisse der Mathematik und ihrer Grenzgebiete* [*Results in Mathematics and Related Areas*]. Springer-Verlag, Berlin, 1978. With appendices by D.B.A. Epstein, J.-P. Bourguignon, L. Bérard-Bergery, M. Berger and J.L. Kazdan.

[2] J. Bourgain. Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations. *Geom. Funct. Anal.*, 3(2):107–156, 1993.

[3] N. Burq, P. Gérard, and N. Tzvetkov. Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds. *Amer. J. Math.*, 126(3):569–605, 2004.

[4] Nicolas Burq. Mesures semi-classiques et mesures de défaut. *Astérisque*, (245):Exp. No. 826, 4, 167–195, 1997. Séminaire Bourbaki, Vol. 1996/97.

[5] Nicolas Burq and Maciej Zworski. Control theory and high energy eigenfunctions. In *Journées "Équations aux Dérivées Partielles"*, pages Exp. No. XIII, 10. École Polytech., Palaiseau, 2004.

[6] Nicolas Burq and Maciej Zworski. Geometric control in the presence of a black box. *J. Amer. Math. Soc.*, 17(2):443–471 (electronic), 2004.

[7] Rémi Carles, Clotilde Fermanian-Kammerer, Norbert J. Mauser, and Hans Peter Stimming. On the time evolution of Wigner measures for Schrödinger equations. *Commun. Pure Appl. Anal.*, 8(2):559–585, 2009.

[8] Yves Colin de Verdière. Sur le spectre des opérateurs elliptiques à bicaractéristiques toutes périodiques. *Comment. Math. Helv.*, 54(3):508–522, 1979.

[9] B. Dehman, P. Gérard, and G. Lebeau. Stabilization and control for the nonlinear Schrödinger equation on a compact surface. *Math. Z.*, 254(4):729–749, 2006.

[10] J.J. Duistermaat and V.W. Guillemin. The spectrum of positive elliptic operators and periodic bicharacteristics. *Invent. Math.*, 29(1):39–79, 1975.

[11] Lawrence C. Evans and Maciej Zworski. *Lectures on semiclassical analysis*. 2009. Avalaible at: http://www.math.berkeley.edu/ zworski/semiclassical.pdf.

[12] C. Fermanian Kammerer. Propagation and absorption of concentration effects near shock hypersurfaces for the heat equation. *Asymptot. Anal.*, 24(2):107–141, 2000.

[13] Clotilde Fermanian-Kammerer. Mesures semi-classiques 2-microlocales. *C. R. Acad. Sci. Paris Sér. I Math.*, 331(7):515–518, 2000.

[14] Clotilde Fermanian-Kammerer and Patrick Gérard. Mesures semi-classiques et croisement de modes. *Bull. Soc. Math. France*, 130(1):123–168, 2002.

[15] P. Gérard. Mesures semi-classiques et ondes de Bloch. In *Séminaire sur les Équations aux Dérivées Partielles,* 1990–1991, pages Exp. No. XVI, 19. École Polytech., Palaiseau, 1991.

[16] Patrick Gérard and Éric Leichtnam. Ergodic properties of eigenfunctions for the Dirichlet problem. *Duke Math. J.*, 71(2):559–607, 1993.

[17] Patrick Gérard, Peter A. Markowich, Norbert J. Mauser, and Frédéric Poupaud. Homogenization limits and Wigner transforms. *Comm. Pure Appl. Math.*, 50(4):323–379, 1997.

[18] Victor Isakov. *Inverse problems for partial differential equations*, volume 127 of *Applied Mathematical Sciences*. Springer, New York, second edition, 2006.

[19] S. Jaffard. Contrôle interne exact des vibrations d'une plaque rectangulaire. *Portugal. Math.*, 47(4):423–429, 1990.

[20] G. Lebeau. Contrôle de l'équation de Schrödinger. *J. Math. Pures Appl.* (9), 71(3):267–291, 1992.

[21] G. Lebeau. Équation des ondes amorties. In *Algebraic and geometric methods in mathematical physics* (*Kaciveli,* 1993), volume 19 of *Math. Phys. Stud.*, pages 73–109. Kluwer Acad. Publ., Dordrecht, 1996.

[22] J.-L. Lions. Exact controllability, stabilization and perturbations for distributed systems. *SIAM Rev.*, 30(1):1–68, 1988.

[23] Pierre-Louis Lions and Thierry Paul. Sur les mesures de Wigner. *Rev. Mat. Iberoamericana*, 9(3):553–618, 1993.

[24] Fabricio Macià. Semiclassical measures and the Schrödinger flow on Riemannian manifolds. *Nonlinearity*, 22(5):1003–1020, 2009.

[25] Fabricio Macià. High-frequency propagation for the Schrödinger equation on the torus. *J. Funct. Anal.*, 258(3):933–955, 2010.

[26] André Martinez. *An introduction to semiclassical and microlocal analysis.* Universitext. Springer-Verlag, New York, 2002.

[27] Jeremy Marzuola. Eigenfunctions for partially rectangular billiards. *Comm. Partial Differential Equations*, 31(4-6):775–790, 2006.

[28] Luc Miller. *Propagation d'ondes semi-classiques à travers une interface et mesures 2-microlocales*. Ph. D. Thesis. École Polytechnique, Palaiseau, 1996.

[29] Francis Nier. A semi-classical picture of quantum scattering. *Ann. Sci. École Norm. Sup.* (4), 29(2):149–183, 1996.

[30] Christopher D. Sogge. *Fourier integrals in classical analysis*, volume 105 of *Cambridge Tracts in Mathematics.* Cambridge University Press, Cambridge, 1993.

[31] Christof Sparber, Peter A. Markowich, and Norbert J. Mauser. Wigner functions versus WKB-methods in multivalued geometrical optics. *Asymptot. Anal.*, 33(2):153–187, 2003.

[32] A. Zygmund. On Fourier coefficients and transforms of functions of two variables. *Studia Math.*, 50:189–201, 1974.

Fabricio Macià
Universidad Politécnica de Madrid
DEBIN, ETSI Navales
Avda. Arco de la Victoria s/n.
E-28040 Madrid, Spain
e-mail: fabricio.macia@upm.es

Operator Theory:
Advances and Applications, Vol. 216, 291–315

Optimality of the Asymptotic Behavior of the Energy for Wave Models

Michael Reissig

Abstract. In the present paper we study the behavior of different energies to wave equations with a propagation speed which depends on a shape function and an oscillating function. Our goal is to describe how far we are away from a generalized energy conservation law. We shall explain by an instability argument in which sense our results are sharp. Finally, we study possible interactions of oscillations in coefficients and describe lower bounds for the blow-up rate of the energy for $t \to \infty$.

Mathematics Subject Classification (2000). Primary 35L05, 35L15; Secondary 35B05, 35B35.

Keywords. Wave models, oscillations, time-dependent coefficients, Floquet theory, Ljapunov functional.

1. Introduction

Let us consider the strictly hyperbolic Cauchy problem

$$D_t^2 u - b^2(t) D_x^2 u = 0,\ u(0,x) = u_0(x),\ D_t u(0,x) = u_1(x) \tag{1.1}$$

under the assumption $0 < b_0 \le b(t) \le b_1$. In general one cannot expect the conservation of wave energy. But if we assume for the oscillating behavior of b

$$|b^{(k)}(t)| \le C_k (1+t)^{-k} \text{ for } k = 1, 2, \tag{1.2}$$

then the so-called *generalized energy conservation* holds, that is, the condition $C_0 \mathbb{E}(u;0) \le \mathbb{E}(u;t) \le C_1 \mathbb{E}(u;0)$ holds for the wave energy (see [12]). In [6] and [7] it was shown that one can get some benefit of higher regularity of b, namely $b \in C^m$ or $b \in C^\infty$ or b from a Gevrey space, if one assumes a so-called *stabilization condition*. In this way the conditions (1.2) can be weakened for $k = 1, 2$. In [2] one can find an example of a coefficient $b = b(t)$ which oscillating behavior does not allow boundedness of the wave energy for $t \to \infty$. This hints to a blow-up behavior of the energy for $t \to \infty$. Finally, the paper [9] is devoted to the Cauchy problem

$$D_t^2 u - \lambda^2(t) b^2(t) D_x^2 u = 0,\ u(0,x) = u_0(x),\ D_t u(0,x) = u_1(x) \tag{1.3}$$

with an increasing *shape function* $\lambda = \lambda(t)$. By using C^m regularity of the coefficient and the idea of stabilization the goal is to prove the two-sided estimate

$$C_0 \le \frac{1}{\lambda(t)} \mathbb{E}_\lambda(u;t) \le C_1, \tag{1.4}$$

where the constants C_0 and C_1 depend on the data and where

$$\mathbb{E}_\lambda(u;t) := \frac{1}{2} \int_{\mathbb{R}} \Big(\lambda^2(t) |D_x u(t,x)|^2 + |D_t u(t,x)|^2 \Big) dx. \tag{1.5}$$

The goals of the paper are the following:

- We are interested in the blow-up behavior of energies to solutions to (1.3), we shall describe the blow-up rate and make a proposal to prove optimality of the blow-up rate.
- We will explain the interaction of oscillations between coefficients $a = a(t)$ and $b = b(t)$ on the blow-up rate of the energy by the aid of the Cauchy problem

$$D_t^2 u + 2\lambda(t) a(t) D_{xt}^2 u - \lambda^2(t) b^2(t) D_x^2 u = 0,\ u(0,x) = u_0(x),\ D_t u(0,x) = u_1(x).$$

The content of the paper is as follows: In Sections 2 and 3 we derive upper bounds for the growth of different energies. In Section 2 we assume data $(u_0, u_1) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$. In Section 3 we assume data $(u_0, u_1) \in \dot{H}^1(\mathbb{R}) \times L^2(\mathbb{R})$. The necessary steps for a hyperbolic WKB analysis in the phase space are explained. In Section 4 we study the optimality of our approach. First we discuss in which sense do we understand optimality. All is reduced to the estimate of the fundamental solution in the phase space. An instability argument and an effective estimate for the elastic energy yield optimality. The considerations in Section 4 base on [5]. Finally, in Section 5 we explain possible results for the interaction of oscillations. Here, the description of the interplay between Ljapunov and energy function is the main tool. The considerations in Section 5 generalize those ones from [8].

2. Asymptotic behavior of the energy

We are interested in the Cauchy problem

$$D_t^2 u - \lambda^2(t) b^2(t) D_x^2 u = 0,\ u(0,x) = u_0(x),\ D_t u(0,x) = u_1(x). \tag{2.1}$$

For the shape function $\lambda = \lambda(t)$ we assume the following conditions:

(A1) $\lambda(0) > 0$, $\lambda'(t) > 0$ for $t \in [0,\infty)$ together with the estimates

$$\lambda'(t) \sim \frac{\lambda^2(t)}{\Lambda(t)},\ |D_t^k\lambda(t)| \lesssim \lambda(t)\Big(\frac{\lambda(t)}{\Lambda(t)}\Big)^k \text{ for } k=2, \tag{2.2}$$

$$t + \frac{C}{\sqrt{\lambda(t)}} \text{ is strict increasing with a positive } C \text{ and for large } t. \tag{2.3}$$

Here $\Lambda(t) = 1 + \int_0^t \lambda(s)ds$ is a primitive of $\lambda(t)$.

For the oscillating function $b = b(t)$ we assume the following conditions:

(A2) $0 < b_0 \le b(t) \le b_1$ for $t \in [0,\infty)$ together with the estimates

$$|D_t^k b(t)| \lesssim \Big(\frac{\lambda(t)}{\Lambda(t)}\nu(t)\Big)^k \text{ for } k=1,2. \tag{2.4}$$

Here $\nu = \nu(t)$, $t \in [0,\infty)$, is a positive and monotonously increasing continuous function which measures the oscillating behavior of $b(t)$. In this paper we assume

(A3) $\nu(t) \lesssim \log\Lambda(t)$ for large t, that is, we *exclude very fast oscillations.* Moreover, $\nu = \nu(t) = f(\Lambda(t))$. Here the function $f = f(r)$ fulfils $|f'(r)| \le \frac{C_0}{r}$ on an interval $[r_0,\infty)$.

Theorem 2.1. *Assume the conditions* (A1) *to* (A3)*. Then the solution to* (2.1) *for data* $u_0 \in H^1(\mathbb{R})$ *and* $u_1 \in L^2(\mathbb{R})$ *satisfies*

$$\mathbb{E}_\lambda(u;t) \le C_0 \exp(C_1\nu(t))\lambda(t)(\|u_0\|_{H^1}^2 + \|u_1\|_{L^2}^2). \tag{2.5}$$

The positive constants C_0 *and* C_1 *are independent of the data and of* $t \in [0,\infty)$*.*

Proof. The proof bases on a precise WKB-analysis for the solutions to

$$D_t^2 v - \lambda^2(t)b^2(t)\xi^2 v = 0,\ v(0,\xi) = \hat{u}_0(\xi),\ D_t v(0,\xi) = \hat{u}_1(\xi). \tag{2.6}$$

Definition 2.2. We divide the extended phase space $\{(t,\xi) \in [0,\infty) \times \mathbb{R}_\xi\}$ into the pseudo-differential zone

$$Z_{\text{pd}}(N) = \{(t,\xi) : \Lambda(t)|\xi| \le N\},$$

the middle zone

$$Z_{\text{mid}}(N) = \{(t,\xi) : N \le \Lambda(t)|\xi| \le N\nu(t)\},$$

and the hyperbolic zone

$$Z_{\text{hyp}}(N) = \{(t,\xi) : N\nu(t) \le \Lambda(t)|\xi|\}.$$

We define the function $t_\xi^{(1)} = t_\xi^{(1)}(|\xi|)$ as the solution of $\Lambda(t)|\xi| = N$ and the function $t_\xi^{(2)} = t_\xi^{(2)}(|\xi|)$ as the solution of $\Lambda(t)|\xi| = N\nu(t)$. Due to (A3) the second function is well defined.

Lemma 2.3. *Let us assume* (A1) *and* (A2). *Then for all* $t \in [0, t_\xi^{(1)}]$ *the following estimates hold for the solution to* (2.6):

$$\lambda(t)|\xi||v(t,\xi)| \lesssim \frac{\lambda(t)}{\Lambda(t)}|v(0,\xi)| + \frac{t\lambda(t)}{\Lambda(t)}|D_t v(0,\xi)|,$$
$$|D_t v(t,\xi)| \lesssim \frac{\lambda(t)}{\Lambda(t)}|v(0,\xi)| + |D_t v(0,\xi)|.$$

Proof. Introducing $V(t,\xi) := (\lambda(t)|\xi|v, D_t v)^T$ we transform (2.6) into the following system

$$D_t V = AV := \begin{pmatrix} \frac{D_t\lambda(t)}{\lambda(t)} & \lambda(t)|\xi| \\ \lambda(t)b^2(t)|\xi| & 0 \end{pmatrix} V.$$

We are interested to estimate for $t \in [0, t_\xi^{(1)}]$ the entries $E_{kl} = E_{kl}(t,s,\xi)$, $k,l = 1,2$, of the fundamental solution to $D_t - A$, that is, the solution to

$$D_t E(t,s,\xi) = A(t,\xi)E(t,s,\xi), \quad E(s,s,\xi) = I.$$

We obtain the following system of integral equations:

$$E_{11}(t,0,\xi) = \frac{\lambda(t)}{\lambda(0)} + i|\xi|\lambda(t)\int_0^t E_{21}(s,0,\xi)ds,$$
$$E_{21}(t,0,\xi) = i|\xi|\int_0^t \lambda(s)b^2(s)E_{11}(s,0,\xi)ds,$$
$$E_{12}(t,0,\xi) = i|\xi|\lambda(t)\int_0^t E_{22}(s,0,\xi)ds,$$
$$E_{22}(t,0,\xi) = 1 + i|\xi|\int_0^t \lambda(s)b^2(s)E_{12}(s,0,\xi)ds.$$

Setting the equation for E_{11} into the equation for E_{21} we have

$$E_{21}(t,0,\xi) = \frac{i|\xi|}{\lambda(0)}\int_0^t \lambda^2(s)b^2(s)ds - |\xi|^2\int_0^t\Big(\int_\theta^t \lambda^2(s)b^2(s)ds\Big)E_{21}(\theta,0,\xi)d\theta.$$

Using the monotonicity of λ and the definition of the pseudo-differential zone the last integral equation for E_{21} allows the estimate

$$|E_{21}(t,0,\xi)| \le C_0|\xi|\lambda(t)\Lambda(t) \le C_0\lambda(t).$$

Using this estimate in the integral equation for E_{11} implies

$$|E_{11}(t,0,\xi)| \le C_0\lambda(t).$$

The integral equation for E_{22} gives the estimate $|E_{22}(t,0,\xi)| \le C_0$. Using this estimate in the integral equation for E_{12} brings $|E_{12}(t,0,\xi)| \le C_0 t\lambda(t)|\xi|$. Summarizing we have shown, here we need again the definition of the pseudo-differential zone,

$$\begin{aligned}\lambda(t)|\xi||v(t,\xi)| &\lesssim \lambda(t)|\xi|(|v(0,\xi)|+t|D_tv(0,\xi)|)\\ &\lesssim \frac{\lambda(t)}{\Lambda(t)}|v(0,\xi)|+\frac{t\lambda(t)}{\Lambda(t)}|D_tv(0,\xi)|,\\ |D_tv(t,\xi)| &\lesssim \lambda(t)|\xi||v(0,\xi)|+|D_tv(0,\xi)|\\ &\lesssim \frac{\lambda(t)}{\Lambda(t)}|v(0,\xi)|+|D_tv(0,\xi)|.\end{aligned}$$

This completes the proof. □

In the following we use C, C_0 and C_1 as universal constants.

Lemma 2.4. *Let us assume* (A1) *to* (A3)*. Then for all* $t \in [t_\xi^{(1)}, t_\xi^{(2)}]$ *the following estimate holds for the solution to* (2.6)*:*

$$\lambda(t)|\xi||v(t,\xi)|+|D_tv(t,\xi)| \lesssim \exp(C_1\nu(t))\frac{\lambda(t_\xi^{(1)})}{\Lambda(t_\xi^{(1)})}\Big(|v(0,\xi)|+t_\xi^{(1)}|D_tv(0,\xi)|\Big).$$

Proof. Introducing $V(t,\xi) := (\lambda(t)|\xi|v, D_tv)^T$ we transform (2.6) into the following system

$$D_tV = AV := \begin{pmatrix} \frac{D_t\lambda(t)}{\lambda(t)} & \lambda(t)|\xi| \\ \lambda(t)b^2(t)|\xi| & 0 \end{pmatrix} V.$$

In the following statement we give a representation of the fundamental solution to the last system.

Lemma 2.5. *The solution to the system*

$$D_tE(t,s,\xi) = A(t,\xi)E(t,s,\xi), \quad E(s,s,\xi) = I,$$

is given by the matrizant representation

$$E(t,s,\xi) = I + \sum_{k=1}^{\infty} i^k \int_s^t A(t_1,\xi)\int_s^{t_1} A(t_2,\xi)\cdots\int_s^{t_{k-1}} A(t_k,\xi)dt_k\cdots dt_1.$$

From Lemma 2.5 we conclude the following estimate for the fundamental solution:

$$\|E(t,t_\xi^{(1)},\xi)\| \le \exp\Big(\int_{t_\xi^{(1)}}^t \|A(s,\xi)\|ds\Big) \text{ for all } t\in[t_\xi^{(1)},t_\xi^{(2)}].$$

The monotonic behavior of λ and (A2) imply $\|A(t,\xi)\| \lesssim \lambda(t)|\xi| + \frac{\lambda(t)}{\Lambda(t)}$. Consequently, we have for $t \in [t_\xi^{(1)}, t_\xi^{(2)}]$ the estimate

$$\|E(t,t_\xi^{(1)},\xi)\| \le \exp\Big(C_1\Big(\log\frac{\Lambda(t)}{\Lambda(t_\xi^{(1)})}+N\nu(t)\Big)\Big) \le \exp\big(C_1\nu(t)\big).$$

Finally, $V(t,\xi) = E(t,t_\xi^{(1)},\xi)V(t_\xi^{(1)},\xi)$ and Lemma 2.3 yield the desired estimates. □

Remark 2.6. In the pseudo-differential zone and in the middle zone we have to take into consideration the properties of the shape function (besides $0 < b_0 \le b(t) \le b_1$) for the derivation of energy estimates. The middle zone is, in general, larger than the pseudo-differential zone. So, through the zone definition the influence of the function describing the oscillations is given. Moreover, we learn that the assumption $u_0 \in H^1$ is important. If we would only assume $u_0 \in \dot{H}^1$, then the energy inequality becomes worse (see Section 3).

Lemma 2.7. *Let us assume* (A1) *to* (A3)*. Then for all* $t \in [t_\xi^{(2)}, \infty)$ *the following estimate holds for the solution to* (2.6) :

$$\begin{aligned} &\lambda(t)|\xi||v(t,\xi)| + |D_t v(t,\xi)| \\ &\quad \lesssim \exp(C_1\nu(t)) \frac{\sqrt{\lambda(t)}}{\sqrt{\lambda(t_\xi^{(2)})}} \Big(\lambda(t_\xi^{(2)})|\xi||v(t_\xi^{(2)},\xi)| + |D_t v(t_\xi^{(2)},\xi)|\Big). \end{aligned}$$

Proof. Introducing $V(t,\xi) := (\lambda(t)b(t)|\xi|v, D_t v)^T$ we transform (2.6) into the following system

$$D_t V - \begin{pmatrix} 0 & \lambda(t)b(t)|\xi| \\ \lambda(t)b(t)|\xi| & 0 \end{pmatrix} V - \begin{pmatrix} \frac{D_t\lambda(t)}{\lambda(t)} + \frac{D_t b(t)}{b(t)} & 0 \\ 0 & 0 \end{pmatrix} V = 0.$$

Choosing $M^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ and $V = M^{-1}V_0$ we get the following system after the first step of diagonalization

$$D_t V_0 - \begin{pmatrix} -\lambda(t)b(t)|\xi| & 0 \\ 0 & \lambda(t)b(t)|\xi| \end{pmatrix} V_0 - \frac{1}{2}\Big(\frac{D_t\lambda(t)}{\lambda(t)} + \frac{D_t b(t)}{b(t)}\Big) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} V_0 = 0.$$

To understand the philosophy of the second step of diagonalization we use the following symbol classes in the hyperbolic zone.

Definition 2.8. We define the following classes of symbols in the hyperbolic zone $Z_{\mathrm{hyp}}(N)$:

$$\begin{aligned} S_{N,l}\{m_1,m_2,m_3\} = \{&a(t,\xi) \in C^l(Z_{\mathrm{hyp}}(N)) : \\ &|D_t^k D_\xi^\alpha a(t,\xi)| \le C_{k,\alpha}|\xi|^{m_1-|\alpha|}\lambda(t)^{m_2}\Big(\frac{\lambda(t)\nu(t)}{\Lambda(t)}\Big)^{m_3+k} \\ &\text{for all } k \le l \text{ and for all multi-indices } \alpha\}. \end{aligned}$$

Defining the matrices

$$D_0 := \begin{pmatrix} -\lambda(t)b(t)|\xi| & 0 \\ 0 & \lambda(t)b(t)|\xi| \end{pmatrix},\ B_0 := -\frac{1}{2}\Big(\frac{D_t\lambda(t)}{\lambda(t)} + \frac{D_t b(t)}{b(t)}\Big)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

then after the first step of diagonalization we obtain $D_tV_0 - D_0V_0 + B_0V_0 = 0$ with $D_0 \in S_{N,2}\{1,1,0\}$ and $B_0 \in S_{N,1}\{0,0,1\}$. During the second step of diagonalization we need the following rules of the classes of symbols:

- $S_{N,l}\{m_1, m_2, m_3\} \subset S_{N,l}\{m_1 + k, m_2 + k, m_3 - k\}$ for $k \geq 0$;
- if $a \in S_{N,l}\{m_1, m_2, m_3\}$ and $b \in S_{N,l}\{k_1, k_2, k_3\}$, then $ab \in S_{N,l}\{m_1 + k_1, m_2 + k_2, m_3 + k_3\}$;
- if $a \in S_{N,l}\{m_1, m_2, m_3\}$, then $D_t^k a \in S_{N,l-k}\{m_1, m_2, m_3 + k\}$ for all $k \leq l$, and $D_\xi^\alpha a \in S_{N,l}\{m_1 - |\alpha|, m_2, m_3\}$ for all multi-indices α;
- if $a(t,\xi) \in S_{N,0}\{-1,-1,2\}$, then $\big|\int_{t_\xi^{(2)}}^t a(s,\xi)ds\big| \leq C\nu(t_\xi^{(2)})$ for all $(t,\xi) \in Z_{\text{hyp}}(N)$. To prove this we need (A3), the decreasing behavior of $\frac{\nu(t)}{\Lambda(t)}$ together with the definition of the hyperbolic zone.

Now let us explain the second step of diagonalization. Here we follow the procedure of the asymptotic theory of ordinary differential equations. Namely, we look for a matrix $N_1(t,\xi)$ having the representation $N_1(t,\xi) := I + N^{(1)}(t,\xi)$. To define $N^{(1)}$ we need $B^{(0)} := B_0$, $F^{(0)} := \text{diag}B^{(0)}$ and the characteristic roots $\tau_k := (-1)^k\lambda(t)b(t)|\xi|$, $k = 1, 2$. Then we define

$$N_{qr}^{(1)} := \frac{B_{qr}^{(0)}}{\tau_q - \tau_r}, q \neq r,\ N_{qq}^{(1)} := 0,$$
$$B^{(1)} := (D_t - D_0 + B_0)(I + N^{(1)}) - (I + N^{(1)})(D_t - D_0 + F^{(0)}).$$

According to the properties of symbols we have $N^{(1)} \in S_{N,1}\{-1,-1,1\}$ and $F^{(0)} \in S_{N,1}\{0,0,1\}$. For $B^{(1)}$ we obtain the relation

$$B^{(1)} = B_0 + [N^{(1)}, D_0] - F^{(0)} + D_t N^{(1)} + B_0 N^{(1)} - N^{(1)}F^{(0)}.$$

The construction principle implies that the sum of the first three terms vanishes, hence $B^{(1)} \in S_{N,0}\{-1,-1,2\}$. Finally, let us define

$$R_1 = N_1^{-1}\big((D_t - D_0 + B_0)(I + N^{(1)}) - (I + N^{(1)})(D_t - D_0 + F^{(0)})\big).$$

But this means $R_1 = N_1^{-1}B^{(1)} \in S_{N,0}\{-1,-1,2\}$. From the construction we have $N^{(1)} \in S_{N,1}\{-1,-1,1\}$. Due to the definition of symbols this means $|N_{qr}^{(1)}| \leq \frac{C_1}{N}$. Consequently, for N large enough $\|N_1 - I\| < \frac{1}{2}$ in $Z_{\text{hyp}}(N)$ implies the invertibility of N_1.

Setting $V_0(t,\xi) =: N_1(t,\xi)V_1(t,\xi)$ we obtain from the above construction the system $D_tV_1 - D_0V_1 + \text{diag}B_0V_1 + R_1V_1 = 0$. We shall study this system for $t \in [t_\xi^{(2)}, \infty)$. The goal is to estimate the fundamental solution $E = E(t, t_\xi^{(2)}, \xi)$. We find the fundamental solution in the form $E(t,s,\xi) = E_2(t,s,\xi)Q(t,s,\xi)$. Here E_2 is the fundamental solution to $D_t - D_0 + \text{diag}B_0$, that is,

$$\begin{cases} E_2^{(11)}(t,s,\xi) = \exp\Big(-i\int_s^t \lambda(\tau)b(\tau)|\xi|d\tau + \int_s^t \frac{\partial_\tau\lambda(\tau)}{2\lambda(\tau)}d\tau + \int_s^t \frac{\partial_\tau b(\tau)}{2b(\tau)}d\tau\Big), \\ E_2^{(22)}(t,s,\xi) = \exp\Big(i\int_s^t \lambda(\tau)b(\tau)|\xi|d\tau + \int_s^t \frac{\partial_\tau\lambda(\tau)}{2\lambda(\tau)}d\tau + \int_s^t \frac{\partial_\tau b(\tau)}{2b(\tau)}d\tau\Big), \\ E_2^{(12)}(t,s,\xi) = E_2^{(21)}(t,s,\xi) = 0. \end{cases}$$

Then Q satisfies

$$D_tQ + E_2(s,t,\xi)R_1(t,\xi)E_2(t,s,\xi)Q = 0,\ Q(s,s,\xi) = I.$$

From the representation of E_2 we conclude

$$\|E_2(s,t,\xi)R_1(t,\xi)E_2(t,s,\xi)\| \lesssim \|R_1(t,\xi)\|.$$

The matrizant representation yields

$$\|Q(t,s,\xi)\| \le \exp\Big(\int_s^t \|R_1(\tau,\xi)\|d\tau\Big) \le \exp(C_1\nu(s)).$$

Summarizing the following estimate holds for the fundamental solution:

$$\|E(t,t_\xi^{(2)},\xi)\| \le C_0 \exp\Big(C_1\nu(t_\xi^{(2)})\Big)\sqrt{\frac{\lambda(t)}{\lambda(t_\xi^{(2)})}}.$$

Taking account of

$$V_1(t,\xi) = E(t,t_\xi^{(2)},\xi)V_1(t_\xi^{(2)},\xi),\ V_0 = N_1V_1,\ V = M^{-1}V_0,$$

we get, finally,

$$\begin{aligned}|V(t,\xi)| &\le C_0 \exp\Big(C_1\nu(t_\xi^{(2)})\Big)\sqrt{\frac{\lambda(t)}{\lambda(t_\xi^{(2)})}}|V(t_\xi^{(2)},\xi)| \\ &\le C_0 \exp\Big(C_1\nu(t)\Big)\sqrt{\frac{\lambda(t)}{\lambda(t_\xi^{(2)})}}|V(t_\xi^{(2)},\xi)|.\end{aligned}$$

These inequalities give the estimates we wanted to prove. □

Remark 2.9. In the hyperbolic zone we have to take account of the oscillating behavior of the coefficient, too. For the desired estimate the assumption $u_0 \in \dot{H}^1$ is of importance.

From Lemmas 2.3, 2.4 and 2.7 we immediately derive the following statement:

Corollary 2.10. *Let us assume* (A1) *to* (A3). *Then the solution to* (2.6) *satisfies the following estimates*:

$$\begin{aligned}&\lambda(t)|\xi||v(t,\xi)| + |D_tv(t,\xi)| \\ &\quad\le C_0\Big(\frac{\lambda(t)}{\Lambda(t)}|v(0,\xi)| + \frac{t\lambda(t)}{\Lambda(t)}|D_tv(0,\xi)|\Big),\ t\in[0,t_\xi^{(1)}], \\ &\lambda(t)|\xi||v(t,\xi)| + |D_tv(t,\xi)| \\ &\quad\le C_0\exp(C_1\nu(t))\frac{\lambda(t_\xi^{(1)})}{\Lambda(t_\xi^{(1)})}\Big(|v(0,\xi)| + t_\xi^{(1)}|D_tv(0,\xi)|\Big),\ t\in[t_\xi^{(1)},t_\xi^{(2)}], \\ &\lambda(t)|\xi||v(t,\xi)| + |D_tv(t,\xi)| \\ &\quad\le C_0\exp(C_1\nu(t))\frac{\sqrt{\lambda(t)}}{\sqrt{\lambda(t_\xi^{(2)})}}\frac{\lambda(t_\xi^{(1)})}{\Lambda(t_\xi^{(1)})}\Big(|v(0,\xi)| + t_\xi^{(1)}|D_tv(0,\xi)|\Big),\ t\in[t_\xi^{(2)},\infty).\end{aligned}$$

The desired statement follows immediately from Corollary 2.10 together with the estimate

$$\frac{t\lambda(t)}{\Lambda(t)} \leq C\sqrt{\lambda(t)}.$$

Here we use the assumptions (2.2) and (2.3) from (A1). This completes the proof of Theorem 2.1. □

Remark 2.11. The statement of Theorem 2.1 was proved in the case $\nu(t) \equiv 0$, that is, for very slow oscillations in [9]. There one can also find an estimate of the energy to below, both together yield a result about *generalized energy conservation.*

3. Parameter dependent Cauchy problems

The goal of this section is to derive another energy estimate for a parameter dependent family of Cauchy problems. Now we assume only $u_0 \in \dot{H}^1$ and $u_1 \in L^2$. We consider with $t_0 \in [0, \infty)$ the family of Cauchy problems

$$D_t^2 u - \lambda^2(t) b^2(t) D_x^2 u = 0,\ u(t_0, x) = u_0(x),\ D_t u(t_0, x) = u_1(x). \tag{3.1}$$

Defining $\lambda_{t_0}(t) := \lambda(t + t_0)$, $\nu_{t_0}(t) := \nu(t + t_0)$ and the energy

$$\mathbb{E}_{\lambda,t_0}(u;t) := \frac{1}{2} \int_{\mathbb{R}} \Big(\lambda_{t_0}^2(t) |D_x u(t,x)|^2 + |D_t u(t,x)|^2\Big) dx, \tag{3.2}$$

our goal is to prove the following statement.

Theorem 3.1. *Assume the conditions* (A1) *to* (A3)*. Then the solution to* (3.1) *for data* $u_0 \in \dot{H}^1(\mathbb{R})$ *and* $u_1 \in L^2(\mathbb{R})$ *satisfies*

$$\mathbb{E}_{\lambda,t_0}(u;t) \leq C_0 \exp(C_1 \nu_{t_0}(t)) \frac{\lambda_{t_0}^2(t)}{\lambda_{t_0}^2(0)} \mathbb{E}_{\lambda,t_0}(u;0). \tag{3.3}$$

The positive constants C_0 *and* C_1 *are independent of the data and of* $t_0, t \in [0, \infty)$.

Proof. The proof follows the lines of the proof to Theorem 2.1. Instead of (3.1) we study

$$D_t^2 u - \lambda_{t_0}^2(t) b_{t_0}^2(t) D_x^2 u = 0,\ u(0, x) = u_0(x),\ D_t u(0, x) = u_1(x). \tag{3.4}$$

We define two zones

$$Z_{\mathrm{pd}}(N) := \{(t, \xi) \in [0, \infty) \times \mathbb{R} : \Lambda_{t_0}(t)|\xi| \leq N\nu_{t_0}(t)\},$$
$$Z_{\mathrm{hyp}}(N) := \{(t, \xi) \in [0, \infty) \times \mathbb{R} : \Lambda_{t_0}(t)|\xi| \geq N\nu_{t_0}(t)\},$$

where $\Lambda_{t_0}(t) := 1 + \int_0^t \lambda_{t_0}(s) ds$ and N is a positive large constant.

In the pseudo-differential zone $Z_{\mathrm{pd}}(N)$ we define the micro-energy $V(t, \xi) := (\lambda_{t_0}(t)|\xi| v, D_t v)^T$ and transform

$$D_t^2 v - \lambda_{t_0}^2(t) b_{t_0}^2(t) \xi^2 v = 0$$

to the following system of first order:

$$D_tV = \begin{pmatrix} 0 & \lambda_{t_0}(t)|\xi| \\ \lambda_{t_0}(t)b_{t_0}^2(t)|\xi| & 0 \end{pmatrix} V + \frac{D_t\lambda_{t_0}(t)}{\lambda_{t_0}(t)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} V.$$

Then we can follow the proof to Lemma 2.4 and obtain the estimate

$$|V(t,\xi)| \le C_0 \frac{\lambda_{t_0}(t)}{\lambda_{t_0}(0)} \exp(C_1\nu_{t_0}(t))|V(0,\xi)|. \tag{3.5}$$

In the hyperbolic zone $Z_{\mathrm{hyp}}(N)$ we carry out again a diagonalization procedure consisting of two steps. Defining the micro-energy $V(t,\xi) := (\lambda_{t_0}(t)b_{t_0}(t)|\xi|v, D_tv)^T$ we obtain the following system of first order:

$$D_tV = \begin{pmatrix} 0 & \lambda_{t_0}(t)b_{t_0}(t)|\xi| \\ \lambda_{t_0}(t)b_{t_0}(t)|\xi| & 0 \end{pmatrix} V + \frac{D_t(\lambda_{t_0}(t)b_{t_0}(t))}{2\lambda_{t_0}(t)b_{t_0}(t)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} V.$$

Then we can follow the proof of Lemma 2.7. There are no new essential difficulties, if we take into consideration that $\Lambda(t+t_0) \ge \Lambda_{t_0}(t)$ and the assumption (A2) is satisfied for $b_{t_0}(t)$ with constants which are independent of t_0. In the hyperbolic zone $Z_{\mathrm{hyp}}(N)$ we obtain the estimate

$$|V(t,\xi)| \le C_0 \exp(C_1\nu_{t_0}(t)) \frac{\sqrt{\lambda_{t_0}(t)}}{\sqrt{\lambda_{t_0}(t_\xi)}} |V(t_\xi,\xi)|, \tag{3.6}$$

where the constants C_0 and C_1 are independent of t_0 and $t_\xi(t_0)$ is defined by $\Lambda_{t_0}(t_\xi)|\xi| = N\nu(t_\xi)$. From (3.5) and (3.6) we conclude

$$\begin{aligned} &\lambda_{t_0}(t)|\xi||v(t,\xi)| + |D_tv(t,\xi)| \\ &\quad \le C_0 \exp(C_1\nu_{t_0}(t)) \frac{\lambda_{t_0}(t)}{\lambda_{t_0}(0)} \Big(\lambda_{t_0}(0)|\xi||v(0,\xi)| + |D_tv(0,\xi)|\Big) \end{aligned}$$

with constants C_0 and C_1 which are independent of $t_0 \in [0,\infty)$. The last inequality implies the statement which we wanted to prove. □

4. In which sense do we have optimality?

In this section we want to discuss the question if the energy estimate from Theorem 3.1 is optimal. Here we follow the discussion from [5]. First we have to develop a strategy which yields an understanding of optimality. For this reason we use the estimate of the elastic energy from Theorem 3.1.

Corollary 4.1. *Assume the conditions* (A1) *to* (A3)*. Then the solution to* (3.1) *for data* $u_0 \in \dot{H}^1(\mathbb{R})$ *and* $u_1 \equiv 0$ *satisfies*

$$\|D_xu(t,\cdot)\|_{L^2} \le C_0 \exp(C_1\nu_{t_0}(t))\|D_xu(t_0,\cdot)\|_{L^2}. \tag{4.1}$$

The positive constants C_0 *and* C_1 *are independent of the data and of* $t_0, t \in [0,\infty)$.

Instead of (3.1) let us consider the family of Cauchy problems

$$D_t^2 u - \lambda^2(t) b_k^2(t) D_x^2 u = 0,\ u(t_0,x) = u_{0.k}(x),\ D_t u(t_0,x) = 0. \tag{4.2}$$

Our goal is to apply an instability argument.

Theorem 4.2. *Let us consider the family of Cauchy problems* (4.2)*. Let $b = b(t)$ be a non-constant,* 1*-periodic, positive and smooth function which is constant in a neighborhood of $t = 0$. Then there exist a function λ which satisfies* (A1)*, a family of coefficients $\{b_k(t)\}_k$ which satisfies* (A2) *with constants C_0 and C_1 which are independent of k and with a function $\nu = \nu(t) = f(\Lambda(t))$ satisfying* (A3)*, and, finally, a family of data $\{u_{0,k}\} \in \dot{H}^1(\mathbb{R})$ which are prescribed for $t = t_k^{(1)}$ such that the following estimate for the elastic energy holds:*

$$\|D_x u(t_k^{(2)}, \cdot)\|_{L^2} \geq C_0 \exp(C_1 \nu(t_k^{(1)})) \|D_x u(t_k^{(1)}, \cdot)\|_{L^2}.$$

Here $\{t_k^{(1)}\}_k$ and $\{t_k^{(2)}\}_k$ are two sequences which tend to infinity. The constants C_0 and C_1 are independent of k.

Proof. The proof generalizes ideas from [3]. We divide it into several steps.

Step 1: *Sequences of parameters and intervals*

We use sequences of parameters

(C1) $\{t_k\}_k$, $\{t_k'\}_k$, $\{t_k''\}_k$ and $\{\delta_k\}_k$ tending to ∞,
(C2) $\{h_k\}_k$ and $\{\rho_k\}_k$ with the property $h_k\rho_k \to \infty$ for $k \to \infty$.

Finally we need three sequences of intervals

- $\{I_k\}_k$, $\{I_k'\}_k$ and $\{I_k''\}_k$, which are defined as follows:

$$I_k = \Big[t_k - \frac{\rho_k}{2}, t_k + \frac{\rho_k}{2}\Big],\quad I_k' = \Big[t_k' - \frac{\rho_k}{2}, t_k' + \frac{\rho_k}{2}\Big],\quad t_k' := t_k + \rho_k,$$
$$I_k'' = \Big[t_k'' - \frac{\rho_k}{2}, t_k'' + \frac{\rho_k}{2}\Big],\quad t_k'' := t_k - \rho_k.$$

The intervals I_k, I_k', I_k'' should belong to $[0,\infty)$. For this reason we assume
(C3) $\rho_k \leq \frac{1}{2} t_k$ for $k \to \infty$.

Step 2: *Construction of a family of coefficients*

We choose a monotonous increasing function $\mu \in C^\infty(\mathbb{R})$ with

$$\mu(r) = \begin{cases} 0, & r \in (-\infty, -\frac{1}{3}], \\ 1, & r \in [\frac{1}{3}, +\infty), \end{cases}$$

and define the family of coefficients $\{a_k = a_k(t)\}_k$ with $a_k = \lambda^2(t) b_k^2(t)$ as follows:

$$a_k(t) = \begin{cases} \lambda^2(t), & t \in\ [0,\infty) \setminus (I_k' \cup I_k \cup I_k''); \\ \delta_k b^2(h_k(t - t_k)), & t \in I_k; \\ \delta_k b(0)^2 \Big(1 - \mu\Big(\frac{t - t_k'}{\rho_k}\Big)\Big) + \lambda^2(t)\mu\Big(\frac{t - t_k'}{\rho_k}\Big), & t \in I_k'; \\ \delta_k b(0)^2 \mu\Big(\frac{t - t_k''}{\rho_k}\Big) + \lambda^2(t)\Big(1 - \mu\Big(\frac{t - t_k''}{\rho_k}\Big)\Big), & t \in I_k''. \end{cases}$$

Here $b = b(t)$ is a non-constant, 1-periodic, smooth and positive function which is constant in a small neighborhood of $t = 0$. To guarantee that the coefficients are in $C^2(\mathbb{R})$, that means in particular, that they are C^2 if we transfer from I_k to I'_k and I''_k we assume

$$\textbf{(C4)} \qquad \frac{h_k \rho_k}{2} \in \mathbb{N}.$$

Step 3: *Choice of parameters*

We choose the sequences $\{\rho_k\}_k$, $\{\delta_k\}_k$ and $\{h_k\}_k$ as follows:

$$\rho_k = \varepsilon \frac{\Lambda(t_k)}{\lambda(t_k)}, \ \delta_k = \lambda^2(t_k), \ h_k = 2\frac{\lambda(t_k)}{\varepsilon\Lambda(t_k)}\left[\nu(t_k)\right],$$

here $\varepsilon > 0$ is small. Then the conditions (C1), (C2) and (C4) are satisfied if $\{t_k\}_k$ tends to infinity. The condition (C3) is satisfied if we assume

$$\textbf{(C5)} \qquad \frac{\Lambda(t_k)}{\lambda(t_k)} = O(t_k) \text{ for } k \to \infty.$$

Step 4: *Estimates for* b_k

To explain corresponding properties to $\{b_k\}_k$ we assume

$$\textbf{(C6)} \qquad d_0 \le \inf_k \frac{\lambda(t_k)}{\lambda(t_k \pm \frac{4}{3}\rho_k)} \le \sup_k \frac{\lambda(t_k)}{\lambda(t_k \pm \frac{4}{3}\rho_k)} \le d_1$$

with positive constants d_0 and d_1.

Condition (C6) implies

$$0 < b_0 \le \inf_{t\in[0,\infty)} b_k(t) \le \sup_{t\in[0,\infty)} b_k(t) \le b_1 < \infty,$$

where the constants b_0 and b_1 are independent of k. It remains to prove the following inequalities on the set $I_k \cup I'_k \cup I''_k$:

$$|b'_k(t)| \le C_0 \frac{\lambda(t)}{\Lambda(t)}\nu(t), \ \ |b''_k(t)| \le C_0 \Big(\frac{\lambda(t)}{\Lambda(t)}\nu(t)\Big)^2,$$

where the constants C_0 are independent of k.

We have to study $b_k = b_k(t)$ on the interval $[t_k - \frac{4}{3}\rho_k, t_k + \frac{4}{3}\rho_k]$. The relation $\frac{\lambda'(t)}{\lambda(t)} \sim \frac{\lambda(t)}{\Lambda(t)}$ follows from (2.2), $\sqrt{\delta_k} \sim \lambda(t_k)$ gives the choice of parameters. From (C6) it follows $\lambda(t) \sim \lambda(t_k)$ on $[t_k - \frac{4}{3}\rho_k, t_k + \frac{4}{3}\rho_k]$. Now we show $\Lambda(t) \sim \Lambda(t_k)$ on the interval $[t_k, t_k + \frac{4}{3}\rho_k]$. We have to prove this equivalence only for $t = t_k + \frac{4}{3}\rho_k$. On the one hand $\Lambda(t_k) \le \Lambda(t)$, on the other hand we have with (C6)

$$\begin{aligned}\Lambda(t) &= \int_0^t \lambda(s)ds + 1 = \int_0^{t_k} \lambda(s)ds + \int_{t_k}^t \lambda(s)ds + 1 \\ &\lesssim \Lambda(t_k) + \lambda(t_k)\frac{4}{3}\rho_k \lesssim \Lambda(t_k) + \lambda(t_k)\frac{\Lambda(t_k)}{\lambda(t_k)} \lesssim \Lambda(t_k).\end{aligned}$$

Finally, we show $\nu(t_k) \sim \nu(t)$ on the interval $[t_k - \frac{4}{3}\rho_k, t_k]$. To prove this equivalence we assume

$$\textbf{(C7)} \qquad \frac{\lambda(t)}{\Lambda(t)}\frac{\Lambda(t_k)}{\lambda(t_k)} \le C \quad \text{for all} \quad t \in [t_k - \tfrac{4}{3}\rho_k, t_k].$$

The monotonicity of ν implies $\nu(t) \le \nu(t_k)$. The relation $\nu(t_k) \sim \nu(t)$ we obtain from the assumption $\nu(t) = f(\Lambda(t))$ with $|f'(r)| \le C\frac{1}{r}$. Hence, $\nu'(t) = f'(\Lambda(t))\lambda(t)$ gives $|\nu'(t)| \le C\frac{\lambda(t)}{\Lambda(t)}$. It remains to prove $\nu(t_k) \sim \nu(t)$ only for $t = t_k - \frac{4}{3}\rho_k$. We use

$$\begin{aligned}|\nu(t_k)| &\le \big|\nu\big(t_k - \frac{4}{3}\rho_k\big)\big| + \big|\nu\big(t_k - \frac{4}{3}\rho_k\big) - \nu(t_k)\big| \\ &\le \big|\nu\big(t_k - \frac{4}{3}\rho_k\big)\big| + \big|\nu'(\tilde{t_k})\big(\frac{4}{3}\rho_k\big)\big|.\end{aligned}$$

Together with (C7) we have

$$\left|\nu'(\tilde{t_k})\frac{4}{3}\rho_k\right| \le C\frac{\lambda(\tilde{t_k})}{\Lambda(\tilde{t_k})}\frac{\Lambda(t_k)}{\lambda(t_k)} \le C.$$

Consequently, $\nu(t_k) \lesssim \nu(t_k - \frac{4}{3}\rho_k)$, $\nu(t_k) \lesssim \nu(t)$ on $[t_k - \frac{4}{3}\rho_k, t_k]$, respectively. We did not prove $\Lambda(t) \sim \Lambda(t_k)$ on the interval $[t_k - \frac{4}{3}\rho_k, t_k]$ and $\nu(t_k) \sim \nu(t)$ on $[t_k, t_k + \frac{4}{3}\rho_k]$. On these intervals we use the monotonic behavior of the functions ν and $\frac{1}{\Lambda}$. Summarizing we estimate as follows:

$$\frac{\lambda(t_k)}{\Lambda(t_k)}\nu(t_k) \lesssim \frac{\lambda(t)}{\Lambda(t_k)}\nu(t_k) \lesssim \frac{\lambda(t)}{\Lambda(t)}\nu(t)$$

if we use the equivalences and the monotonic behavior.

Now we are able to estimate the derivatives of b_k. From

$$b_k'(t) = -\frac{\lambda'(t)}{\lambda^2(t)}\sqrt{\delta_k}b(h_k(t-t_k)) + h_k\frac{1}{\lambda(t)}\sqrt{\delta_k}b'(h_k(t-t_k))$$

the above estimates and the choice of parameters imply

$$\begin{aligned}|b_k'(t)| &\le C_0\frac{\lambda(t_k)}{\Lambda(t_k)}|b(h_k(t-t_k))| + h_k|b'(h_k(t-t_k))|, \\ |b_k'(t)| &\le C_0\left(\frac{\lambda(t_k)}{\Lambda(t_k)} + \frac{\lambda(t_k)}{\Lambda(t_k)}\nu(t_k)\right) \le C_0\frac{\lambda(t)}{\Lambda(t)}\nu(t) \text{ on } I_k\end{aligned}$$

with a constant C_0 which is independent of k. For the second derivative we obtain

$$\begin{aligned}|b_k''(t)| &\le C_0\left(\frac{\lambda(t)}{\Lambda(t)}h_k + h_k^2\right) \le C_0\left(\Big(\frac{\lambda(t)}{\Lambda(t)}\Big)^2\nu(t) + \Big(\frac{\lambda(t)}{\Lambda(t)}\nu(t)\Big)^2\right) \\ &\le C_0\Big(\frac{\lambda(t)}{\Lambda(t)}\nu(t)\Big)^2 \text{ on } I_k,\end{aligned}$$

where C_0 is independent of k. On the other both intervals I_k' and I_k'' we proceed in a similar way. As a conclusion we obtain on the set $I_k \cup I_k' \cup I_k''$:

$$|b_k'(t)| \le C_0\frac{\lambda(t)}{\Lambda(t)}\nu(t), \quad |b_k''(t)| \le C_0\Big(\frac{\lambda(t)}{\Lambda(t)}\nu(t)\Big)^2,$$

where the constant C_0 is independent of k.

Step 5: *Choice of data and estimates*

Let $\chi = \chi(r) \in [0,1]$ be from $C_0^\infty(\mathbb{R})$, where $\chi \equiv 1$ for $|r| \le 1$ and $\chi \equiv 0$ for $|r| \ge 2$. We choose for large k the following data:

$$u_{0,k}(x) = \exp\Big(i\frac{h_k}{\sqrt{\delta_k}}x\xi\Big)\chi\Big(\frac{x}{\nu^2(t_k)P_k}\Big), \quad u_{1,k}(x) \equiv 0 \quad \text{for all } x \in \mathbb{R},$$

where

$$P_k = 2\pi\frac{\sqrt{\delta_k}}{h_k\xi} \sim \Lambda(t_k)(\nu(t_k))^{-1},$$

here ξ will be chosen later. The elastic energy of the data $u_{0,k}$ is estimated as follows:

$$\begin{aligned}\|\partial_x u_{0,k}(\cdot)\|^2_{L^2(\mathbb{R})} &\le C_0\Big(\frac{h_k}{\sqrt{\delta_k}} + \frac{1}{\nu^2(t_k)P_k}\Big)^2\nu^2(t_k)P_k \\ &\le C_0\Big(\frac{h_k}{\sqrt{\delta_k}} + \frac{1}{\Lambda(t_k)\nu(t_k)}\Big)^2\nu^2(t_k)P_k.\end{aligned}$$

Taking account of $\frac{1}{\Lambda(t_k)\nu(t_k)} = o(\frac{h_k}{\sqrt{\delta_k}})$ it holds

$$\|\partial_x u_{0,k}(\cdot)\|_{L^2(\mathbb{R})} \le C_0\Big(\frac{h_k}{\sqrt{\delta_k}}\Big)\nu(t_k)\sqrt{P_k}.$$

Step 6: *Cauchy problems on* I_k

We study the following Cauchy problems on I_k:

$$u_{tt} - \delta_k b^2(h_k(t-t_k))u_{xx} = 0,\ u(t_k,x) = u_{0,k}(x),\ u_t(t_k,x) = 0.$$

Later we are interested in the unique solution $u_k = u_k(t_k + \frac{\rho_k}{2}, x)$ on the set $\{|x| \le P_k\}$. The solution on this set will be influenced by the data on the set $\{|x| \le P_k + \frac{\rho_k\sqrt{\delta_k}}{2}\}$. From $\rho_k\sqrt{\delta_k} = O(\Lambda(t_k))$ and $P_k = \Lambda(t_k)(\nu(t_k))^{-1}$ we know that we need the knowledge about the data on the set $\{|x| \le O(\nu(t_k)P_k)\}$. On this set the initial data $u_{0,k}$ has the representation $u_{0,k}(x) = \exp\big(i\frac{h_k}{\sqrt{\delta_k}}x\xi\big)$. The change of variables $s = h_k(t-t_k)$, $v(s,x) := u(t,x)$ transfers the above Cauchy problems into

$$v_{ss} - \frac{\delta_k}{h_k^2}b^2(s)v_{xx} = 0,\ v(0,x) = u_{0,k}(x),\ v_s(0,x) = 0,\ s \in \Big[-\frac{h_k\rho_k}{2}, \frac{h_k\rho_k}{2}\Big],$$

where we assume for the data $u_{0,k}$ the representation $u_{0,k}(x) = \exp\big(i\frac{h_k}{\sqrt{\delta_k}}x\xi\big)$. Then there exists a uniquely determined solution $u_k = u_k(s,x)$ in the form $u_k(s,x) = u_{0,k}(x)w(s)$, where $w = w(s)$ solves the Cauchy problem

$$w''(s) + \xi^2b^2(s)w(s) = 0,\ w(0) = 1,\ w'(0) = 0,\ s \in \Big[-\frac{h_k\rho_k}{2}, \frac{h_k\rho_k}{2}\Big].$$

Step 7: *A lemma from Floquet theory*

To derive an estimate for $u_k = u_k(t_k + \frac{\rho_k}{2}, x)$ on the set $\{|x| \le P_k\}$ we apply the following lemma.

Lemma 4.3. *Let $w = w(t)$ be a solution to*

$$w_{tt} + \xi^2 b^2(t) w = 0,\ w(0) = 1,\ w_t(0) = 0.$$

Here ξ^2 is from an instability interval for the function $b = b(t)$ which we suppose to be non-constant, 1-periodic, positive and smooth. Then the solution $w = w(t)$ satisfies the asymptotic relation $|w(M)| \sim |\mu_0|^M$ with $|\mu_0| > 1$ for all sufficiently large $M \in \mathbb{N}$.

The statement of Lemma 4.3 follows from the following basic lemma of Floquet theory.

Lemma 4.4. ([4], [11]) *Let the coefficient $b = b(t)$ be a non-constant, 1-periodic, positive and smooth function. Then there exists a positive $\lambda_0 := \xi^2$ such that the fundamental matrix $X = X(t, t_0)$ to the system*

$$d_t X = \begin{pmatrix} 0 & -\lambda_0 b(t)^2 \\ 1 & 0 \end{pmatrix} X, \quad X(t_0, t_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

has the following property:

The matrix $X(1, 0)$ has the eigenvalues μ_0 and μ_0^{-1} with $|\mu_0| > 1$.

The application of Lemma 4.3 yields for $\{|x| \le P_k\}$ after backward transformation

$$u_k\Big(t_k + \frac{\rho_k}{2}, x\Big) = \exp\Big(i\frac{h_k}{\sqrt{\delta_k}} x\xi\Big) w\Big(\frac{\rho_k h_k}{2}\Big), \quad u_k(t_k, x) = \exp\Big(i\frac{h_k}{\sqrt{\delta_k}} x\xi\Big) w(0),$$

with

$$\Big| w\Big(\frac{\rho_k h_k}{2}\Big)\Big| \sim |\mu_0|^{\frac{\rho_k h_k}{2}}.$$

Step 8: *Verification*

We choose $t_k^{(1)} = t_k$ and $t_k^{(2)} = t_k + \frac{\rho_k}{2}$. The two sequences $\{t_k^{(1)}\}_k$ and $\{t_k^{(2)}\}_k$ tend to infinity. It holds the estimate

$$\big\|\partial_x u_k(t_k^{(2)}, \cdot)\big\|_{L^2(\mathbb{R})} \ge \big\|\partial_x u_k(t_k^{(2)}, \cdot)\big\|_{L^2(\{|x| \le P_k\})} \ge C_0\Big(\frac{h_k}{\sqrt{\delta_k}}\Big)\sqrt{P_k}|\mu_0|^{\frac{\rho_k h_k}{2}}.$$

This estimate is derived as in Step 5. Moreover, we know from Step 5

$$\big\|\partial_x u_k(t_k^{(1)}, \cdot)\big\|_{L^2(\mathbb{R})} \le C_0\Big(\frac{h_k}{\sqrt{\delta_k}}\Big)\nu(t_k^{(1)})\sqrt{P_k}.$$

Setting $a = \log|\mu_0| > 0$, $|\mu_0| > 1$ and from $\frac{h_k \rho_k}{2} \sim \nu(t_k^{(1)})$ it follows

$$|\mu_0|^{\frac{\rho_k h_k}{2}} \sim \exp\big(a\ \nu(t_k^{(1)})\big).$$

Combining both estimates we conclude

$$\big\|\partial_x u_k(t_k^{(2)}, \cdot)\big\|_{L^2(\mathbb{R})} \ge C_0 \exp\big(C_1 \nu(t_k^{(1)})\big)\big\|\partial_x u_k(t_k^{(1)}, \cdot)\big\|_{L^2(\mathbb{R})}.$$

The constants C_0 and C_1 are independent of k. This completes the proof to Theorem 4.2. □

4.1. How to interpret optimality?

We choose the sequences $\{t_k^{(1)}\}_k := \{t_k\}_k$ and $\{t_k^{(2)}\}_k := \{t_k + \frac{\rho_k}{2}\}_k$. We consider the Cauchy problem (4.2) with $u_1 \equiv 0$. Then from Corollary 4.1 and from Theorem 4.2 we conclude the estimates

$$\|\partial_x u_k(t_k^{(2)},\cdot)\|_{L^2(\mathbb{R})} \le C_0 \exp\big(C_1\nu\big(2t_k + \frac{\rho_k}{2}\big)\big)\|\partial_x u_k(t_k^{(1)},\cdot)\|_{L^2(\mathbb{R})},$$
$$\|\partial_x u_k(t_k^{(2)},\cdot)\|_{L^2(\mathbb{R})} \ge C_0 \exp\big(C_1\nu\big(t_k\big)\big)\|\partial_x u_k(t_k^{(1)},\cdot)\|_{L^2(\mathbb{R})}.$$

If we assume the condition

(C8) $\nu(t) \sim \nu(3t)$ for large t,

then we obtain the following statement after using the condition (C3) and the assumptions for ν.

Corollary 4.5. *The estimate of the elastic energy from Corollary* 4.1 *is sharp under the assumption* (C8)*.*

4.2. Examples

Let us finally give some examples which satisfy the conditions from Theorem 4.2 and the condition $\nu(t) \sim \nu(3t)$ for all large t.

Example. (potential growth of λ)
Let us choose $\lambda(t) = (1+t)^l$, $l > 0$, and $\nu(t) = \log^{[n]} t$, $n \ge 1$, for large t. Here $\log^{[n]}$ means the n times application of log. Then λ and ν satisfy (A1) and (A3). In this case we choose $t_k := 2^k$, $\rho_k := 2^{k-3}$, $\delta_k := 2^{2kl}$ and $h_k := 2^{4-k}[\nu(2^k)]$. Here $[\nu(2^k)]$ is the integer part of $\nu(2^k)$. There are no difficulties to show that the conditions (C1) to (C7) are satisfied. To show (C8) we use the induction principle and derive

$$\lim_{t\to\infty} \frac{\log^{[n]}(3t)}{\log^{[n]} t} = 1 \text{ for all } n \ge 0.$$

Example. (exponential growth of λ)
Let us choose $\lambda(t) = \exp t$ and $\nu(t) = \log^{[n]} \exp t$, $n \ge 1$, for large t. Then λ and ν satisfy (A1) and (A3). In this case we choose $t_k := k$, $\rho_k := 1$, $\delta_k := \exp(2k)$ and $h_k := 2[\nu(2^k)]$. There are no difficulties to show that the conditions (C1) to (C7) are satisfied. To show (C8) we use the argument from the previous example.

Example. (super-exponential growth of λ)
Let us choose $\Lambda(t) = \exp^{[m]} t$, $m \ge 2$, and $\nu(t) = \log^{[n]} \exp^{[m]} t$ for large t. Here $\exp^{[m]}$ means the m times application of exp. Then λ and ν satisfy (A1) and (A3). In this case we choose $t_k := k$ and ρ_k, δ_k and h_k with $\varepsilon = 1$ as it is proposed in Step 3 of the proof to Theorem 4.2. There are no difficulties to show that the conditions (C1) to (C5) are satisfied. The condition (C5) is even better, namely, $\frac{\Lambda(t_k)}{\lambda(t_k)} = o(t_k)$ for $k \to \infty$. To show (C8) we use the assumption $n \ge m$ and apply the argument from the previous examples. It remains to discuss the conditions

(C6) and (C7). These conditions follow from the equivalences $\lambda(t) \sim \lambda(t_k)$ and $\Lambda(t) \sim \Lambda(t_k)$ on the interval $[t_k, t_k + \frac{\rho_k}{2}]$. Here we use

$$\rho_k = \frac{1}{\exp^{[m-1]}(t_k)\exp^{[m-2]}(t_k)\cdots\exp^{[2]}(t_k)\exp^{[1]}(t_k)},$$

$$\begin{aligned}\exp^{[l]}\Big(t_k + \frac{\rho_k}{2}\Big) &= \exp^{[l-1]}\Big(\exp k \exp^{\frac{\rho_k}{2}}\Big) \le \exp^{[l]}(k)\exp^{[l-1]}\Big(\frac{1}{\exp^{[l-1]}(k)}\Big)\\ &\le \exp^{[l]}(k)\exp^{[l-1]}\Big(\frac{1}{\exp^{[l-1]}(1)}\Big)\\ &\sim \exp^{[l]}(k) = \exp^{[l]}(t_k) \text{ for } l \in \mathbb{N} \text{ and } l \le m.\end{aligned}$$

5. Interaction of oscillations

In this section we study instead of (2.1) the Cauchy problem

$$\begin{cases} D_t^2 u + 2\lambda(t)b(t)D_{xt}^2 u - \lambda^2(t)a^2(t)D_x^2 u = 0, \\ u(0,x) = u_0(x),\ D_t u(0,x) = u_1(x). \end{cases} \tag{5.1}$$

The oscillating functions $a = a(t)$ and $b = b(t)$ satisfy the assumption

(A4) $0 < a_0 \le a(t) \le a_1$, $0 < b_0 \le b(t) \le b_1$ for $t \in [0,\infty)$ together with the estimates

$$|D_t^k a(t)| + |D_t^k b(t)| \lesssim \Big(\frac{\lambda(t)}{\Lambda(t)}\nu(t)\Big)^k \text{ for } k = 1,2. \tag{5.2}$$

This condition implies the strict hyperbolicity of (5.1). Moreover, the characteristic roots to the strict hyperbolic operators from (2.1) and (5.1) have the same properties, that is, the influence of the shape function and the oscillating functions on the characteristic roots is the same. Nevertheless, there is a big difference coming from *interactions of the oscillating functions* $a(t)$ and $b(t)$. To explain this difference is the goal of this section. For this reason we are interested in the following two cases:

Case 1: The shape function $\lambda(t) = (1+t)^l$, $l > 0$. The function $\nu(t) = \log^{[n+1]}\Lambda(t)$, $n \in \mathbb{N}$, for large t. Both functions satisfy (A1) and (A3).

Then from the proof of Theorem 2.1 we conclude the following statement for the fundamental solution $E = E(t,s,\xi)$ to the system (which is related to (2.6) and the estimate of its fundamental solution gives energy estimates to (2.1))

$$D_t V = \begin{pmatrix} \frac{D_t\lambda(t)}{\lambda(t)} & \lambda(t)|\xi| \\ \lambda(t)a^2(t)|\xi| & 0 \end{pmatrix} V.$$

Corollary 5.1. *Under the assumptions* (A1) *to* (A3) *it holds*

$$\|E(t,0,\xi)\| \le C_0\lambda(t)\exp(C_1\nu(t)).$$

If we assume (A4), then possible interactions of oscillations may lead to the following result for the fundamental solution $E = E(t,s,\xi)$ to the system

$$D_tV = \begin{pmatrix} \frac{D_t\lambda(t)}{\lambda(t)} & \lambda(t)|\xi| \\ \lambda(t)a^2(t)|\xi| & -2\lambda(t)b(t)\xi \end{pmatrix} V.$$

This system is related to

$$\begin{cases} D_t^2 v + 2\lambda(t)b(t)\xi D_t v - \lambda^2(t)a^2(t)\xi^2 v = 0, \\ v(0,\xi) = \hat{u}_0(\xi),\ D_t v(0,\xi) = \hat{u}_1(\xi), \end{cases} \tag{5.3}$$

and estimates for the fundamental solution explain a possible behavior of the energy of solutions to (5.1).

Theorem 5.2. *There exist coefficients $a(t)$ and $b(t)$ satisfying* (A4) *and a sequence $\{t_j\}_j$ which tends to infinity such that*

$$\|E(t_j,0,\xi)\| \geq C_0 \exp\big(C_1 \log\Lambda(t_j)\nu(t_j)\big) \textit{ for all } j \geq j_0(\xi).$$

Case 2: The shape function $\lambda(t) = \Lambda'(t)$ with $\Lambda(t) = \exp^{[m]} t$. The function $\nu(t) = (\log\Lambda(t))^\gamma$, $\gamma \in [0,1]$, for large t. Both functions satisfy (A1) and (A3).

Similar to Corollary 5.1 we have the following statement:

Corollary 5.3. *Under the assumptions* (A1) *to* (A3) *it holds for the fundamental solution $E = E(t,s,\xi)$ to the system*

$$D_tV = \begin{pmatrix} \frac{D_t\lambda(t)}{\lambda(t)} & \lambda(t)|\xi| \\ \lambda(t)a^2(t)|\xi| & 0 \end{pmatrix} V$$

the estimate

$$\|E(t,0,\xi)\| \leq C_0\lambda(t) \exp\big(C_1(\log\Lambda(t))^\gamma\big),\ \gamma \in [0,1],\ \textit{for large } t.$$

On the other hand we are able to show the following result:

Theorem 5.4. *There exist coefficients $a(t)$ and $b(t)$ satisfying* (A4) *and a sequence $\{t_j\}_j$ which tends to infinity such that*

$$\|E(t_j,0,\xi)\| \geq C_0 \exp\big(C_1(\log\Lambda(t_j))^{\gamma+1}\big) \textit{ for all } j \geq j_0(\xi).$$

5.1. Proof of Theorem 5.2

We divide the proof into several steps.

Step 1: *Consideration in the pseudo-differential zone*

Let us devote to (5.3). In the pseudo-differential zone $Z_{\mathrm{pd}}(N) = \{(t,\xi) \in [0,\infty) \times \mathbb{R} : \Lambda(t)|\xi| \leq N\nu(t)\}$ we define the micro-energy $V(t,\xi) := (\lambda(t)|\xi|v, D_tv)^T$. Then following the proof to Lemma 2.5 we get

$$\|E(t,0,\xi)\| \leq C_0 \exp(N\nu(t))\lambda(t) \text{ for all } t \in [0,t_\xi]. \tag{5.4}$$

Step 2: Diagonalization and elliptic transformation

Now we explain the WKB analysis in the hyperbolic zone $Z_{\text{hyp}}(N) = \{(t,\xi) \in [0,\infty) \times \mathbb{R} : \Lambda(t)|\xi| \geq N\nu(t)\}$. First we use the transformation

$$w(t,\xi) := \exp\left(i\xi \int_0^t \lambda(s)b(s)ds\right) v(t,\xi).$$

Then the equation from (5.3) is transformed to

$$D_t^2 w - \lambda^2(t)(b^2(t) + a^2(t))\xi^2 w - D_t(\lambda(t)b(t))\xi w = 0.$$

Setting $c^2(t) := b^2(t) + a^2(t)$ we define the micro-energy

$$W(t,\xi) := (\lambda(t)c(t)\xi w, D_t w)^T.$$

Then we obtain

$$D_t W = \begin{pmatrix} 0 & \lambda(t)c(t)\xi \\ \lambda(t)c(t)\xi & 0 \end{pmatrix} W + \begin{pmatrix} \frac{D_t(\lambda(t)c(t))}{\lambda(t)c(t)} & 0 \\ \frac{D_t(\lambda(t)b(t))}{\lambda(t)c(t)} & 0 \end{pmatrix} W.$$

We apply two steps of the diagonalization procedure which was introduced in the proof to Lemma 2.7. After the first step of diagonalization we have

$$\begin{aligned} D_t W_1 = {} & \lambda(t)c(t)\xi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} W_1 + \frac{D_t(\lambda(t)c(t))}{2\lambda(t)c(t)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} W_1 \\ & + \frac{1}{2\lambda(t)c(t)} \begin{pmatrix} D_t(\lambda(t)b(t)) & D_t(\lambda(t)(c(t)+b(t))) \\ D_t(\lambda(t)(c(t)-b(t))) & -D_t(\lambda(t)b(t)) \end{pmatrix} W_1. \end{aligned}$$

Introducing $\tau_\pm(t,\xi) = \pm\lambda(t)c(t)\xi + \frac{D_t(\lambda(t)c(t))}{2\lambda(t)c(t)}$ the last system can be written in the following form: $D_t W_1 - D_0(t,\xi)W_1 - B(t)W_1 = 0$ with the matrices

$$\begin{aligned} D_0(t,\xi) &:= \begin{pmatrix} \tau_+(t,\xi) & 0 \\ 0 & \tau_-(t,\xi) \end{pmatrix}, \\ B(t) &:= \frac{1}{2\lambda(t)c(t)} \begin{pmatrix} D_t(\lambda(t)b(t)) & D_t(\lambda(t)(c(t)+b(t))) \\ D_t(\lambda(t)(c(t)-b(t))) & -D_t(\lambda(t)b(t)) \end{pmatrix}. \end{aligned}$$

After the second step of diagonalization procedure we obtain the following system

$$D_t W_2 - D_0(t,\xi)W_2 - \Phi(t)W_2 - B_2(t,\xi)W_2 = 0,$$

where

$$\begin{aligned} \Phi(t) &= \frac{1}{2\lambda(t)c(t)} \begin{pmatrix} D_t(\lambda(t)b(t)) & 0 \\ 0 & -D_t(\lambda(t)b(t)) \end{pmatrix}, \\ \|B_2(t,\xi)\| &\leq \frac{C}{\lambda(t)|\xi|} \left(\frac{\lambda(t)}{\Lambda(t)}\nu(t)\right)^2 \end{aligned}$$

for all $(t,\xi) \in Z_{\text{hyp}}(N)$. Finally, we carry out an elliptic transformation by the aid of

$$M_3(t,\xi) = \exp\left(\int_{t_\xi}^t \frac{c'(s)}{c(s)}ds\right) \begin{pmatrix} \exp\left(\int_{t_\xi}^t i\tau_+(s,\xi)ds\right) & 0 \\ 0 & \exp\left(\int_{t_\xi}^t i\tau_-(s,\xi)ds\right) \end{pmatrix}.$$

After setting $W_3 := M_3 W_2$ we conclude

$$\partial_t W_3 - \frac{\lambda'(t)}{2\lambda(t)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} W_3 - \frac{1}{2c(t)\lambda(t)} \begin{pmatrix} \partial_t(\lambda(t)b(t)) & 0 \\ 0 & -\partial_t(\lambda(t)b(t)) \end{pmatrix} W_3 - B_3 W_3 = 0,$$

where

$$\|B_3(t,\xi)\| \le C \frac{1}{\lambda(t)|\xi|} \Big(\frac{\lambda(t)}{\Lambda(t)} \nu(t) \Big)^2.$$

Step 3: *Ljapunov via energy function*

We define the Ljapunov function $S = S(t,\xi)$ and the energy function $E = E(t,\xi)$ by

$$S(t,\xi) := -\lambda(t)^2 |y_1(t,\xi)|^2 + \lambda(t)^2 |y_2(t,\xi)|^2,$$
$$E(t,\xi) := \lambda(t)^2 |y_1(t,\xi)|^2 + \lambda(t)^2 |y_2(t,\xi)|^2,$$

where $Y = (y_1, y_2)^T$. The new vector-function Y arises from the transformation

$$Y(t,\xi) := \begin{pmatrix} \Theta(t,\xi) & 0 \\ 0 & \Theta(t,\xi)^{-1} \end{pmatrix} W_3(t,\xi), \quad \Theta(t,\xi) := \exp\Big(\int_{t_\xi}^t \theta(s,\xi) ds \Big).$$

If we assume

(C9) $|\Theta(t,\xi)| \le C, \ |\Theta(t,\xi)^{-1}| \le C$ for all $(t,\xi) \in Z_{\text{hyp}}(N)$,

then we conclude with this auxiliary function

$$\partial_t Y - \frac{\lambda'(t)}{2\lambda(t)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} Y - \begin{pmatrix} \frac{\partial_t(\lambda(t)b(t))}{2c(t)\lambda(t)} + \theta & 0 \\ 0 & -\frac{\partial_t(\lambda(t)b(t))}{2c(t)\lambda(t)} - \theta \end{pmatrix} Y$$
$$-QY = 0, \qquad \|Q(t,\xi)\| \le C \frac{1}{\lambda(t)|\xi|} \Big(\frac{\lambda(t)}{\Lambda(t)} \nu(t) \Big)^2.$$

Now we estimate the Ljapunov function via the energy function. It holds

$$\begin{aligned} \partial_t S = {} & \frac{2\lambda'(t)}{\lambda(t)} S \\ & - 2\lambda(t)^2 \Re\Big(y_1, \frac{\lambda'(t)}{2\lambda(t)} \overline{y_1} + \Big(\frac{\partial_t(\lambda(t)b(t))}{2c(t)\lambda(t)} + \theta \Big) \overline{y_1} + \overline{Q_{11}}\,\overline{y_1} + \overline{Q_{12}}\,\overline{y_2} \Big) \\ & + 2\lambda(t)^2 \Re\Big(y_2, \frac{\lambda'(t)}{2\lambda(t)} \overline{y_2} - \Big(\frac{\partial_t(\lambda(t)b(t))}{2c(t)\lambda(t)} + \theta \Big) \overline{y_2} + \overline{Q_{21}}\,\overline{y_1} + \overline{Q_{22}}\,\overline{y_2} \Big) \\ \ge {} & 3 \frac{\lambda'(t)}{\lambda(t)} S - \Big(\frac{\partial_t(\lambda(t)b(t))}{c(t)\lambda(t)} + \theta + \|Q\| \Big) E. \end{aligned}$$

If we assume for the function

(C10) $\zeta(t,\xi) := \frac{\partial_t(\lambda(t)b(t))}{c(t)\lambda(t)} + \theta(t,\xi) + \|Q(t,\xi)\| \leq 0$,

then the last inequality implies immediately

$$\partial_t S \geq 3\frac{\lambda'(t)}{\lambda(t)}S - \Big(\frac{\partial_t(\lambda(t)b(t))}{c(t)\lambda(t)} + \theta + \|Q\|\Big)S.$$

As a consequence we may conclude

$$S(t,\xi) \geq S(t_\xi,\xi)\exp\Big(\int_{t_\xi}^t \Big(3\frac{\lambda'(s)}{\lambda(s)} - \frac{\partial_s(\lambda(s)b(s))}{c(s)\lambda(s)} - \theta - \|Q\|\Big)ds\Big).$$

This allows to estimate the Ljapunov function in $Z_{\text{hyp}}(N)$.

Step 4: *Choice of coefficients*

We define the sequence $\{t_j\}_j$ with $t_j = \exp\big(j\frac{1}{\log^{[n]} j}\big)$. It is clear that $t_j \to \infty$ for $j \to \infty$. If we define the sequence $\{d_j\}_j$ with $d_j := \frac{t_j - t_{j-1}}{4}$, then $d_j \sim t_j\frac{1}{\log^{[n+1]} t_j}$. We define the coefficients $a = a(t)$ and $b = b(t)$ from $C^\infty[1,\infty)$ in the following way:

$$a(t) := \int_{t_{j-1}}^t \int_{t_{j-1}}^{s_1} \chi_j(s_2)ds_2 ds_1 + 1 \qquad \text{for} \quad t \in [t_{j-1},t_j],$$

$$b(t) := \begin{cases} a(t-d_j) & \text{for } t \in [t_{j-1}+d_j, t_j], \\ \text{monotone decreasing for} & t \in [t_{j-1}, t_{j-1}+d_j]. \end{cases}$$

They have the following behavior:

$$a(t) := \begin{cases} 1 & \text{on} & [t_{j-1}, t_{j-1}+d_j], \\ \text{monotone increasing} & \text{on} & [t_{j-1}+d_j, t_{j-1}+2d_j], \\ 2 & \text{on} & [t_{j-1}+2d_j, t_{j-1}+3d_j], \\ \text{monotone decreasing} & \text{on} & [t_{j-1}+3d_j, t_j], \end{cases}$$

$$b(t) := \begin{cases} \text{monotone decreasing} & \text{on} & [t_{j-1}, t_{j-1}+d_j], \\ 1 & \text{on} & [t_{j-1}+d_j, t_{j-1}+2d_j], \\ \text{monotone increasing} & \text{on} & [t_{j-1}+2d_j, t_{j-1}+3d_j], \\ 2 & \text{on} & [t_{j-1}+3d_j, t_j]. \end{cases}$$

The functions $\chi_j = \chi_j(t)$ are defined as follows:

$$\chi_j(t) := \begin{cases} 0 & \text{for} & t \in [t_{j-1}, t_{j-1}+d_j], \\ 32d_j^{-3}(t-(t_{j-1}+d_j)) & \text{for} & t \in [t_{j-1}+d_j, t_{j-1}+\frac{5}{4}d_j], \\ -32d_j^{-3}(t-(t_{j-1}+\frac{3}{2}d_j)) & \text{for} & t \in [t_{j-1}+\frac{5}{4}d_j, t_{j-1}+\frac{7}{4}d_j], \\ 32d_j^{-3}(t-(t_{j-1}+2d_j)) & \text{for} & t \in [t_{j-1}+\frac{7}{4}d_j, t_{j-1}+2d_j], \\ 0 & \text{for} & t \in [t_{j-1}+2d_j, t_{j-1}+3d_j], \\ -32d_j^{-3}(t-(t_{j-1}+3d_j)) & \text{for} & t \in [t_{j-1}+3d_j, t_{j-1}+\frac{13}{4}d_j], \\ 32d_j^{-3}(t-(t_{j-1}+\frac{7}{2}d_j)) & \text{for} & t \in [t_{j-1}+\frac{13}{4}d_j, t_{j-1}+\frac{15}{4}d_j], \\ -32d_j^{-3}(t-(t_{j-1}+4d_j)) & \text{for} & t \in [t_{j-1}+\frac{15}{4}d_j, t_j]. \end{cases}$$

Then straightforward calculations imply

$$\max_{t\in[t_{j-1},t_j]} |a'(t)|^2 = \max_{t\in[t_{j-1},t_j]} |b'(t)|^2 \lesssim d_j^{-2},$$
$$\max_{t\in[t_{j-1},t_j]} |a''(t)| = \max_{t\in[t_{j-1},t_j]} |b''(t)| \lesssim d_j^{-2}.$$

There exist positive constants q_0 and q_1 independent of j such that

$$\int_{t_{j-1}}^{t_{j-1}+d_j} \frac{b'(s)}{c(s)}ds = \int_{t_{j-1}}^{t_{j-1}+d_j} \frac{b'(s)}{\sqrt{1+b(s)^2}}ds = -\log\Big(\frac{2+\sqrt{5}}{1+\sqrt{2}}\Big) =: -q_1,$$
$$\int_{t_{j-1}+2d_j}^{t_{j-1}+3d_j} \frac{b'(s)}{c(s)}ds = \int_{t_{j-1}+2d_j}^{t_{j-1}+3d_j} \frac{b'(s)}{\sqrt{4+b(s)^2}}ds = \log\Big(\frac{2+2\sqrt{2}}{1+\sqrt{5}}\Big) =: q_0$$

with $q_0 < q_1$.

Step 5: *Definition of* θ *and assumptions* (C9) *and* (C10).

We can find a positive real number δ_0 such that $p_0 := \frac{\delta_0+q_0}{q_1} \in (0,1)$. We define with a large j_0 a sequence $\{p_j(\xi)\}_{j\geq j_0}$ by

$$p_j(\xi) := \frac{K}{q_1\lambda(t_j)|\xi|}\int_{t_{j-1}}^{t_j}\Big(\frac{1}{s}\log^{[n+1]} s\Big)^2 ds + \frac{1}{q_1}\int_{t_{j-1}}^{t_j}\frac{b(s)}{c(s)}\frac{\lambda'(s)}{\lambda(s)}ds + \frac{q_0}{q_1}.$$

On the one hand we have with the definition of t_j

$$\int_{t_{j-1}}^{t_j}\frac{b(s)}{c(s)}\frac{\lambda'(s)}{\lambda(s)}ds \leq \int_{t_{j-1}}^{t_j}\frac{b_1}{c_0}\,\frac{\lambda'(s)}{\lambda(s)}ds \sim (\log t_j - \log t_{j-1})$$
$$\leq \frac{1}{4}\delta_0 \quad \text{for} \quad j \geq j_0(\delta_0).$$

On the other hand by taking into account of the definition of $Z_{\text{hyp}}(N)$ it holds

$$\frac{K}{\lambda(t_j)|\xi|}\int_{t_{j-1}}^{t_j}\Big(\frac{1}{s}\log^{[n+1]} s\Big)^2 ds \leq \frac{K}{\lambda(t_j)|\xi|}(t_j-t_{j-1})\Big(\frac{1}{t_{j-1}}\log^{[n+1]} t_{j-1}\Big)^2$$
$$\leq \frac{KC}{\lambda(t_j)|\xi|}(t_j-t_{j-1})\Big(\frac{1}{t_j}\log^{[n+1]} t_j\Big)^2 \leq \frac{KCd_j}{\lambda(t_j)|\xi|}\Big(\frac{1}{t_j}\log^{[n+1]} t_j\Big)^2$$
$$\leq \frac{KC}{\lambda(t_j)|\xi|}t_j(\log^{[n+1]} t_j)^{-1}\Big(\frac{1}{t_j}\log^{[n+1]} t_j\Big)^2 \leq \frac{KC}{|\xi|}\frac{1}{\Lambda(t_j)}\log^{[n+1]}\Lambda(t_j)$$
$$\leq \frac{KC}{N} \leq \frac{1}{2}\delta_0 \quad \text{for} \quad j \geq j_0(\delta_0).$$

Hence, $|p_j(\xi)| \leq p_0$ for all large $j \geq j_0$. We introduce the notation $q(t) := \frac{b'(t)}{c(t)}$ and as usual the non-negative part $[q(t)]_+$ and the non-positive part $[q(t)]_-$ of $q(t)$. We have $\int_{t_{j-1}}^{t_j} q(s)ds = q_0 - q_1 < 0$. Let us define the function $\theta(t,\xi)$ in the following way:

$$\theta(t,\xi) := -([q(t)]_+ + p_j(\xi)[q(t)]_-) - \frac{K}{\lambda(t_j)|\xi|}\Big(\frac{1}{t}\log^{(n+1)} t\Big)^2 - \frac{b(t)}{c(t)}\frac{\lambda'(t)}{\lambda(t)}$$

for $t \in [t_{j-1}, t_j]$.

Then we may use

$$\int_{t_{j-1}}^{t_j} \theta(t,\xi)dt = 0, \qquad \Big|\int_{t_{j-1}}^{t} \theta(s,\xi)ds\Big| \le \delta_0 \quad \text{for} \quad t \in [t_{j-1},t_j).$$

Therefore we obtain $\exp(-2\delta_0) \le |\Theta(t,\xi)| \le \exp(2\delta_0)$. Thus, the assumption (C9) is satisfied. Finally, we may conclude with $\lambda(t) \sim \lambda(t_j)$ on the interval $[t_{j-1},t_j]$ and with K sufficiently large

$$\begin{aligned}\zeta(t,\xi) &\le (1-p_j(\xi))[q(t)]_-) - \frac{K-C}{\lambda(t)|\xi|}\Big(\frac{1}{t}\log^{[n+1]} t\Big)^2 \\ &\le (1-p_0)[q(t)]_- \le 0.\end{aligned}$$

This gives (C10).

Step 6: *Verification*

For large $t \in [t_{j-1},t_j]$ we conclude

$$\int_{t_\xi}^{t} -\zeta(s,\xi)ds \ge q_1(1-p_0)(j-j_0(\xi)).$$

As a consequence we obtain

$$S(t,\xi) \ge S(t_\xi,\xi)\exp\Big(\int_{t_\xi}^{t} 3\frac{\lambda'(s)}{\lambda(s)}ds\Big)\exp(Cq_1(1-p_0)(j-j_0(\xi))).$$

Taking into consideration

$$j \sim (\log t_j)(\log^{[n+1]} t_j), \;\; j_0(\xi) \;\sim (\log t_\xi)(\log^{[n+1]} t_\xi)$$

we conclude

$$S(t_j,\xi) \ge S(t_\xi,\xi)\exp\Big(C(\log t_j)(\log^{[n+1]} t_j) - C(\log t_\xi)(\log^{[n+1]} t_\xi)\Big).$$

We define $S(t_\xi,\xi)$ in such a way that it coincides with $E(t_\xi,\xi)$. Then the last inequality yields

$$E(t_j,\xi) \ge E(t_\xi,\xi)\exp\Big(C(\log t_j)(\log^{[n+1]} t_j) - C(\log t_\xi)(\log^{[n+1]} t_\xi)\Big).$$

After backward transformation we get

$$|V(t_j,\xi)| \ge |V(t_\xi,\xi)|\exp\Big(C(\log t_j)(\log^{[n+1]} t_j) - C(\log t_\xi)(\log^{[n+1]} t_\xi)\Big).$$

Then we solve the backward Cauchy problem in $Z_{\text{pd}}(N)$ and obtain from (5.4)

$$|V(0,\xi)| \le C\exp\Big(N(\log^{[n+1]} t_\xi)\Big)|V(t_\xi,\xi)|.$$

Summarizing we have proved

$$|E(t_j,0,\xi)| \ge C_0\exp\Big(C_1(\log t_j)(\log^{[n+1]} t_j)\Big) \text{ for all } j \ge j_0(\xi).$$

In this estimate it is allowed to choose $j \to \infty$. So, the desired estimate of Theorem 5.2 is proved.

5.2. Proof of Theorem 5.4

We define the sequence $\{t_j\}_j$ implicitly by

$$\Lambda(t_j) = \exp(j^{\frac{1}{\gamma+1}}).$$

We have

$$t_j = \log^{[m-1]} j^{\frac{1}{\gamma+1}}, \;\; d_j = \frac{1}{4}(t_j - t_{j-1}).$$

Consequently,

$$\begin{aligned} \frac{1}{d_j} &\sim (\log^{[m-2]} \tilde{j}^{\frac{1}{\gamma+1}})(\log^{[m-3]} \tilde{j}^{\frac{1}{\gamma+1}}) \cdots (\log \tilde{j}^{\frac{1}{\gamma+1}})\tilde{j} \\ &\sim (\log^{[m-2]} j^{\frac{1}{\gamma+1}})(\log^{[m-3]} j^{\frac{1}{\gamma+1}}) \cdots (\log j^{\frac{1}{\gamma+1}})j \end{aligned}$$

for large j and $\tilde{j} \in (j-1, j)$. On the other hand

$$\begin{aligned} \frac{\lambda(t_j)}{\Lambda(t_j)}(\log \Lambda(t_j))^\gamma &= (\exp^{[m-1]} t_j)^{\gamma+1} \exp^{[m-2]} t_j \cdots \exp t_j \\ &\sim j(\log j^{\frac{1}{\gamma+1}}) \cdots (\log^{[m-3]} j^{\frac{1}{\gamma+1}})(\log^{[m-2]} j^{\frac{1}{\gamma+1}}). \end{aligned}$$

This shows, that the definition of $\{t_j\}_j$ and the construction of $a = a(t)$ and $b = b(t)$ as in Step 4 of the previous proof give the desired oscillating behavior. So we can follow all the steps as in the proof to Theorem 5.2. As a consequence we conclude

$$S(t_j, \xi) \geq S(t_\xi, \xi) \left(\frac{\lambda(t_j)}{\lambda(t_\xi)}\right)^3 \exp\left(C(\log \Lambda(t_j))^{\gamma+1} - C(\log \Lambda(t_\xi))^{\gamma+1}\right)$$

for all large t_j. Taking into consideration $\lambda(t_j) = o(\exp(C(\log \Lambda(t_j))^{\gamma+1}))$ the same approach as in Case 1 implies the desired estimate to below from Theorem 5.4.

Remark 5.5. To produce slower oscillations we define $\{t_j\}_j$ in the following way (c.f. with Case 1):

$$\Lambda(t_j) = \exp\left(j \frac{1}{\log^{[n+1]} j}\right) \qquad \text{for large } \; j.$$

In this way we may understand the interaction of oscillations for shape functions increasing not slower than exponential growth and with oscillations which are very close to very slow oscillations.

6. Concluding remark

The recent papers [1] and [13] are devoted to energy estimates (even to $L^p - L^q$ decay estimates) for solutions to strictly hyperbolic systems. To understand the new effects coming from systems itself it would be interesting to generalize the results from [10] to the long time behavior of energies to 2 by 2 hyperbolic systems by using the results of this paper.

References

[1] M. D'Abbicco, S. Lucente, G. Taglialatela, *$L^p - L^q$ estimates for regularly hyperbolic systems*, Advances Differential Equations **14** (2009) 9-10, 801–834.

[2] F. Colombini, *Energy estimates at infinity for hyperbolic equations with oscillating coefficients*, J. Differential Equations **231** (2006) 598–610.

[3] F. Colombini, E. De Giorgi, S. Spagnolo, *Sur les équations hyperboliques avec des coefficients qui ne dépendent que du temps*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **6** (1979), 511–559.

[4] M.S.P. Eastham, *The spectral theory of periodic differential equations*, Scottish Academic Press, Edinburgh, 1973.

[5] R. Helbig, *Optimalitätsaussagen für Oszillationen in Wellenmodellen*, Diploma thesis, Freiberg, 2009.

[6] F. Hirosawa, *On the asymptotic behavior of the energy for the wave equations with time depending coefficients*, Math. Ann. **339** (2007) 819–839.

[7] F. Hirosawa, *Energy estimates for wave equations with time dependent propagation speeds in the Gevrey class*, J. Differential Equations **248** (2010) 2972–2993.

[8] F. Hirosawa and M. Reissig, *About the optimality of oscillations in non-Lipschitz coefficients for strictly hyperbolic equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci.(5) **3** (2004) 589–608.

[9] F. Hirosawa and J. Wirth, *Generalised energy conservation law for wave equations with variable propagation speed*, J. Math. Anal. Appl. **358** (2009) 1, 56–74.

[10] T. Kinoshita and M. Reissig, *The Log-effect for* 2 *by* 2 *hyperbolic systems*, J. Differential Equations **248** (2010) 470–500.

[11] W. Magnus und St. Winkler, *Hill's equation*, Interscience Publishers, New York-London-Sydney, 1966.

[12] M. Reissig and J. Smith, *$L^p - L^q$ estimate for wave equation with bounded time-dependent coefficient*, Hokkaido Mathematical Journal **34** (2005) 541–586.

[13] M. Ruzhansky and J. Wirth, *Dispersive estimates for T-dependent hyperbolic systems*, Rend. Sem. Mat. Univ. Pol. Torino **66** (2008) 4, 339–349.

Michael Reissig
Institute of Applied Analysis
TU Bergakademie Freiberg
D-09596 Freiberg, Germany
e-mail: reissig@math.tu-freiberg.de

Operator Theory:
Advances and Applications, Vol. 216, 317–330

On Singular Systems of Parabolic Functional Equations

László Simon

Abstract. We consider systems consisting of an initial-boundary value problem for second-order quasilinear parabolic equation and an initial value problem for first-order ordinary differential equation where both equations contain functional dependence on the unknown functions.

Mathematics Subject Classification (2000). Primary 35R10.

Keywords. Parabolic functional equation, singular system, monotone type operator.

1. Introduction

In this paper we shall consider initial-boundary value problems for the system

$$D_t u - \sum_{i=1}^{n} D_i[a_i(t,x,u(t,x),Du(t,x)+g(w(t,x))Dw(t,x);u,w)] \\ + a_0(t,x,u(t,x),Du(t,x)+g(w(t,x)Dw(t,x));u,w) = G, \tag{1.1}$$

$$D_t w = F(t,x;u,w) \text{ in } Q_T = (0,T)\times\Omega \subset \mathbb{R}^{n+1}, \quad T\in(0,\infty) \tag{1.2}$$

where $D_t = \frac{\partial}{\partial t}$, $D_i = \frac{\partial}{\partial x_i}$, $D = (D_1, D_2, \dots, D_n)$, the functions

$$a_i : Q_T \times \mathbb{R}^{n+1} \times L^{p_1}(0,T;V_1) \times L^{p_2}(Q_T) \to \mathbb{R}$$

(with a closed linear subspace V_1 of the Sobolev space $W^{1,p_1}(\Omega)$, $2 \le p_i < \infty$) satisfy conditions which are generalizations of the usual conditions for quasilinear parabolic differential equations, considered by using the theory of monotone type operators (see, e.g., [3], [7], [14], [16]) but the equation (1.1) is not uniformly parabolic in the sense, analogous to the linear case (see also [13]). Further,

$$F : Q_T \times L^{p_1}(0,T;V_1) \times L^{p_2}(Q_T) \to \mathbb{R}$$

This work was supported by the Hungarian National Foundation for Scientific Research under grant OTKA T 049819.

satisfies a Lipschitz condition. It will be proved existence of weak solutions in Q_T. In the first part of the paper the case $g = 0$ will be considered and in the second part the general case will be considered.

Such problems with $g = 0$ arise, e.g., when considering diffusion and transport in porous media with variable porosity, see [4], [8]. In [8] J.D. Logan, M.R. Petersen, T.S. Shores considered and numerically studied a nonlinear system, consisting of a parabolic, an elliptic and an ODE which describes reaction-mineralogy-porosity changes in porous media. System (1.1), (1.2) with $g = 0$ is the particular case when the pressure is assumed to be constant. This case was studied in [12] when a_i satisfy modified (in some sense more special) conditions, by using different arguments.

The case of general g was motivated by non-Fickian diffusion in viscoelastic polymers and by spread of morphogens (see [9], [10], [11]). In [2], [5] similar degenerate systems of parabolic differential equations were considered without functional dependence. This general case is studied also in [15], by using different method, if certain modified conditions are satisfied. The modified conditions are in some sense weaker and in some sense stronger than the assumptions in the present paper. Here Schauder's fixed point theorem is applied while in [15] the existence theorem on pseudomonotone operators is directly applied.

2. Case $g = 0$

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain having the uniform C^1 regularity property (see [1]) and $p_i \geq 2$ be real numbers $(i = 1, 2)$. Denote by $W^{1,p_1}(\Omega)$ the usual Sobolev space of real-valued functions with the norm

$$\|u\| = \left[\int_\Omega (|Du|^{p_1} + |u|^{p_1}) \right]^{1/p_1}.$$

Let $V_1 \subset W^{1,p_1}(\Omega)$ be a closed linear subspace containing $C_0^\infty(\Omega)$. Denote by $L^{p_1}(0,T;V_1)$ the Banach space of the set of measurable functions $u : (0,T) \to V_1$ such that $\|u\|_{V_1}^p$ is integrable and define the norm by

$$\|u\|_{L^{p_1}(0,T;V_1)}^{p_1} = \int_0^T \|u(t)\|_{V_1}^{p_1} dt.$$

For the sake of brevity we denote $L^{p_1}(0,T;V_1)$ by X_1^T. The dual space of X_1^T is $L^{q_1}(0,T;V_1^\star)$ where $1/p_1 + 1/q_1 = 1$ and $V_1^\star$ is the dual space of V_1 (see, e.g., [7], [14], [16]). Further, let $X^T = X_1^T \times L^{p_2}(Q_T)$.

On functions a_i we assume:

(A$_1$) The functions $a_i : Q_T \times \mathbb{R}^{n+1} \times X^T \to \mathbb{R}$ satisfy the Carathéodory conditions for arbitrary fixed $(u, w) \in X^T$ $(i = 0, 1, \dots, n)$.

(A$_2$) There exist $0 < \delta \leq 1$ and bounded (nonlinear) operators

$$g_1 : L^{p_1}(0,T;W^{1-\delta,p_1}(\Omega)) \times L^{p_2}(Q_T) \to \mathbb{R}^+,$$
$$k_1 : L^{p_1}(0,T;W^{1-\delta,p_1}(\Omega)) \times L^{p_2}(Q_T) \to L^{q_1}(Q_T)$$

such that k_1 is continuous,

$$|a_i(t,x,\zeta_0,\zeta;u,w)| \le g_1(u,w)[|\zeta_0|^{p_1-1} + |\zeta|^{p_1-1}] + [k_1(u,w)](t,x), \quad i=0,1,\dots,n$$

for a.e. $(t,x) \in Q_T$, each $(\zeta_0,\zeta) \in \mathbb{R}^{n+1}$ and $(u,w) \in X_\delta^T$ where we use the notations

$$X_{1,\delta}^T = L^{p_1}(0,T;W^{1-\delta,p_1}(\Omega)), \quad X_\delta^T = X_{1,\delta}^T \times L^{p_2}(Q_T).$$

(A$_3$)
$$\sum_{i=0}^n [a_i(t,x,\zeta_0,\zeta;u,w) - a_i(t,x,\zeta_0^\star,\zeta^\star;u,w)](\zeta_i - \zeta_i^\star) \ge g_2(u,w)[|\zeta_0-\zeta_0^\star|^{p_1} + |\zeta-\zeta^\star|^{p_1}], \quad t \in (0,T]$$

where

$$g_2(u,w) \ge \frac{c_2}{1+\|(u,w)\|_{X_\delta^T}^{\sigma^\star}} \tag{2.1}$$

with some constants $c_2 > 0$, $0 \le \sigma^\star < p_1 - 1$.

(A$_4$) There exists a nonlinear operator $k_2 : X_\delta^T \to L^1(Q_T)$ such that

$$\sum_{i=0}^n a_i(t,x,\zeta_0,\zeta;u,w)\zeta_i \ge g_2(u,w)[|\zeta_0|^{p_1} + |\zeta|^{p_1}] - [k_2(u,w)](t,x)$$

for a.e. $(t,x) \in Q_T$, all $(\zeta_0,\zeta) \in \mathbb{R}^{n+1}$, $(u,w) \in X_\delta^T$ and

$$\|k_2(u,w)\|_{L^1(Q_T)} \le c_3 \left(\|(u,w)\|_{X_\delta^T}^{\sigma} + 1\right) \tag{2.2}$$

with some nonnegative constant $\sigma < p_1 - \sigma^\star$.

(A$_5$) If $(u_k,w_k) \to (u,w)$ in X_δ^T then for $i=0,1,\dots,n$, a.e. $(t,x) \in Q_T$, all (ζ_0,ζ) in $\mathbb{R}^{n+1}$

$$a_i(t,x,\zeta_0,\zeta;u_k,w_k) \to a_i(t,x,\zeta_0,\zeta;u,w),$$

for a subsequence.

Example. Conditions (A$_1$)–(A$_5$) are satisfied if, e.g.,

$$a_i(t,x,\zeta_0,\zeta;u,w) = b(H_1(u),H_2(w))\zeta_i|\zeta|^{p_1-2}, \quad i=1,2,\dots,n,$$
$$a_0(t,x,\zeta_0,\zeta;u,w) = b(H_1(u),H_2(w))\zeta_0|\zeta_0|^{p_1-2} + b_0(F_0(u),G_0(w))$$

and b, b_0 are continuous functions, satisfying with some positive constants c_3, c_4

$$b(\theta_1,\theta_2) \ge \frac{c_3}{1+|(\theta_1,\theta_2)|^{\sigma^\star}} \qquad \text{where } 0 \le \sigma^\star < p_1 - 1,$$
$$|b_0(\theta_1,\theta_2)| \le c_4|(\theta_1,\theta_2)|^\lambda \qquad \text{where } 0 \le \lambda < p_1 - 1 - \sigma^\star.$$

Finally,

$$H_1 : L^{p_1}(0,T;W^{1-\delta,p_1}(\Omega)) \to C(\overline{Q_T}), \quad F_0 : L^{p_1}(0,T;W^{1-\delta,p_1}(\Omega)) \to L^{p_1}(Q_T),$$
$$H_2 : L^{p_2}(Q_T) \to C(\overline{Q_T}), \qquad G_0 : L^{p_2}(Q_T) \to L^{p_1}(Q_T)$$

are continuous linear operators. (See [13].) If $\sigma^\star = 0$ and b is bounded, H_1, H_2 may be such as F_0, G_0, respectively.

Now we formulate assumptions on $F : Q_T \times X^T \to \mathbb{R}$.

(F_1) For each fixed $(u, w) \in X^T$, $F(\cdot; u, w) \in L^{p_2}(Q_T)$.

(F_2) F satisfies the following (global) Lipschitz condition: there exists a constant K such that for each $t \in (0, T]$, $(u, w), (u^\star, w^\star) \in X^T$ we have

$$\|F(\cdot; u, w) - F(\cdot; u^\star, w^\star)\|^{p_2}_{L^{p_2}(Q_t)} \le K\left[\|u - u^\star\|^{p_2}_{X^t_{1,\delta}} + \|w - w^\star\|^{p_2}_{L^{p_2}(Q_t)}\right].$$

Definition. We define operator $A : X^T \to (X_1^T)^\star$ by

$$\begin{aligned}[A(u, w), v] &= \int_0^T \langle A(u, w)(t), v(t)\rangle dt \\ &= \int_{Q_T} \Big\{\sum_{i=1}^n a_i(t, x, u(t, x), Du(t, x); u, w) D_i v \\ &\qquad + a_0(t, x, u(t, x), Du(t, x); u, w) v\Big\} dtdx,\end{aligned}$$

$(u, w) \in X^T$, $v \in X_1^T$ where the brackets $\langle\cdot,\cdot\rangle$, $[\cdot,\cdot]$ mean the dualities in spaces $V_1^\star, V_1$; $(X_1^T)^\star, X_1^T$, respectively.

Theorem 2.1. *Assume* (A_1)–(A_5) *and* (F_1), (F_2). *Then for any* $G \in (X_1^T)^\star$, $u_0 \in L^2(\Omega)$ *and* $w_0 \in L^{p_2}(\Omega)$ *there exists* $u \in X_1^T$, $w \in L^{p_2}(Q_T)$ *such that* $D_t u \in (X_1^T)^\star$, $D_t w \in L^{p_2}(Q_T)$,

$$D_t u + A(u, w) = G, \quad u(0) = u_0, \tag{2.3}$$

$$D_t w = F(t, x; u, w) \text{ for a.e. } (t, x) \in Q_T, \quad w(0) = w_0. \tag{2.4}$$

Before the proof of this theorem we prove two lemmas. Define (with fixed $(u, w) \in X^T$) the operator $A_{u,w} : X_1^T \to (X_1^T)^\star$ by

$$\begin{aligned}[A_{u,w}(\tilde{u}), v] = \int_{Q_T} \Big\{&\sum_{i=1}^n a_i(t, x, \tilde{u}(t, x), D\tilde{u}(t, x); u, w) D_i v \\ &+ a_0(t, x, \tilde{u}(t, x), D\tilde{u}(t, x); u, w) v\Big\} dtdx,\end{aligned}$$

$\tilde{u}, v \in X_1^T$.

Lemma 2.2. *Assume* (A_1)–(A_5). *Then for arbitrary* $(u, w) \in X_\delta^T$, $G \in (X_1^T)^\star$, $u_0 \in L^2(\Omega)$ *there exists a unique solution* $\tilde{u} \in X_1^T$ *of*

$$D_t \tilde{u} + A_{u,w}(\tilde{u}) = G, \quad \tilde{u}(0) = u_0. \tag{2.5}$$

If (u_k, w_k) *is a bounded sequence in* X_δ^T *then for the sequence* $(\tilde{u}_k)$ *of solutions of* (2.5) *with* $(u, w) = (u_k, w_k)$ *we have:* $(\tilde{u}_k)$ *is bounded in* X_1^T, $(D_t \tilde{u}_k)$ *is bounded in* $(X_1^T)^\star$. *Further, if* $(u_k, w_k) \to (u, w)$ *in* X_δ^T *then* $(\tilde{u}_k) \to \tilde{u}$ *in* X_1^T.

Proof. By (A_1)–(A_4) operator $A_{u,w} : X_1^T \to (X_1^T)^\star$ is bounded, demicontinuous, strictly monotone and coercive (see [3], [7], [14] or [16]), thus for any fixed $(u,w) \in X_\delta^T$ there exists a unique solution $\tilde{u} \in X_1^T$ of (2.5).

Further, if (u_k, w_k) is a bounded sequence in X_δ^T the by (A_4) we have for the solution $\tilde{u}_k$ of (2.5) (by $p_1 \geq 2$, $W^{1,p_1}(\Omega)$ is continuously imbedded into $L^2(\Omega)$):

$$\begin{aligned}[G, \tilde{u}_k] = & \int_0^T \langle (D_t \tilde{u}_k)(t), \tilde{u}_k(t) \langle dt + [A_{u_k,w_k}(\tilde{u}_k), \tilde{u}_k] \\ & \geq \frac{1}{2} \|\tilde{u}_k(T)\|^2_{L^2(\Omega)} - \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} \\ & \quad + \frac{c_2}{1 + \|(u_k, w_k)\|^{\sigma^\star}_{X_\delta^T}} \|\tilde{u}_k\|^{p_1}_{X_1^T} - c_3 \left(\|(u_k, w_k)\|^\sigma_{X_\delta^T} + 1 \right).\end{aligned} \tag{2.6}$$

Since

$$|[G, \tilde{u}_k] \leq \|G\|_{(X_1^T)^\star} \|\tilde{u}_k\|_{X_1^T},$$

$p_1 \geq 2$, we obtain from (2.6) the boundedness of $(\tilde{u}_k)$ in X_1^T if (u_k, w_k) is bounded in X_δ^T. Thus by (A_2), (2.5), $(D_t \tilde{u}_k)$ is bounded in $(X_1^T)^\star$.

Finally, if $(u_k, w_k) \to (u, w)$ in X_δ^T then by (2.5) one obtains

$$D_t \tilde{u}_k + A_{u_k,w_k}(\tilde{u}_k) = D_t \tilde{u} + A_{u,w}(\tilde{u})$$

thus

$$\begin{aligned}&[A_{u_k,w_k}(\tilde{u}_k) - A_{u_k,w_k}(\tilde{u}), \tilde{u}_k - \tilde{u}] \\ &\quad = -[D_t(\tilde{u}_k - \tilde{u}), \tilde{u}_k - \tilde{u}] + [A_{u,w}(\tilde{u}), -A_{u_k,w_k}(\tilde{u}), \tilde{u}_k - \tilde{u}],\end{aligned}$$

hence by (A_3), $[D_t(\tilde{u}_k - \tilde{u}), \tilde{u}_k - \tilde{u}] \geq 0$,

$$\begin{aligned}\|\tilde{u}_k - \tilde{u}\|^{p_1}_{X_1^T} & \leq \frac{1 + \|(u_k, w_k)\|^{\sigma^\star}_{X_\delta^T}}{c_2} [A_{u,w}(\tilde{u}) - A_{u_k,w_k}(\tilde{u}), \tilde{u}_k - \tilde{u}] \\ & \leq \text{const} \|A_{u,w}(\tilde{u}) - A_{u_k,w_k}(\tilde{u})\|_{(X_1^T)^\star} \|\tilde{u}_k - \tilde{u}\|_{X_1^T}\end{aligned} \tag{2.7}$$

and by (A_5), (A_2) and Vitali's theorem

$$\lim_{k \to \infty} \|A_{u,w}(\tilde{u}) - A_{u_k,w_k}(\tilde{u})\|_{(X_1^T)^\star} = 0,$$

for a subsequence. Consequently, it holds for the original sequence, too. (Assuming that it is not true, one gets a contradiction.) Thus inequality (2.7) and $p_1 \geq 2$ imply that $(\tilde{u}_k) \to \tilde{u}$ in X_1^T. □

Lemma 2.3. *Let* (F_1), (F_2) *be satisfied and assume that*

$$(\tilde{u}_k) \to \tilde{u} \text{ in } L^{p_1}(0, T; W^{1-\delta, p_1}(\Omega)).$$

Further, with a given $w_0 \in L^{p_2}(\Omega)$ *define the sequence* $(\tilde{w}_k)$ *by*

$$\begin{aligned}\tilde{w}_k(t,x) = w_0(x) + \int_0^t F(\tau, x, ; \tilde{u}_k, \tilde{w}_{k-1}) d\tau, \\ k = 1, 2, \ldots, \quad w_0(t,x) = w_0(x).\end{aligned} \tag{2.8}$$

Then $(\tilde{w}_k) \to \tilde{w}$ *in* $L^{p_2}(Q_T)$ *and* $\tilde{w}$ *satisfies*

$$\tilde{w}(t,x) = w_0(x) + \int_0^t F(\tau, x, ; \tilde{u}, \tilde{w}) d\tau \ \text{a.e. in } Q_T. \tag{2.9}$$

The solution $\tilde{w}$ *of* (2.9) *is unique. Further, if* $(u_j^\star) \to u^\star$ *in* $L^{p_1}(0,T; W^{1-\delta,p_1}(\Omega))$ *then for the solution* $\tilde{w} = w_j^\star$ *of* (2.9) *with* $\tilde{u} = u_j^\star$ *we have* $(w_j^\star) \to w^\star$ *in* $L^{p_2}(Q_T)$ *where* $w^\star$*is the solution of* (2.9) *with* $\tilde{u} = u^\star$.

Proof. By (2.8) and (F_2)

$$\begin{aligned}|\tilde{w}_{j+1} - \tilde{w}_{k+1}|^{p_2} &= \left| \int_0^t [F(\tau, x; \tilde{u}_{j+1}, \tilde{w}_j) - F(\tau, x; \tilde{u}_{k+1}, \tilde{w}_k)] d\tau \right|^{p_2} \\ &\le t^{\frac{p_2}{q_2}} \int_0^t |F(\tau, x; \tilde{u}_{j+1}, \tilde{w}_j) - F(\tau, x; \tilde{u}_{k+1}, \tilde{w}_k)|^{p_2} d\tau,\end{aligned}$$

hence

$$\begin{aligned}&\int_\Omega |\tilde{w}_{j+1} - \tilde{w}_{k+1}|^{p_2} dx \le t^{\frac{p_2}{q_2}} \int_{Q_t} |F(\tau, x; \tilde{u}_{j+1}, \tilde{w}_j) - F(\tau, x; \tilde{u}_{k+1}, \tilde{w}_k)|^{p_2} d\tau dx \\ &\quad \le K t^{\frac{p_2}{q_2}} \left[\|\tilde{u}_{j+1} - \tilde{u}_{k+1}\|^{p_2}_{X^t_{1,\delta}} + \|\tilde{w}_j - \tilde{w}_k\|^{p_2}_{L^{p_2}(Q_t)} \right], \\ &\int_{Q_{\tilde{t}}} |\tilde{w}_{j+1} - \tilde{w}_{k+1}|^{p_2} dt dx \\ &\quad \le K \left[\int_0^{\tilde{t}} t^{\frac{p_2}{q_2}} \|\tilde{u}_{j+1} - \tilde{u}_{k+1}\|^{p_2}_{X^t_{1,\delta}} dt + \int_0^{\tilde{t}} t^{\frac{p_2}{q_2}} \|\tilde{w}_j - \tilde{w}_k\|^{p_2}_{L^{p_2}(Q_t)} dt \right] \\ &\quad \le K T^{\frac{p_2}{q_2} - 1} \frac{\tilde{t}^2}{2!} \left[\|\tilde{u}_{j+1} - \tilde{u}_{k+1}\|^{p_2}_{X^T_{1,\delta}} + \|\tilde{w}_j - \tilde{w}_k\|^{p_2}_{L^{p_2}(Q_T)} \right].\end{aligned}$$

Further,

$$\begin{aligned}&\int_{Q_{\tilde{t}}} |\tilde{w}_{j+2} - \tilde{w}_{k+2}|^{p_2} dt dx \\ &\quad \le K \left[\int_0^{\tilde{t}} t^{\frac{p_2}{q_2}} \|\tilde{u}_{j+2} - \tilde{u}_{k+2}\|^{p_2}_{X^t_{1,\delta}} dt + \int_0^{\tilde{t}} t^{\frac{p_2}{q_2}} \|\tilde{w}_{j+1} - \tilde{w}_{k+1}\|^{p_2}_{L^{p_2}(Q_t)} dt \right] \\ &\quad \le K T^{\frac{p_2}{q_2} - 1} \frac{\tilde{t}^2}{2!} \|\tilde{u}_{j+2} - \tilde{u}_{k+2}\|^{p_2}_{X^T_{1,\delta}} \\ &\qquad + K T^{\frac{p_2}{q_2}} \int_0^{\tilde{t}} K T^{\frac{p_2}{q_2} - 1} \frac{t^2}{2!} \left[\|\tilde{u}_{j+1} - \tilde{u}_{k+1}\|^{p_2}_{X^T_{1,\delta}} + \|\tilde{w}_j - \tilde{w}_k\|^{p_2}_{L^{p_2}(Q_T)} \right] dt \\ &\quad \le K T^{\frac{p_2}{q_2} - 1} \frac{\tilde{t}^2}{2!} \|\tilde{u}_{j+2} - \tilde{u}_{k+2}\|^{p_2}_{X^T_{1,\delta}} \\ &\qquad + K^2 \left(T^{\frac{p_2}{q_2}} \right)^2 \frac{\tilde{t}^3}{3!} \left[\|\tilde{u}_{j+1} - \tilde{u}_{k+1}\|^{p_2}_{X^T_{1,\delta}} + \|\tilde{w}_j - \tilde{w}_k\|^{p_2}_{L^{p_2}(Q_T)} \right].\end{aligned}$$

By induction we obtain (with some constant $\tilde{c}$)

$$\int_{Q_{\tilde{t}}} |\tilde{w}_{j+m} - \tilde{w}_{k+m}|^{p_2} dtdx \leq \sum_{l=1}^{m} \tilde{c}^{l+1} \frac{\tilde{t}^{l+1}}{(l+1)!} \|\tilde{u}_{j+m+1-l} - \tilde{u}_{k+m+1-l}\|^{p_2}_{X^T_{1,\delta}} + \tilde{c}^{m+1} \frac{\tilde{t}^{m+1}}{(m+1)!} \|\tilde{w}_j - \tilde{w}_k\|^{p_2}_{L^{p_2}(Q_T)}. \tag{2.10}$$

In a similar way, one estimates $\|\tilde{w}_k\|_{L^{p_2}(Q_{\tilde{t}})}$: by (2.8)

$$|\tilde{w}_k(t,x)|^{p_2} \leq p_2 \left[t^{\frac{p_2}{q_2}} \int_0^t |F(\tau, x; \tilde{u}_k, \tilde{w}_{k-1})|^{p_2} d\tau + |w_0(x)|^{p_2} \right]$$

and by (F_1), (F_2)

$$\|F(\cdot; \tilde{u}_k, \tilde{w}_{k-1})\|^{p_2}_{L^{p_2}(Q_t)} \leq p_2 K \left[\|\tilde{u}_k\|^{p_2}_{X^t_{1,\delta}} + \|\tilde{w}_{k-1}\|^{p_2}_{L^{p_2}(Q_t)} \right] + p_2 \|F(\cdot; 0, 0)\|^{p_2}_{L^{p_2}(Q_t)}.$$

Consequently,

$$\int_\Omega |\tilde{w}_k(t,x)|^{p_2} dx \leq p_2^2 K t^{\frac{p_2}{q_2}} \left[\|\tilde{u}_k\|^{p_2}_{X^t_{1,\delta}} + \|\tilde{w}_{k-1}\|^{p_2}_{L^{p_2}(Q_t)} \right] + C^\star$$

where the constant $C^\star$is independent of k, t, x thus for all $\tilde{t} \in (0, T]$ we have

$$\int_{Q_{\tilde{t}}} |\tilde{w}_k|^{p_2} dtdx \leq p_2^2 K T^{\frac{p_2}{q_2} - 1} \left[\int_0^{\tilde{t}} t \|\tilde{u}_k\|^{p_2}_{X^T_{1,\delta}} dt + \int_0^{\tilde{t}} t \|\tilde{w}_{k-1}\|^{p_2}_{L^{p_2}(Q_T)} dt \right] + C^\star \tilde{t}.$$

By using induction, one obtains

$$\int_{Q_{\tilde{t}}} |\tilde{w}_k|^{p_2} dtdx \leq \sum_{l=1}^{k} \tilde{c}^{l+1} \frac{\tilde{t}^{l+1}}{(l+1)!} \left[\|\tilde{u}_{k+1-l}\|^{p_2}_{X^T_{1,\delta}} dt + 1 \right] \tag{2.11}$$

which implies the boundedness of $\|\tilde{w}_k\|_{L^{p_2}(Q_T)}$ since $\|\tilde{u}_{k+1-l}\|_{X^T_{1,\delta}}$ is bounded.

Therefore, by (2.10) $(\tilde{w}_k)$ is a Cauchy sequence in $L^{p_2}(Q_T)$ thus there exists $\tilde{w} \in L^{p_2}(Q_T)$ such that $\tilde{w}_k \to \tilde{w}$ in $L^{p_2}(Q_T)$. From (2.8) and (F_2) we obtain (2.9) as $k \to \infty$. The uniqueness of the solution of (2.9) follows from (F_2) in a standard way: similarly to the proof of (2.10), one obtains for the solutions $\tilde{w}_1, \tilde{w}_2$ of (2.9):

$$\int_{Q_{\tilde{t}}} |\tilde{w}_1 - \tilde{w}_2|^{p_2} dtdx \leq \tilde{c}^{m+1} \frac{\tilde{t}^{m+1}}{(m+1)!} \|\tilde{w}_1 - \tilde{w}_2\|^{p_2}_{L^{p_2}(Q_T)}$$

which implies $\tilde{w}_1 = \tilde{w}_2$ a.e.

Finally, if $(u^\star_j) \to u^\star$ in $X^T_{1,\delta}$ then for the (unique) solutions $\tilde{w} = w^\star_j$ and $\tilde{w} = w^\star$ of (2.9) with $\tilde{u} = u^\star_j$, $\tilde{u} = u^\star$, respectively, we have according to (2.10)

$$\int_{Q_{\tilde{t}}} |w^\star_{j+m} - w^\star|^{p_2} dtdx \leq \sum_{l=1}^{m} \tilde{c}^{l+1} \frac{\tilde{t}^{l+1}}{(l+1)!} \|u^\star_{j+m+1-l} - u^\star\|^{p_2}_{X^T_{1,\delta}} + \tilde{c}^{m+1} \frac{\tilde{t}^{m+1}}{(m+1)!} \|w^\star_j - w^\star\|^{p_2}_{L^{p_2}(Q_T)}.$$

which implies

$$\lim_{j\to\infty} \|w_j^\star - w^\star\|_{L^{p_2}(Q_T)} = 0$$

since by (2.11) $(w_j^\star)$ is bounded in $L^{p_2}(Q_T)$. □

Proof of Theorem 2.1. According to Lemma 2.2, for arbitrary $(u,w) \in X_\delta^T$ there exists a unique solution $\tilde{u} \in X_1^T$ of

$$D_t\tilde{u} + A_{u,w}(\tilde{u}) = G, \quad \tilde{u}(0) = u_0. \tag{2.12}$$

Further, by Lemma 2.3, for arbitrary $w_0 \in L^{p_2}(\Omega)$ there exists a unique solution $\tilde{w} \in L^{p_2}(Q_T)$ of

$$\tilde{w}(t,x) = w_0(x) + \int_0^t F(\tau, x, ; \tilde{u}, \tilde{w})d\tau. \tag{2.13}$$

Define mapping $\Phi : X_\delta^T \to X_\delta^T$ by $\Phi(u,w) = (\tilde{u}, \tilde{w})$ where $\tilde{u}, \tilde{w}$ are solutions of (2.12) and (2.13).

According to Lemmas 2.2, 2.3 the mapping Φ is continuous. Further, Φ is compact. Because, if (u_k, w_k) is bounded in X_δ^T then, by Lemma 2.2, the sequence $(\tilde{u}_k)$ of solutions of (2.12) with $(u,w) = (u_k, w_k)$ is bounded in X_1^T and $(D_t\tilde{u}_k)$is bounded in $(X_1^T)^\star$. Since $W^{1-\delta,p_1}(\Omega)$ is compactly imbedded in $W^{1,p_1}(\Omega)$, there is a subsequence $(\tilde{u}_{k_l})$ of $(\tilde{u}_k)$ which is convergent in $X_{1,\delta}^T$ (see, e.g., [7], [14]). Thus by Lemma 2.3 $(\tilde{w}_{k_l})$ is convergent in $L^{p_2}(Q_T)$, whence $(\tilde{u}_{k_l}, \tilde{w}_{k_l})$ is convergent in X_δ^T.

Finally, we show that there is a closed ball B_R in X_δ^T such that $\Phi(B_R) \subset B_R$. Assume that

$$\|(u,w)\|_{X_\delta^T} \le r. \tag{2.14}$$

Then by inequality (2.6) we obtain for the solution $\tilde{u}$ of (2.12)

$$\frac{1}{2}\|u_0\|^2_{L^2(\Omega)} + \|G\|_{(X_1^T)^\star}\|\tilde{u}\|_{X_1^T} \ge \frac{c_2}{1 + \|(u,w)\|^{\sigma^\star}_{X_\delta^T}}\|\tilde{u}\|^{p_1}_{X_1^T} - c_3\|(u,w)\|^{\sigma}_{X_\delta^T},$$

thus

$$\frac{c_2}{2r^{\sigma^\star}}\|\tilde{u}\|^{p_1}_{X_1^T} - \|G\|_{(X_1^T)^\star}\|\tilde{u}\|_{X_1^T} - \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} \le c_3 r^\sigma \text{ if } r \ge 1,$$

i.e.,

$$\|\tilde{u}\|^{p_1}_{X_1^T}\left[\frac{c_2}{2r^{\sigma^\star}} - \frac{\|G\|_{(X_1^T)^\star}}{\|\tilde{u}\|^{p_1-1}_{X_1^T}}\right] \le c_3 r^\sigma + \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} \text{ if } r \ge 1.$$

Consequently, if

$$\|\tilde{u}\|_{X_1^T} \ge \|G\|^{\frac{1}{p_1-1}}_{(X_1^T)^\star}\left(\frac{4r^{\sigma^\star}}{c_2}\right)^{\frac{1}{p_1-1}} = \left(\|G\|_{(X_1^T)^\star}\frac{4}{c_2}\right)^{\frac{1}{p_1-1}} r^{\frac{\sigma^\star}{p_1-1}}$$

then (for sufficiently large r)

$$\|\tilde{u}\|^{p_1}_{X_1^T}\frac{c_2}{4r^{\sigma^\star}} \le c_4 r^\sigma, \text{ i.e., } \|\tilde{u}\|_{X_1^T} \le \left(\frac{4c_4}{c_2}\right)^{1/p_1} r^{\frac{\sigma+\sigma^\star}{p_1}}.$$

Thus

$$\|\tilde{u}\|_{X^T_{1,\delta}} \leq \|\tilde{u}\|_{X^T_1} \leq \text{const } r^\rho \text{ where } \rho = \max\left\{\frac{\sigma^\star}{p_1 - 1}, \frac{\sigma + \sigma^\star}{p_1}\right\} < 1. \tag{2.15}$$

Further, by using an estimate, analogous to (2.11), we obtain that, assuming (2.14), for the solution $\tilde{w}$ of (2.13) we have by (2.15)

$$\|\tilde{w}\|_{L^{p_2}(Q_T)} \leq \text{const } r^\rho. \tag{2.16}$$

Since $\rho < 1$, from (2.15), (2.16) we obtain that for sufficiently large R,

$$\|(u, w)\|_{X^T_\delta} \leq R \text{ implies } \|\Phi(u, w)\|_{X^T_\delta} = \|(\tilde{u}, \tilde{w})\|_{X^T_\delta} \leq R.$$

Thus Schauder's fixed point theorem implies that Φ has a fixed point $(u, w) \in X^T_\delta$:

$$\Phi(u, w) = (u, w).$$

Consequently, $u \in X^T_1$ and $(u, w) \in X^T$ satisfies (2.3), (2.4). □

Remark. If some Lipschitz conditions are satisfied with respect to the "functional variables" u, w in a_i, one can prove uniqueness of the solution.

3. Case $g \neq 0$

Now we shall consider equations (1.1), (1.2) with a bounded, continuous function g. This problem will be transformed to the case $g = 0$, considered in Section 1, with $p = p_1 = p_2 \geq 2$. Let $f = \int g$.

Define

$$\tilde{X}^T = L^p(0, T; W^{1,p}(\Omega)) \times L^p(0, T; W^{1,p}(\Omega))$$

and operator $A : \tilde{X}_T \to (X^T_1)^\star$ for $(u, w) \in \tilde{X}^T$, $v \in X^T_1$ by

$$\begin{aligned}
[A(u, w), v] &= \int_0^T \langle A(u, w)(t), v(t)\rangle dt \\
&= \int_{Q_T} \left\{\sum_{i=1}^n a_i(t, x, u(t, x), Du(t, x) + g(w(t, x))Dw(t, x); u, w)D_i v\right\} dtdx \\
&\quad + \int_{Q_T} a_0(t, x, u(t, x), Du(t, x) + g(w(t, x))Dw(t, x); u, w)v dtdx.
\end{aligned}$$

Further, assume

(F_3) F has the form

$$F(t, x; u, w) = F_1(t, x, [h(u)](t, x), w(t, x))$$

where F_1 is continuously differentiable with respect to the last three variables, the partial derivatives are bounded and either $h(u) = u$ or

$$h : L^p(Q_T) \to L^p(0, T; W^{1,p}(\Omega))$$

is a continuous linear operator such that $h(u) \in L^p(0,T;C^1(\overline{\Omega}))$ for all $u \in L^p(Q_T)$. Further, there exists $c_0 > 0$ such that

$$F_1(t,x,\zeta_0,\eta)\eta < 0 \text{ if } |\eta| \geq c_0. \tag{3.1}$$

Remark. In the second case $h(u)$ may have, e.g., the form

$$[h(u)](t,x) = \int_{Q_t} H(t,x,\tau,\xi)u(\tau,\xi)d\tau d\xi$$

(with a "sufficiently good" function H).

Theorem 3.1. *Assume that* (A_1)–(A_5) *and* (F_1)–(F_3) *are satisfied with* $p_1 = p_2 = p > 2$, $q_1 = q$, $\delta = 1$, $\sigma^\star < p-2$ *such that for the operators* g_1, k_1, g_2, k_2 *in* (A_2)–(A_4) *we have*

$$g_1(u,w)^q \leq \text{const}\, g_2(u,w), \quad k_1(u,w)^q \leq \text{const}\, k_2(u,w).$$

Further, g is a bounded, continuous function. Then for any $G \in (X_1^T)^\star$, $u_0 \in L^2(\Omega)$, $w_0 \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$ *there exists* $(u,w) \in \tilde{X}^T$ *such that* $u + f(w) \in L^p(0,T;V_1)$,

$$D_t u \in (X_1^T)^\star, \quad D_t w \in L^p(Q_T),$$
$$D_t u + A(u,w) = G, \quad u(0) = u_0, \tag{3.2}$$
$$D_t w = F(t,x;u,w) \textit{ for a.e. } (t,x) \in Q_T, \quad w(0) = w_0. \tag{3.3}$$

Proof. Instead of u introduce the new unknown function $\tilde{u}$ by

$$\tilde{u}(t,x) = u(t,x) + f(w(t,x)) \quad (\text{where } f = \int g). \tag{3.4}$$

By using the formulas

$$D_t\tilde{u} = D_t u + f'(w)D_t w, \quad D\tilde{u} = Du + f'(w)Dw \tag{3.5}$$

we obtain that $(u,w) \in \tilde{X}^T$ is a solution of (3.2), (3.3) if and only if $(\tilde{u},w) \in \tilde{X}^T$ satisfies

$$D_t\tilde{u} + \tilde{A}(\tilde{u},w) = G, \quad \tilde{u}(0) = u_0 + f(w_0(x)), \tag{3.6}$$
$$D_t w = F(t,x;\tilde{u} - f(w), w), \quad w(0) = w_0 \tag{3.7}$$

where

$$[\tilde{A}(\tilde{u},w),v]$$
$$= \int_{Q_T} \left\{ \sum_{i=1}^n a_i(t,x,\tilde{u}(t,x) - f(w(t,x)), D\tilde{u}(t,x); \tilde{u} - f(w), w) D_i v \right\} dtdx$$
$$+ \int_{Q_T} \{a_0(t,x,\tilde{u} - f(w), D\tilde{u}; \tilde{u} - f(w), w) - f'(w)F(t,x;\tilde{u} - f(w), w)\}\, v dtdx.$$

First we show that by Theorem 2.1 there is a solution $(\tilde{u},w) \in X^T$ of (3.6), (3.7) (such that $D_t w \in L^p(Q_T)$). Then we prove that $w \in L^p(0,T;W^{1,p}(\Omega))$, hence $(\tilde{u},w) \in \tilde{X}^T$ and thus with $u = \tilde{u} - f(w)$, (u,w) satisfies (3.2), (3.3).

Since $w_0 \in L^\infty(\Omega)$, by assumption (3.1), a solution of (3.3) satisfies

$$\|w\|_{L^\infty(Q_T)} \leq \max\left\{\|w_0\|_{L^\infty(\Omega)}, c_0\right\} = \tilde{c}_0.$$

Let $\psi \in C_0^\infty(\mathbb{R})$ such that $\psi(\eta) = 1$ for $|\eta| \leq \tilde{c}_0$ and define function f_1 by $f_1(\eta) = f(\eta)\psi(\eta)$. Then f_1 is bounded and consider (3.2), (3.3) with f_1 instead of f.

Since a_i satisfy (A_1)–(A_5) with $\delta = 1$, f_1, f_1' are bounded and continuous, functions $\tilde{a}_i$ (defining operator $\tilde{A}$ with f_1, instead of f)

$$\begin{aligned}\tilde{a}_i(t,x,\zeta_0,\zeta;\tilde{u},w) &= a_i(t,x,\zeta_0 - f_1(w(t,x)),\zeta;\tilde{u} - f_1(w),w), \quad i = 1,\dots,n\\ \tilde{a}_0(t,x,\zeta_0,\zeta;\tilde{u},w) &= a_0(t,x,\zeta_0 - f_1(w(t,x)),\zeta;\tilde{u} - f_1(w),w)\\ &\quad - f_1'(w)F(t,x;\tilde{u} - f_1(w),w)\end{aligned}$$

satisfy (A_1)–(A_5), too.

Clearly, (A_1)–(A_3), (A_5) are satisfied. Now we show that (A_4) is satisfied, too. By using the assumptions of our theorem and Young's inequality, we obtain for any $\varepsilon > 0$

$$\begin{aligned}&\sum_{i=0}^n \tilde{a}_i(t,x,\zeta_0,\zeta;\tilde{u},w)\zeta_i\\ &\quad= \sum_{i=0}^n a_i(t,x,\zeta_0 - f_1(w(t,x)),\zeta;\tilde{u} - f_1(w),w)\zeta_i + f_1'(w)F(t,x;\tilde{u} - f_1(w),w)\zeta_0\\ &\quad\geq g_2(\tilde{u} - f_1(w),w)[|\zeta_0 - f_1(w(t,x))|^p + |\zeta|^p] - [k_2(\tilde{u} - f_1(w),w)](t,x)\\ &\qquad + a_0(t,x,\zeta_0 - f_1(w(t,x)),\zeta;\tilde{u} - f_1(w),w)f_1(w(t,x))\\ &\qquad + f_1'(w)F(t,x;\tilde{u} - f_1(w),w)\zeta_0\\ &\quad\geq \tilde{g}_2(\tilde{u},w)[|\zeta_0|^p + |\zeta|^p] - [\tilde{k}_2(\tilde{u},w)](t,x)\\ &\qquad - \varepsilon\left\{g_1(\tilde{u} - f_1(w),w)(|\zeta_0|^{p-1} + |\zeta|^{p-1}) + [k_1(\tilde{u} - f_1(w)](t,x)\right\}^q - C_1(\varepsilon)\\ &\qquad - \varepsilon g_2(h(\tilde{u}) - h(f(w)),w)|\zeta_0|^p - C_2(\varepsilon)\frac{|F_1(t,x,h(\tilde{u}) - h(f_1(w)),w)|^q}{g_2(h(\tilde{u}) - h(f(w)),w)^{q/p}}\end{aligned}$$

where $\tilde{g}_2$, $\tilde{k}_2$ satisfy inequalities which are analogous to (2.1), (2.2), respectively (because f_1 is bounded) and $C_1(\varepsilon)$, $C_2(\varepsilon)$ are constants, depending on ε. Since by (F_3)

$$|F_1(t,x,h(\tilde{u}) - h(f_1(w)),w)|^q \leq \text{const}\ [|h(\tilde{u}) - h(f_1(w))|^q + |w|^q + |F_1(t,x,0,0)|^q],$$

we obtain

$$\begin{aligned}&\int_{Q_T} \frac{|F_1(t,x,h(\tilde{u}) - h(f_1(w)),w)|^q}{g_2(h(\tilde{u}) - h(f(w)),w)^{q/p}} dtdx\\ &\quad\leq \text{const} \int_{Q_T} |h(\tilde{u})|^q dtdx \cdot \left[\int_{Q_T} |\tilde{u}|^p dtdx\right]^{\frac{\sigma^\star}{(p-1)p}} + \text{const}\\ &\quad\leq \text{const}\, \|\tilde{u}\|_{L^p(Q_T)}^{\frac{p}{p-1} + \frac{\sigma^\star}{p-1}} + \text{const}\end{aligned}$$

where

$$\frac{p}{p-1} + \frac{\sigma^\star}{p-1} = \frac{p+\sigma^\star}{p-1} < p - \sigma^\star,$$

because $\sigma^\star < p-2$, thus $\max\left\{\frac{p+\sigma^\star}{p-1}, \sigma\right\} < p - \sigma^\star$ where σ is the constant in (A_4). Consequently, choosing sufficiently small $\varepsilon > 0$, we obtain (A_4) for functions $\tilde{a}_i$.

Further, it is easy to show that function $\tilde{F}$, defined by

$$\tilde{F}(t,x;\tilde{u},w) = F(t,x;\tilde{u} - f_1(w), w) = F_1(t,x,h(\tilde{u}) - h(f_1(w)), w)$$

satisfies (F_1), (F_2) because f_1' is bounded. Thus, by Theorem 2.1 there is a solution $(\tilde{u}, w) \in X^T$ of (3.6), (3.7) with f_1, instead of f, since $f(w_0) \in L^\infty(\Omega)$. As

$$\|w\|_{L^\infty(Q_T)} \le \tilde{c}_0,$$

$(\tilde{u}, w)$ satisfies (3.6), (3.7) with f, too.

Now we show that $w \in L^p(0,T;W^{1,p}(\Omega))$. According to (3.7)

$$D_t w(t,x) = F_1(t,x,[h(\tilde{u})](t,x) - [h(f(w))](t,x), w(t,x)), \quad w(0) = w_0 \tag{3.8}$$

Since $\tilde{u} \in L^p(0,T;V_1)$, there exists a sequence of functions $\tilde{u}_l \in C^{0,1}([0,T] \times \overline{\Omega})$ (i.e., continuously differentiable functions with respect to x) which converges to $\tilde{u}$ in $L^p(0,T;V_1)$. Further, by $w_0 \in W^{1,p}(\Omega)$ there is a sequence of functions $w_{0l} \in C^1(\overline{\Omega})$ such that $(w_{0l}) \to w_0$ in $W^{1,p}(\Omega)$.

In the case when $h(u) = u$, denote by $\Phi(t,x,\xi,\lambda)$ the solution of the Cauchy problem

$$\dot{w}_l = F_1(t,x,\lambda - f(w_l), w_l), \quad w_l(0) = \xi.$$

Then

$$w_l(t,x) = \Phi(t,x,w_{0l}(x),\tilde{u}_l(t,x))$$

satisfies

$$w_l(t,x) = w_{0l}(x) + \int_0^t F_1(\tau,x,\tilde{u}_l(\tau,x) - f(w_l(\tau,x)), w_l(t,x))d\tau.$$

By the assumption on function F_1 and the differentiability of the characteristic function Φ we obtain that w_l is continuously differentiable with respect to x and

$$\partial_{x_j} w_l(t,x) = \partial_{x_j}\Phi(t,x,w_{0l}(x),\tilde{u}_l(t,x))$$

$$+\partial_3\Phi(t,x,w_{0l}(x),\tilde{u}_l(t,x))\partial_{x_j}w_{0l}(x) + \partial_4\Phi(t,x,w_{0l}(x),\tilde{u}_l(t,x))\partial_{x_j}\tilde{u}_l(t,x).$$

Further,

$$y(t) = \partial_3\Phi(t,x,\xi,\lambda)\eta + \partial_4\Phi(t,x,\xi\lambda)\zeta$$

is a solution of the Cauchy problem for the following linear differential equation:

$$\dot{y}(t) = \partial_{w_l}F_1(t,x,\lambda - f(w_l), w_l)y(t) + \partial_\lambda F_1(t,x,\lambda - f(w_l), w_l)\zeta, \tag{3.9}$$

$$y(0) = \eta.$$

Since the last three partial derivatives of F_1 and the derivative of f are bounded, by using the formula for the solution of (3.9), it is easy to show that $\partial_3\Phi, \partial_4\Phi$ are bounded. One obtains similarly that $\partial_{x_j}\Phi$ is bounded, too.

Since $w_{0l} \to w_0$ in $W^{1,p}(\Omega)$ and $(\tilde{u}_l) \to \tilde{u}$ in $L^p(0,T;V_1)$, by Vitali's theorem we obtain from the above formulas that the sequences $(\partial_{x_j} w_l)$ are convergent in $L^p(Q_T)$. On the other hand, according to Lemma 2.3, $(w_l) \to w$ in $L^p(Q_T)$. Therefore, $w \in L^p(0,T;W^{1,p}(\Omega))$.

If the second assumption is satisfied on $h(u)$ in (F_3), then denote by $\tilde{\Phi}(t,x,\xi,\lambda)$ the solution of the Cauchy problem

$$\dot{w}_l = F_1(t,x,\lambda,w_l), \quad w_l(0) = \xi.$$

Then the solution of (3.8) with $\tilde{u} = \tilde{u}_l$, $w_0 = w_{0l}$:

$$w_l(t,x) = \tilde{\Phi}(t,x,w_{0l}(x),[h(\tilde{u}_l)](t,x) - h[f(w_l)](t,x))$$

and $w_l \in L^p(0,T;C^1(\overline{\Omega}))$ since

$$\begin{aligned}\partial_{x_j} w_l(t,x) = {} & \partial_{x_j}\tilde{\Phi}(t,x,w_{0l}(x),[h(\tilde{u}_l)](t,x) - h[f(w_l)](t,x)) \\ & + \partial_3\tilde{\Phi}(t,x,w_{0l}(x),[h(\tilde{u}_l)](t,x) - h[f(w_l)](t,x))\partial_{x_j} w_{0l}(x) \\ & + \partial_4\tilde{\Phi}(t,x,w_{0l}(x),[h(\tilde{u}_l)](t,x) \\ & - h[f(w_l)](t,x))\{\partial_{x_j} h(\tilde{u}_l)(t,x) - \partial_{x_j}[h(f(w_l))](t,x)\}\end{aligned}$$

and by (F_3), $\partial_{x_j}\Phi, \partial_3\Phi, \partial_4\Phi$ derivatives are bounded. (See the case $h(u) = u$.) Since $(w_{0l}) \to w_0$ in $W^{1,p}(\Omega)$, $(\tilde{u}_l) \to \tilde{u}$ in $L^p(0,T;V_1)$, and by Lemma 2.3 $(w_l) \to w$ in $L^p(Q_T)$, by (F_3) we obtain, similarly to the case $h(u) = u$, that the sequence $(\partial_{x_j} w_l)$ is convergent in $L^p(Q_T)$. Thus $w \in L^p(0,T;W^{1,p}(\Omega))$. □

References

[1] R.A. Adams, *Sobolev spaces.* Academic Press, New York – San Francisco – London, 1975.

[2] H. Amann, Highly degenerate quasilinear parabolic systems. *Annali della Scuola Normale Superiore di Pisa Ser.* 4, **18** (1991), 136–166.

[3] J. Berkovits, V. Mustonen, Topological degree for perturbations of linear maximal monotone mappings and applications to a class of parabolic problems. *Rend. Mat. Ser. VII*, **12** Roma (1992), 597–621.

[4] S. Cinca, Diffusion und Transport in porösen Medien bei veränderlicher Porosität. Diplomawork, Univ. Heidelberg, 2000.

[5] Bei Hu, Jianhua Zhang, Global existence for a class of non-Fickian polymer-penetrant systems. *J. Part. Diff. Eq.* **9** (1996), 193–208.

[6] W. Jäger, L. Simon, On a system of quasilinear parabolic functional differential equations. *Acta Math. Hung.* **112** (2006), 39–55.

[7] J.L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires.* Dunod, Gauthier-Villars, Paris, 1969.

[8] J.D. Logan, M.R. Petersen, T.S. Shores, Numerical study of reaction-mineralogy-porosity changes in porous media. *Appl. Math. Comput.* **127**, (2002), 149–164.

[9] J.H. Merkin, D.J. Needham, B.D. Sleeman, A mathematical model for the spread of morphogens with density dependent chemosensitivity. *Nonlinearity* **18**, (2005), 2745–2773.

[10] B. Rivière, S. Show, Discontinuous Galerkin finite element approximation of nonlinear non-Fickian diffusion in viscoelastic polymers. *SIAM J. Numer. Anal.* **44**, (2006), 2650–2670.

[11] S. Show, J.R. Whiteman, Adaptive space-time finite element solution for Volterra equations arising in viscoelasticity problems. *J. Comp. Appl. Math.* **125** (2000), 337–345.

[12] L. Simon, On a system with a singular parabolic equation. *Folia FSN Univ. Masarykiane Brunensis, Math.* **16** (2007), 149–156.

[13] L. Simon, W. Jäger, On non-uniformly parabolic functional differential equations. *Studia Sci. Math. Hung.* **45** (2008), 285–300.

[14] L. Simon, Application of monotone type operators to parabolic and functional parabolic PDE's. *Handbook on Evolutionary Differential Equations* **4**, Elsevier, 2008, 267–321.

[15] L. Simon, On some singular systems of parabolic functional equations. *Math. Bohem.*, to appear.

[16] E. Zeidler, *Nonlinear functional analysis and its applications II A and II B*. Springer, 1990.

László Simon
Institute of Mathematics
Pázmány P. sétány 1/C
H-1117 Budapest, Hungary
e-mail: simonl@ludens.elte.hu

Operator Theory:
Advances and Applications, Vol. 216, 331–341

Boundary-value Problems for a Class of Third-order Composite Type Equations

O.S. Zikirov

Abstract. In the paper, we study boundary-value problems with the normal derivative for a class of third-order composite type equation with Laplace operator in the main part. We prove the theorems of the existence and uniqueness of classical solution for considered problems. The proof is based on an energy inequality and Fredholm type integral equations.

Mathematics Subject Classification (2000). Primary 35G15; Secondary 35M20.

Keywords. Composite type equation, boundary conditions, Dirichlet problem, energy integrals, singularity, integral equations, Green function, Laplace operator, third-order PDE.

1. Introduction

One of the most widely known method for investigation of boundary value problems is the method of potentials (Green function). It allows to reduce investigation of a boundary value problems for partial differential equations investigation of the corresponding integral equation. It should be noted, this method is applicable not only for stationary problems, but also for problems of the evolutionary form, i.e., initial-boundary and nonlocal problems. It is sufficient to recall the classical method of the Green function to solving boundary value problems for second-order elliptic type equations [1].

However, at the present time this method does not lose its significance and it is widely applied in solving boundary value problems for nonclassical partial differential equations.

The present paper is devoted to the studying of boundary value problems with the normal derivative for the composite type mixed equation of third order

$$\left(\alpha\frac{\partial}{\partial x}+\beta\frac{\partial}{\partial y}\right)(k(y)u_{xx}+u_{yy})+Lu=f(x,y), \tag{1.1}$$

where α and β are given real numbers, moreover $\alpha^2 + \beta^2 \neq 0$, L is the linear second-order differential operator of the form

$$\begin{aligned} Lu \equiv a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} \\ + a_1(x,y)u_x + b_1(x,y)u_y + c_1(x,y)u. \end{aligned} \tag{1.2}$$

The coefficients and right side of equation (1.1) are given real functions.

The correct statement of boundary value problems for (1.1) depends on the sign and values of coefficients α and β. Equations in the form of (1.1) generalize the wide class of composite type equations.

For example, if $\alpha = 1$, $\beta = 0$ and $\alpha = 0$, $\beta = 1$, but $Lu \equiv 0$, then we obtain equations, investigated in works [6], [14] and others.

Investigation of boundary-value problems are interesting on theoretical point of view. Also there are a number of the non-local boundary conditions for evolution problems that have various applications in chemical engineering, thermoelasticity, underground water flow and population dynamics and etc.: see for example [4] or [8].

We remark that equation (1.1) often is called the composite type equation. Boundary-value problems for equations of third order with local and non-local boundary conditions are investigated by L.A. Bougoffa [2], A. Bouziani [3], V.V. Daynyak and V.I. Korzyuk [5], T.D. Dzhuraev [6], T.D. Dzhuraev and O.S. Zikirov [7], A.M. Nakhushev [13], M.S. Salakhitdinov [14], O.S. Zikirov [15] and many references therein.

2. Formulation of the problem and the uniqueness of the solution

In this section, we formulate correct boundary value problems for a composite type linear equation and prove theorems of uniqueness for the solution of stated problems using the method of energetic identities.

Let $k(y)$ $(k(y) > 0)$ be a continuous function in the simply connected domain D bounded by the segment $AB[A(0,0)B(1,0)]$ of the axe x and by the smooth curve σ which lies in the half-plane $y > 0$ with endpoints on the axe x at the points A and B.

Consider in the domain D the composite type third-order equation

$$\left(\alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y}\right)(k(y)u_{xx} + u_{yy}) + Lu = f(x,y), \tag{2.1}$$

where α and β are given real numbers, moreover $\alpha^2 + \beta^2 \neq 0$, L is the second-order linear differential operator of the form (1.2).

Coefficients and the right side of equation (2.1) are given real functions.

Concerning the curve σ, we suppose in addition, it intersects each straight line $x = \text{const}$ only at the one point.

Divide the curve σ on two parts σ_1 and σ_2 in the following way:

$$\sigma_1 = \{(x,y) \in \sigma : \ \alpha x_n + \beta y_n > 0\}, \qquad \sigma_2 = \sigma \backslash \sigma_1, \tag{2.2}$$

where $x_n = \cos(n,x)$, $y_n = \cos(n,y)$, and n is the exterior normal to the σ.

Definition 1. Any classic solution of equation (2.1) called as a solution of this equation, i.e., a function $u(x, y)$, possessing in the domain D continuous partial derivatives up to the third-order inclusively and converting the equation into an identity.

Problem $A^k_{\alpha\beta}$. To find a classic solution $u(x, y)$ of equation (2.1) in the domain D, continuous with its derivatives in the closed domain $\overline{D}$, and satisfying to the following boundary conditions:

a) in case of $0 < \dfrac{\beta}{\alpha} < +\infty$, the conditions

$$u(x,y)\big|_{\sigma} = \varphi_1(x,y),\ (x,y) \in \sigma; \qquad u(x,y)\big|_{AB} = \tau(x),\ 0 \le x \le 1, \quad (2.3)$$

$$\frac{\partial u(x,y)}{\partial n}\Big|_{\sigma_2} = \varphi_2(x,y),\ (x,y) \in \sigma_2; \quad \frac{\partial u(x,y)}{\partial y}\Big|_{AB} = \nu(x),\ 0 < x < 1 \qquad (2.4)$$

hold;

b) in case of $\beta = 0$, conditions (2.3) and the first condition of (2.4) hold;

c) in case of $\alpha = 0$, conditions (2.3) hold in one time with the second condition of (2.4) or with the condition

$$\frac{\partial u(x,y)}{\partial n}\Big|_{\sigma} = \varphi_3(x,y), \qquad (x,y) \in \sigma,$$

here $\varphi_1(x, y)$, $\varphi_2(x, y)$, $\varphi_3(x, y)$, $\tau(x)$, $\nu(x)$ are given functions, moreover $\varphi_1(A) = \tau(0)$, $\varphi_1(B) = \tau(1)$.

One can show, the case of $\alpha\beta < 0$ in the problem $A^k_{\alpha\beta}$ is reduced with the help of exchange $x = 1 - \xi$ or $y = 1 - \eta$ to the case of $\alpha\beta > 0$.

Therefore, without any loss of generality, suppose $\alpha \ge 0$, $\beta \ge 0$.

Assumption 1. *We assume that*

$$a(x,y), \quad b(x,y), \quad c(x,y) \in C^1(\overline{D});$$

$$a_1(x,y), \quad b_1(x,y) \in C(\overline{D}); \quad c_1(x,y) \in C(D);$$

and

$$\frac{\partial^2 a(x,y)}{\partial x^2} \le c_1, \quad \frac{\partial^2 b(x,y)}{\partial x \partial y} \le c_2, \quad \frac{\partial^2 c(x,y)}{\partial y^2} \le c_3;$$

$$\frac{\partial a_1(x,y)}{\partial x} \le c_4, \quad \frac{\partial b_1(x,y)}{\partial y} \le c_5.$$

Assumption 2. *For all* $(x, y) \in D$ *and all* $\xi, \eta \in D$, *we assume that*

1) $a(x, y)\xi^2 + 2b(x, y)\xi\eta + c(x, y)\eta^2 \ge c_6(\xi^2 + \eta^2)$;
2) $a_{xx} + 2b_{xy} + c_{yy} - a_{1x} - b_{1y} + 2c_1 \le -c_7 < 0$.

In Assumptions 1, 2 and in the rest of the paper, we assume that c_j, $(j = 1, \ldots, 7)$ are positive constants.

In this paper, we prove the existence and uniqueness of a classical solution of the problem $A^k_{\alpha\beta}$.

Theorem 2.1. *Let Assumptions 1, 2 be fulfilled. Then classical solution of the Problem $A^k_{\alpha\beta}$ is unique.*

Proof. Let us show that the homogeneous Problem $A^k_{\alpha\beta}$

$$\varphi_1(x,y) = \varphi_2(x,y) = \tau(x) = \nu(x) \equiv 0 \tag{2.5}$$

has only trivial solution. We prove this fact on the base of energetic identities. Multiplying (2.1) by u and integrating the obtained relation by parts in D, we have

$$\iint\limits_D u\left(\alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y}\right)(k(y)u_{xx} + u_{yy})dxdy + \iint\limits_D uLudxdy = 0. \tag{2.6}$$

Transform the integrands in the following way

$$u\left(\alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y}\right)[k(y)u_{xx} + u_{yy}] = \left(\alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y}\right)[uk(y)u_{xx} + uu_{yy}]$$
$$-\frac{1}{2}\left[\left(\alpha\frac{\partial}{\partial x} - \beta\frac{\partial}{\partial y}\right)[k(y)u_x^2 - u_y^2] + \left(\alpha\frac{\partial}{\partial y} - \beta k(y)\frac{\partial}{\partial x}\right)(2u_xu_y)\right];$$

$$u[a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy}]$$
$$= \frac{\partial}{\partial x}\left[auu_x + buu_y - \frac{1}{2}(a_x + b_y)u^2\right] + \frac{\partial}{\partial y}\left[buu_x + cuu_y - \frac{1}{2}(b_x + c_y)u^2\right]$$
$$-\left[\left(a - \frac{1}{2}\beta k'(y)\right)u_x^2 + 2bu_xu_y + cu_y^2)\right] + \frac{1}{2}(a_{xx} + 2b_{xy} + c_{yy})u^2;$$

and

$$u[a_1(x,y)u_x + b_1(x,y)u_y + c_1(x,y)u]$$
$$= \frac{1}{2}\left[\frac{\partial}{\partial x}(a_1u)^2 + \frac{\partial}{\partial y}(b_1u)^2\right] - \frac{1}{2}(a_{1x} + b_{1y} - 2c_1)u^2.$$

Applying the Green formula to integral (2.6) and taking homogeneous boundary conditions into account, we obtain

$$\frac{1}{2}\int\limits_{\sigma+AB}\left[k\cdot(\alpha x_n - \beta y_n)u_x^2 + 2(\alpha y_n + \beta k x_n)u_xu_y + (\beta y_n - \alpha x_n)u_y^2\right]ds$$
$$+\iint\limits_D\left[\left(a(x,y) - \frac{1}{2}\beta k'(y)\right)u_x^2 + 2b(x,y)u_xu_y + c(x,y)u_y^2\right]dxdy$$
$$-\frac{1}{2}\iint\limits_D(a_{xx} + 2b_{xy} + c_{yy} - a_{1x} - b_{1y} + 2c_1)u^2dxdy = 0, \tag{2.7}$$

Since $u(x,y) = 0$ on the boundary of the domain D, we have $\partial u/\partial s = 0$ on $\sigma + AB$ and, therefore on the boundary $\sigma + AB$ of the domain D the following equalities hold $u_x = u_nx_n, \;\; u_y = u_ny_n$.

By virtue of equalities $x_n = y_s$, $y_n = -x_s$, considering homogeneous boundary conditions, from (2.7) we have

$$\int\limits_{\sigma_1} u_n^2(k(y)x_n^2 + y_n^2)(\alpha x_n + \beta y_n)ds$$
$$+ \iint\limits_D \left[\left(a(x,y) - \frac{1}{2}\beta k'(y)\right)u_x^2 + 2b(x,y)u_x u_y + c(x,y)u_y^2\right] dxdy$$
$$- \frac{1}{2}\iint\limits_D (a_{xx} + 2b_{xy} + c_{yy} - a_{1x} - b_{1y} + 2c_1)u^2 dxdy = 0, \tag{2.8}$$

Hence, by conditions of Theorem 2.1, we conclude $u(x,y) \equiv 0$ in $\overline{D}$. □

Remark 1. Uniqueness of the solution of the Problem $A^k_{\alpha\beta}$ for the cases b) and c) can be proved analogously.

3. Existence of the solution for the Problem $A^k_{\alpha\beta}$

In this section, we prove existence of a classic solution for the problem $A^k_{\alpha\beta}$ stated in the previous section. For a solution of the Problem $A^k_{\alpha\beta}$, it is valid the following

Theorem 3.1. *Let all conditions of Theorem* 2.1 *hold and*

$$2b(x,y) = \frac{\beta}{\alpha}\,a(x,y) + \frac{\alpha}{\beta}\,c(x,y), \qquad a_1(x,y) = \frac{\alpha}{\beta}\,b_1(x,y).$$

If functions $\varphi_1{}'(x,y)$, $\varphi_2(x,y)$, $\tau'(x)$, *and* $\nu(x)$ *satisfy to the Hölder condition, then a solution of the Problem* $A^k_{\alpha\beta}$ *exists.*

Proof. We prove Theorem 3.1 for the case a). Let $k(y) \equiv 1$, $\forall\ y \in D$.

1°. Denote by $\omega(s)$ unknown values of the normal derivative of the function $u(x,y)$ on σ_1.

Set

$$\alpha u_x + \beta u_y = v(x,y), \tag{3.1}$$

then, equation (2.1) will have the form

$$\Delta u + A(x,y)u_x + B(x,y)u_y + C(x,y)u = -c_1(x,y)u. \tag{3.2}$$

Here $A(x,y)$, $B(x,y)$, and $C(x,y)$ are known function, and also

$$A(x,y),\ B(x,y),\ C(x,y)\ \in C^1(D)$$

and, furthermore, $C(x,y) \le 0$, $\forall (x,y) \in \overline{D}$.

Taking (2.3)–(2.4) into account for the function $v(x,y)$, we obtain the following boundary conditions

$$v(x,y)\big|_{\sigma} = \mu(s), \qquad v(x,y)\big|_{AB} = \lambda(x), \tag{3.3}$$

where

$$\mu(s) = \begin{cases} \omega(s)(\alpha x_n + \beta y_n) + \varphi_1'(s)(\alpha x'(s) + \beta y'(s)), & s \in \sigma_1, \\ \varphi_2(s)(\alpha x_n + \beta y_n) + \varphi_1'(s)(\alpha x'(s) + \beta y'(s)), & s \in \sigma_2, \end{cases}$$

$$\lambda(x) = \alpha\tau'(x) + \beta\nu(x), \qquad 0 \le x \le 1.$$

The regular solution of equation (3.2), satisfying to conditions (3.3), is represented in the form of [6]

$$v(x,y) = \int\limits_{\sigma} \left[\frac{\partial G}{\partial n}(x,y;s) + \overline{K_0}(x,y;s) + \overline{K_{00}}(x,y;s)\right]\mu(s)ds \qquad (3.4)$$

$$- \iint\limits_{D} \Big[G(x,y;\xi,\eta) + P(x,y;\xi,\eta)\Big] c_1(\xi,\eta)u(\xi,\eta)d\xi d\eta + \Phi(x,y).$$

Where

$$G(x,y;\xi,\eta) = \frac{1}{2\pi}\ln|(x-\xi)^2 + (y-\eta)^2| + g(x,y;\xi,\eta)$$

is the Green function of the Dirichlet problem for the Laplace equation, $g(x,y;\xi,\eta)$ is a regular part of the Green function;

$$\overline{K}_0(x,y;s) = \iint\limits_{D} G(x,y;\xi,\eta)\left[\overline{K}(\xi,\eta;s) + \iint\limits_{D} K(\xi,\eta;\xi',\eta')\overline{K}(\xi',\eta';s)d\xi' d\eta'\right]d\xi d\eta;$$

$$\overline{K}_{00}(x,y;s) = 2\pi \iint\limits_{D}\iint\limits_{D} G(x,y;\xi,\eta)\Gamma_2(\xi,\eta;\xi',\eta')$$

$$\times \left[\overline{K}(\xi',\eta';s) + \iint\limits_{D} K(\xi_1,\eta_1;\xi',\eta')\overline{K}(\xi_1,\eta_1;s)d\xi_1 d\eta_1\right]d\xi' d\eta' d\xi d\eta;$$

$$P(x,y;\xi,\eta) = \iint\limits_{D} G(x,y;\xi,\eta)\Big[K(\xi,\eta;\xi',\eta') + \Gamma_2(xi,\eta;\xi',\eta')$$

$$+ \iint\limits_{D} \Gamma_2(\xi,\eta;\xi_1,\eta_1)K(\xi',\eta';\xi_1,\eta_1)d\xi_1 d\eta_1\Big]d\xi d\eta;$$

$$\overline{K}(x,y;s) = A(x,y)\frac{\partial G^*}{\partial x} + B(x,y)\frac{\partial G^*}{\partial y} + C(x,y)G^*;$$

$$K(x,y;\xi,\eta) = A(x,y)\frac{\partial G(x,y;\xi,\eta)}{\partial x} + B(x,y)\frac{\partial G(x,y;\xi,\eta)}{\partial y}$$

$$+ C(x,y)G(x,y;\xi,\eta); \quad G^* = \frac{\partial G(x,y;s)}{\partial n};$$

$\Gamma_2(x,y;\xi,\eta)$ is the resolvent of the kernel

$$K_2(x,y;\xi,\eta) = \iint\limits_{D} K(x,y;\xi,\eta)K(\xi,\eta;\xi',\eta')d\xi' d\eta';$$

$\Phi(x,y)$ is the known function.

By virtue of the condition $u(x,y) = \varphi_{11}(x,y)$, $(x,y) \in \sigma_1$, from (3.2) we find

$$u(x,y) = \frac{1}{\beta} \int\limits_{f(\beta x - \alpha y)}^{y} v\left(x - \frac{\alpha}{\beta} y + \frac{\alpha}{\beta} t, t\right) dt + \varphi_{11}(x,y), \tag{3.5}$$

here

$$\varphi_1(x,y) = \begin{cases} \varphi_{11}(x,y), & (x,y) \in \sigma_1, \\ \varphi_{12}(x,y), & (x,y) \in \sigma_2, \end{cases}$$

$$f(\beta x - \alpha y) = \frac{1}{2(\alpha^2 + \beta^2)} \Big[\alpha\beta - 2\alpha(\beta x - \alpha y) + \beta \sqrt{\alpha^2 - 4(\beta x - \alpha y)^2 + 4(\beta x - \alpha y)} \Big].$$

Substituting (3.4) into (3.5), after some transformations we obtain the integral equation of the second kind with respect to the function $u(x,y)$:

$$u(x,y) + \frac{1}{\pi} \iint\limits_{D} \mathcal{K}(x,y;\xi,\eta) c_1(\xi,\eta) u(\xi,\eta) u(\xi,\eta) d\xi d\eta = F(x,y). \tag{3.6}$$

Here

$$\mathcal{K}(x,y;\,\xi,\eta) = -\frac{1}{2\pi} \left[\frac{\alpha}{\beta}(x - \xi) + (y - \eta) \right] \ln |(x-\xi)^2 + (y-\eta)^2| + r(x,y;\xi,\eta);$$

$$F(x,y) = \frac{1}{2\pi} \int\limits_{\sigma} \Big\{ \frac{\alpha^2}{2(\alpha^2 + \beta^2)} [\alpha \eta'(s) - \beta \xi'(s)] \times \ln |(x-\xi)^2 + (y-\eta)^2| + r(x,y;s) \Big\} \mu(s) ds + \Psi_1(x,y);$$

$$r(x,y;\,\xi,\eta) = \frac{1}{\beta} \int\limits_{f(\beta x - \alpha y)}^{y} \frac{\partial g}{\partial n} \left(x - \frac{\alpha}{\beta} y + \frac{\alpha}{\beta} t, t;\, \xi, \eta \right) dt,$$

$\Psi_1(x,y)$ is a known continuous function, depending on the Green's function and coefficients of equation (2.1) and the functions $\varphi_1(x,y), \varphi_2(x,y), \tau(x)$ and $\nu(x)$.

Conditions of Theorem 3.1 imposed on given functions allow to assert that $F(x,y) \in C^1(\overline{D})$, and the kernel of the integral equation (3.6) are continuous, moreover first derivatives have singularities not more than logarithmic ones.

Since the solution of the problem $A^1_{\alpha\beta}$ is unique and by virtue of the Fredholm alternative, can conclude that equation (3.6) has the unique solution in the class $C^1(\overline{D})$.

Let $\Gamma(x,y;\xi,\eta)$ be the resolvent of the kernel $\mathcal{K}(x,y;\xi,\eta)$. Then a solution of equation (3.6) can be represented in the form of

$$u(x,y)=\frac{1}{2\pi}\int\limits_{\sigma}\left\{\frac{\alpha^2}{2(\alpha^2+\beta^2)}[\alpha\eta'(s)-\beta\xi'(s)]\right. \\ \left.\times\ln|(x-\xi)^2+(y-\eta)^2|+R(x,y;s)\right\}\mu(s)ds+\Psi_2(x,y), \tag{3.7}$$

where

$$R(x,y;s)=r(x,y;s)+\iint\limits_{D}\Gamma(x,y;\xi_1,\eta_1)\left\{\frac{\alpha^2}{2(\alpha^2+\beta^2)}[\alpha\eta'(s)-\beta\xi'(s)]\right. \\ \left.\times\ln|(\xi_1-\xi)^2+(\eta_1-\eta)^2|+r(\xi_1,\eta_1;s)\right\}d\xi_1 d\eta_1;$$

$$\Psi_2(x,y)=\Psi_1(x,y)+\frac{1}{2\pi}\iint\limits_{D}\Gamma(x,y;\xi,\eta)\Psi_1(\xi,\eta)d\xi d\eta.$$

Formula (3.7) contains as the function $\mu(s)$ the function $\omega(s)$, which is unknown. To define it, it is necessary to pass to the limit turning points (x,y) to the point lying on the arc σ_1. Then we obtain for the unknown function $\omega(s)$ the integral equation of the first kind with the logarithmic singularity in the kernel

$$\frac{1}{2\pi}\frac{\alpha^2}{\alpha^2+\beta^2}\int\limits_{\sigma_1}\Big[\ln|s-s_0|+R(s,s_0)\Big]\omega_*(s)ds=g(s_0), \tag{3.8}$$

where

$$\omega_*(s)=[\alpha\eta'(s)-\beta\xi'(s)]\omega(s);$$

$$g(s_0)=\varphi_1(s_0)-\frac{1}{2\pi}\int\limits_{\sigma}[\ln|s-s_0|+R(s,s_0)]{\varphi_1}'(s)[\alpha\xi'(s)+\beta\eta'(s)]ds \\ -\frac{1}{2\pi}\int\limits_{\sigma_2}[\ln|s-s_0|+R(s,s_0)]\varphi_2(s)[\alpha\xi'(s)+\beta\eta'(s)]ds-\Psi_2(s_0).$$

By virtue or properties of the Green function, one can easily to be sure that the function $R(s,s_0)$ and its first derivatives are continuous. The right side $g(s_0)$ is continuously differentiable function, and $g'(s_0)$ satisfies to the Hölder condition.

2°. Here we give the proof of theorem on existence of the solution of an integral equation of the first kind with the logarithmic singularity in the kernel

$$\int\limits_0^l\ln|s-s_0|\,\omega_*(s)ds=g(s_0),\quad s_0\in\sigma, \tag{3.9}$$

where l is arc length of the curve σ.

The following statement is valid for equation (3.9).

Theorem 3.2. *If* $g(s_0) \in C^{(1,\lambda)}(\sigma)$*, then the solution of the integral equation* (3.9) *exists in the class* $\omega_*(s) \in C^{(0,\lambda)}(\sigma)$, $0 < \lambda < 1$, *and it is given by the formula*

$$\omega_*(s_0) = -\frac{1}{\pi^2\sqrt{s_0(l-s_0)}}\int_0^l \frac{\sqrt{s(l-s)}}{s-s_0}\,g'(s)ds - \frac{1}{\pi^2\sqrt{s_0(l-s_0)}\,(\ln l - 2\ln 2)}\int_0^l \frac{g(s)ds}{\sqrt{s(l_1-s)}}, \tag{3.10}$$

here $C^{(1,\lambda)}(\sigma)$ *and* $C^{(0,\lambda)}(\sigma)$ *are spaces of functions given on* σ *and satisfying to the Hölder condition.*

Proof. Differentiating (3.9), we obtain the singular integral equation

$$\int_0^l \frac{\omega_*(s)ds}{s-s_0}\,ds = {g_1}'(s_0), \tag{3.11}$$

the general solution of which has the form (see, for example, [8])

$$\omega_*(s_0) = -\frac{1}{\pi^2}\frac{1}{\sqrt{s_0(l-s_0)}}\int_0^l \frac{\sqrt{s(l-s)}}{s-s_0}\,{g_1}'(s)ds \pm \frac{C}{\sqrt{s_0(l-s_0)}}. \tag{3.12}$$

Choose the particular solution of equation (3.11) which is also a solution for equation (3.9), (i.e., we choose C).

Equation (3.9) has the solution up to a constant [9], and to select the unique solution, one need to know the value of the integral from the function $\omega_*(s)$ on the segment $[0, l]$.

Multiplying for this (3.9) by $[s_0(l-s_0)]^{-1/2}$ and integrating on the segment $[0, l]$, we obtain

$$\int_0^l \frac{ds_0}{\sqrt{s_0(l-s_0)}}\int_0^l \ln|s-s_0|\,\omega_*(s)ds = \int_0^l \frac{g(s_0)ds_0}{\sqrt{s_0(l-s_0)}}.$$

Then, changing the order of integration in the left side of the last equality (see, for example, [12]), we obtain the equality

$$\int_0^l \omega_*(s)ds\int_0^l \frac{\ln|s-s_0|}{\sqrt{s_0(l-s_0)}}ds_0 = \int_0^l \frac{g(s)ds}{\sqrt{s(l-s)}}. \tag{3.13}$$

By virtue of the known relation [9], we have

$$\int_0^l \frac{\ln|s-s_0|}{\sqrt{s_0(l-s_0)}}\,ds_0 = \pi(\ln l - 2\ln 2).$$

Then (3.13) takes the form

$$\int_0^l \omega_*(s)ds = \frac{1}{\pi(\ln l - 2\ln 2)} \int_0^l \frac{g(s)ds}{\sqrt{s(l-s)}}. \tag{3.14}$$

Following to [10], [11], one can easily to see, the function $\omega_*(s_0)$, defining by equality (3.10), belongs to the class $C^{(0,\lambda)}(\sigma_1),\ 0 < \lambda < 1.$ □

Consider now equation (3.8). Rewrite it, dividing the kernel of the equation on the singular and regular parts, in the form of

$$\int_0^{l_1} \ln|s - s_0|\,\omega_*(s)ds = g_1(s_0), \quad s_0 \in \sigma_1, \tag{3.15}$$

where

$$g_1(s_0) = \frac{2\pi(\alpha^2+\beta^2)}{\alpha^2}\,g(s_0) - \int_0^{l_1} R(s,s_0)\omega_*(s)ds, \tag{3.16}$$

and l_1 is arc length of the curve σ_1.

By condition of Theorem 3.2 and condition (3.14), converting the principal part of integral equation (3.8), we obtain the integral equation of the second kind in the form of

$$\omega_{**}(s_0) + \int_0^{l_1} \frac{M(s,s_0)}{\sqrt{s(l_1-s)}}\,\omega_{**}(s)ds = g_2(s_0), \tag{3.17}$$

where

$$\omega_{**}(s) = \sqrt{s(l_1-s)}\,w_*(s);$$

$$M(s,s_0) = \frac{1}{\pi^2}\int_0^{l_1} \frac{\sqrt{\xi(l_1-\xi)}}{\xi - s_0}\,\frac{\partial R(\xi,s)}{\partial s}\,d\xi - \frac{1}{\ln l_1 - 2\ln 2}\int_0^{l_1} \frac{R(\xi,s)}{\sqrt{\xi(l_1-\xi)}}\,d\xi;$$

$$g_2(s_0) = -\frac{2(\alpha^2+\beta^2)}{\pi\alpha^2}\int_0^{l_1}\left[\frac{\sqrt{s(l_1-s)}}{s-s_0}\,g'(s_1) + \frac{1}{\ln l_1 - 2\ln 2}\,\frac{g(s)}{\sqrt{s(l_1-s)}}\right]ds.$$

As it was shown in [9], [12], one can apply to integral equation (3.17) with the kernel $\dfrac{M(s,s_0)}{\sqrt{s(l_1-s)}}$ the Fredholm alternative on solvability.

Since integral equation (3.17) is equivalent to the problem $A^1_{\alpha\beta}$, a solution of (3.17) exists by virtue of Theorem 3.1. □

Hence, existence of a solution for the problem $A^1_{\alpha\beta}$ for equation (2.1) is proved for the case of $k(y) \equiv 1$.

Remark 2. For cases b) and c) Theorem 3.2 is proved without requirement of the condition (3.10).

Remark 3. Solvability of the problem $A^k_{\alpha\beta}$ in a general case and the cases such as $k(y) = y^m$ and $k(y) = \text{sign}(y)|y|^m$, $m \geq 0$ require an independent investigation.

References

[1] Bitsadze A.V. *Some classes of partial differential equations.* Nauka, Moscow. 1981 (in Russian).

[2] Bougoffa L. A third-order nonlocal problem with nonlocal conditions.//*Internat. J. Math. and Math. Sci.* 34(28), 1501–1507. 2004.

[3] Bouziani A. On the solvability of a nonlocal problem arising in dynamics of moisture transfer.//*Georgian Math. J.* 10(4), 607–627. 2003.

[4] Brekhovskikh L.M., Goncharov V.V. *Introduction to the continuum mechanics.* Nauka, Moskow. 1982 (in Russian).

[5] Daynyak V.V., Korzyuk V.I. Problem of the Dirichlet type for linear differential equation of the third order.// *Differens. uravneniya.* 23(5), 867–872. 1987.

[6] Dzhuraev T.D. *Boundary value problems for the mixed and mixed-composite type equations.* Fan, Tashkent, 1979 (in Russian).

[7] Dzhuraev T.D., Zikirov O.S. On some problems for a partial third order differential equation //*Uzbek math. journal,* 3, 13–25. 2006.

[8] Gabov S.A., Sveshnikov A.G. *Linear problems of non-stationary inner waves.* Nauka, Moscow. 1990 (in Russian).

[9] Gakhov F.D. *Boundary value problems.* Nauka, Moscow. 1977 (in Russian).

[10] Guseinov E.A., Ilinskiy A.S. The integral equations of the first kind with logarithmic singularity in the kernel and their using in the problems of the diffraction by thin screen.// *J. Vich. Math. and Math. Phys.* 27(7), 1050–1057. 1987.

[11] Hapayev M.M. The numerical inversions of some integral equations.// *Differens. uravneniya.* 17(7), 1328–1339. 1981.

[12] Muskheleshvili I.N. *Singular integral equations.* Nauka, Moscow. 1968 (in Russian).

[13] Nakhushev A.M. *Problems with Displacements for Partial Differential Equations.* Nauka, Moskow. 2006 (in Russian).

[14] Salakhitdinov M.S. *Equation of mixed-composite type.* Fan, Tashkent. 1974 (in Russian).

[15] Zikirov O.S. On boundary-value problem for hyperbolic-type equation of the third order.//*Lietuvos Matematikos Rinkinys.* 47(4), 591–603. 2007.

O.S. Zikirov
Faculty of Mechanics and Mathematics
National University of Uzbekistan
VUZgorodok
Tashkent, 100174, Uzbekistan
e-mail: zikirov@yandex.ru

Operator Theory:
Advances and Applications, Vol. 216, 343–367

Shape-morphic Metric, Geodesic Stability

Jean-Paul Zolésio

Communicated by F. Bucci and I. Lasiecka

Abstract. We extend the *Courant* metric in shape analysis to the non smooth family of measurable sets with some Sobolev regularity (this class contains the bounded perimeter sets). The one-to-one flow transformations are replaced by the tube connection concept and the compactness leading to existence for shortest path relays on some BV like perimeter boundedness. This Sobolev perimeter turns to be shape differentiable (in the classical sense, see [4]) so it leads to Sobolev curvature. We derive some stability property for the shortest path achieving the metric. In order to be optimally connected, two such sets can have completely different topologies. We define for each $\epsilon > 0$ a complete pseudo metric in the sense that the triangle axiom is reached up to a multiplicative factor 2^ϵ. With $\epsilon = 0$ we get a metric but we loose some stability properties.

Mathematics Subject Classification (2000). 49Kxx; 51Kxx.

Keywords. Moving domain, topological change, shape metric, Euler equation.

1. Introduction

1.1. Shape and set metrics

The *Courant metric* developed by A.M. Micheletti [19] for smooth domains and extended in [11] to more general setting is obtained as an infimum on all transformations T which are decomposable in $T = (I + h_1) \circ (I + h_2) \circ \cdots \circ (I + h_k)$, the infimum being taken on all h_k and all k. This metric extends for families of submanifolds and geodesic theory can be done using the Eulerian approach developed in [11], [31]. In doing so it appears that the Courant metric can be directly formulated in Eulerian framework. As far as we consider only families of measurable subsets in $D \subset R^N$ ([1]), the transformation T is then *relaxed* by the convection

[1]The analysis could be done without any change in a smooth manifold $M \subset R^N$ by considering Vector fields tangent to M

problem (1.1). Then as we escape to any flow mapping we are able to enlarge the study to families of sets with possible different topologies. We replace the notion of transformation by connecting tubes and the geodesic will be an optimal tube, solution to a variational problem whose vector field V is a solution to the Euler equation. In doing so we also have a non stochastic variational approach for the solution of the Euler equation with some surface tension like term at the boundary of the tube connected to the initial-final condition.

1.2. Tube analysis

We consider a bounded smooth domain $D \subset R^N$. We designate by χ_Ω (or ζ_Ω, or simply ζ) the characteristic function of a measurable subset $\Omega \subset D \subset R^N$. We consider an admissible family $\mathbf{B}_r^p(\Omega)$ of measurable subsets with given measure a (see 2.2). For any pair (Ω_0, Ω_1) in this family we consider the set of connecting tubes $\zeta(t,x) = \zeta(t,x)^2 \in C^0([0,1], L^1(D))$ such that $\zeta(i,x) = \zeta_{\Omega_i}(x)$, $i = 0,1$, and verifying $\forall t \in I$, $\int_D \zeta(t,x)dx = a$, where $I = [0,1]$ will designate the time interval (the final time could be any $\tau > 0$, then we choose $\tau = 1$). The *Eulerian approach* consists in considering the connecting tubes ζ as solutions to the weak convection (1.1) associated to a free divergence speed vector field V([2]): being given $\Omega_i,\ i = 0,1$ subsets in $D \subset R^N$ with $\mathrm{meas}(\Omega_i) = a > 0$,

$$\zeta^2 = \zeta,\ \ \frac{\partial}{\partial t}\zeta + \nabla\zeta.V = 0,\ \ \zeta(i) = \chi_{\Omega_i},\ i = 1,2\ . \tag{1.1}$$

For any such V the problem (1.1) may have no solution or several solutions, so the *product space* tool (see [24]) is to consider the closed *non convex* non empty connecting set:

$$\mathbf{T}(\Omega_0, \Omega_1) = \{(\zeta, V) \in C^0(\bar{I}, L^2(D)) \times L^2_{\mathrm{div}},\ \text{verifying (1.1)}\ \}, \tag{1.2}$$

where

$$L^2_{\mathrm{div}} = \{V \in L^2(I, L^2(D, R^N)),\ \mathrm{div}\, V = 0,\ V.n = 0\ \text{on}\ \ \partial D\ \}$$

1.3. Group like structure

As we have no flow mapping associated with the vector field V (nor a.e. flow, as we don't assume any BV property on V), we nevertheless obtain obvious transitions and inverse elements as follows:

1.3.1. Transition: let $(\zeta^1, V^1) \in \mathbf{T}_r(\Omega^0, \Omega^1)$, $(\zeta^2, V^2) \in \mathbf{T}_r(\Omega^1, \Omega^2)$, then the piecewise defined element

$$\begin{aligned}(\zeta(t), V(t)) &= (\zeta^1(2t), 2\,V^1(2t)), & 0 < t < 1/2,\\ &= (\zeta^2(2t-1), 2\,V^2(2t-1)), & 1/2 < t < 1\end{aligned} \tag{1.3}$$

is an element of $\mathbf{T}_r^p(\Omega_0, \Omega_2)$.

1.3.2. Inverse: The "backward" element

$$(\zeta^-(t), 2\,V^-(t))\ :=\ (\zeta^1(1-t), -\,V^1(1-t))\ \in \mathbf{T}_r^p(\Omega_1, \Omega_0) \tag{1.4}$$

[2] It obviously extends to $\mathrm{div}\, V \in L^2(I \times D)$.

1.3.3. "Algebra": Being given a connecting tube and a smooth vector field $\mathbf{Z}(s,t,x)$, where s is interpreted as a small perturbation parameter, using its flow mapping we construct a "s-perturbed connecting tube". More precisely let $\mathbf{Z}(s,t,x)$ be a smooth free divergence vector field, $\mathbf{Z} \in C^0(\bar{I} \times \bar{I}, C^\infty_{\text{comp}}(D, R^N))$, verifying

$$\mathbf{Z}(s,0,x) = \mathbf{Z}(s,1,x) = 0 \ \text{ in } \ D.$$

Then for all tubes $(\zeta, V) \in \mathbf{T}(\Omega_0, \Omega_1)$ we get a "$s - \mathbf{Z}$ perturbed connecting tube" (see Theorem 5.4 bellow)

$$(\zeta^s, V^s) \in \mathbf{T}(\Omega_0, \Omega_1), \tag{1.5}$$

where

$$\zeta^s = \zeta \circ T_s(\mathbf{Z})^{-1},$$

and

$$V^s = [\ \ D(T_s(\mathbf{Z})).V(t) \ + \ D(T_s(\mathbf{Z}))^{-1}.\frac{\partial}{\partial t}(T_s(\mathbf{Z})) \ \ ] \circ T_s(\mathbf{Z})^{-1}.$$

Remark: the reverse perturbation analysis works for smooth tubes and is not obviously compatible with the connecting concept. We used it for control issues involving moving boundaries in [28], [32], [33]... Let

$$V \in E := W^{1,\infty}(D, R^N) \cap L^2_{\text{div}}(D).$$

Let $(\zeta, V) \in \mathbf{T}_r(\Omega^0, \Omega^1)$ and let V^s be a perturbation of the vector field V, $V^s \in E$, for example $V^s = V + sW$. Then there exists a smooth "transverse" vector field $\mathbf{Z}(s,t,x)$ such that, denoting $Z^t(s,x) = Z(s,t,x)$,

$$\forall t, s \in I, \quad T_s(Z^t) \circ T_t(V) \ = T_t(V^s).$$

Moreover the field $Z(t,x) := \mathbf{Z}(0,t,x)$ is solution to the following evolution problem

$$Z(0) = 0, \ \ \frac{\partial}{\partial t} Z + [Z, V] = W$$

where the Lie bracket is $[Z, V] = DZ.V \ - \ DV.Z$

1.4. The shape morphic metric

We shall consider several energy functionals $\mathbf{E}(V, \zeta)$ associated with several parameters p, ϵ, h, r. The basic idea to derive a metric is to consider $p = 1, \epsilon = 0$. Two main difficulties arose: for existence of geodesics (i.e., compactness results) we need $p > 1$ so we shall deal with pseudo-metric with complete pseudo metric space or simply metric space (with or without existence of geodesic). Also the perimeter term must be replaced by a time capacity term $\theta_{h,r}$ in order to obtain the first metric axiom as any perimeter term never vanishes (for a given volume of the set). The candidate for the *morphic metric* is then in the form

$$\mathbf{d}(\Omega_0, \Omega_1) \ = \text{INF}_{(\zeta,V) \in \mathbf{T}(\Omega_0,\Omega_1)} \ \ \mathbf{F}(\zeta, V),$$

where $\mathbf{F}$ includes an additive regularizing term which is a surface tension like term classically needed in order to make use of the parabolic compactness of tubes:

$$\mathbf{F}(\zeta, V) = \ \mathbf{E}_{\epsilon,p}(\zeta, V) \ + \ \sigma \ \theta_{h,r}(\zeta).$$

And, with $\epsilon \geq 0$, $p \geq 1, \alpha \geq 0$, $\beta > 0$,

$$\mathbf{E}_{\epsilon,p}(\zeta, V) = \int_0^1 \left(\int_D (\alpha\, \zeta(t,x) + \beta)\, |V(t,x)|^p dx \right)^{(1+\epsilon)/p} dt.$$

In order to derive existence results we shall minimize with respect to (ζ, V) in some subset $\mathbf{T}_r(\Omega_0, \Omega_1)$ of $\mathbf{T}(\Omega_0, \Omega_1)$. From necessary condition, the extrema solve the Euler incompressible flow (at $\epsilon = 0, p = 2$):

$$\alpha \geq 0,\, \beta > 0,\ \frac{\partial}{\partial t}(\,(\alpha\zeta + \beta)\, V) \,+\, D(\,(\alpha\zeta + \beta)\, V).V + \nabla \mathbf{P} = \sigma\, \vec{\mathbf{H}}_{h,r}. \tag{1.6}$$

Where $\mathbf{P}$ is a pressure term associated with the free divergence of the vector field V, $\sigma \in R$ is a surface tension-like coefficient while $\vec{\mathbf{H}}_{h,r}$ is a new curvature concept: we introduce the Sobolev perimeter $P_{h,r}(\Omega_t)$ associated with the $\mathbf{H}^r(D))$ norm of $\zeta(t,.)$; its shape gradient will furnish the Sobolev curvature $\vec{\mathbf{H}}_{h,r}$. The advantage of this Sobolev perimeter is that it turns to be shape differentiable under smooth transverse fields perturbations ζ_s and enables us to define the Sobolev curvature for any domain in this new class of Sobolev sets, so that $\vec{\mathbf{H}}_{h,r}$ is the Sobolev curvature of the *interface* associated with the Sobolev perimeter $P_{h,r}(\zeta(t)$. These elements are introduced bellow. Notice that with the choice of the parameters $\alpha = 0$, $\beta = 1$, equation (1.6) is the classical Euler equation for incompressible fluids but with non initial (or final) conditions but with the only condition that the solution V will convect ζ_{Ω_0} onto ζ_{Ω_1} at final time.

The tube approach was introduced in [23], [24] for connecting two given domains whose characteristic functions has some "Sobolev smoothness": $\zeta_i \in \mathbf{H}^r(D)$, for given r such that $0 < r < 1/2$ (this includes the usual finite perimeter sets).

2. Tube variational principle

For measurable subset $Q \subset I \times D \subset R^{N+1}$, we shall write ζ_Q for the characteristic function and denote by $\Omega_t,\ a.e.t \in I$, the measurable subset in D (defined up to a subset with zero measure) such that $\zeta_Q(t,.) = \chi_{\Omega_t}$. We say that Q is a tube when we have some continuity, $\zeta \in C^0(\bar{I}, L^1(D))$, more precisely we will consider *Eulerian* description for the tube and introduce a minimal regularity on the *speed* vector field V, $V \in L^p_{\text{div}}$ in order to insure this continuity. This continuity enables us to consider connecting tubes: being given two measurable subsets in D, a tube Q *connects* Ω_0 and Ω_1 if we have $\zeta(i) = \chi_{\Omega_i},\ i = 0, 1$. We shall consider a framework such that the set $\mathbf{T}(\Omega_0, \Omega_1)$ is non empty. For functional F the optimal solution (ζ, V) will solve a classical Euler equation for incompressible fluid which will not simplify to a Hamilton-Jacobi equation: the field V will not derive from a potential as its curl will not be zero. Indeed the new curvature term that we shall introduce will lead to a generalized curvature term on the boundary of the connecting tube which, in dimension $N = 3$, generates a curl term in the equation.

We adopt the convention that for $r = 0$ the space $\mathbf{H}^r(D))$ stands for the Banach space $BV(D)$, so that for $0 \leq r < 1/2$, $\mathbf{H}^r(D) \subset L^1(D, R^N)$, with continuous and compact inclusion mapping. Notice that from the Luigi Ambrosio's results (see for example [9]), the convection problem is uniquely solved under $L^1(I, BV(D))$ like assumption on the field V. This extra regularity on the vector field V would make the set $\mathbf{T}(V, \Omega_0, \Omega_1) = \{\zeta \ s.t. \ (\zeta, V) \in \mathbf{T}(\Omega_0, \Omega_1) \ \}$ reduced to a single element but would imply some viscosity modelling (e.g., some Navier-Stokes like flow in Eulerian view point). Here we escape to any renormalization benefit, so the solution ζ may be non unique but the regularity $\zeta = \zeta^2 \in L^1(0, 1, \mathbf{H}^r(D))$ will be derived from the variational principle itself (see also [29], [24], [23]).

2.1. Speed vector fields

With $1 \leq p < \infty$, we introduce

$$L^p_{\text{div}} = \{V \in L^p(I \times D, R^N) \ \ s.t. \ \ \text{div}\, V = 0, \quad V.n_D = 0 \ \}.$$

Proposition 2.1. *Let $V \in L^p_{\text{div}}$ and $\zeta = \zeta^2 \in L^\infty(I \times D)$ be solution to*

$$\frac{\partial}{\partial t}\zeta + \nabla\zeta.V = 0,$$

then $\zeta \in C^0(I, L^1(D))$.

Proof. The convection equation implies that: $\zeta_t = \text{div}(\ -\zeta\, V\ ) \in W^{-1,1}(D))$, then

$$\zeta \in C^0(I, W^{-1,1}(D)).$$

And as $\zeta = \zeta^2$, the $L^1(D)$ continuity derives from the following

Lemma 2.2. *Let $\zeta = \zeta^2 \in L^1(I \times D) \cap C^0(I, \mathbf{D}'(D))$, then $\zeta \in C^0(I, L^1(D))$.*

Proof of the lemma. As

$$||\zeta(t+s) - \zeta(t)||_{L^1(D)} \ = \ ||\zeta(t+s) - \zeta(t)||^2_{L^2(D)}\,,$$

then it is enough to show that $\zeta \in C^0(I, L^2(D))$. We begin by establishing the weak $L^2(D)$ continuity: for any element $f \in L^2(D)$, consider

$$\int_D (\zeta(t+s)(x) - \zeta(t)(x))\, f(x) dx = \int_D (\zeta(t+s,x) - \zeta(t,x)\, \phi(x) dx$$
$$+ \int_D (\zeta(t+s,x) - \zeta(t,x))\, (f(x) - \phi(x)) dx.$$

Let be given $r > 0$, by the choice of $\phi \in \mathbf{D}(D)$ (using here the density of $\mathbf{D}(D)$ in $L^2(D)$), we have

$$\left| \int_D (\zeta(t+s,x) - \zeta(t,x))\, (f(x) - \phi(x)) dx \right| \ \leq \ 2 \int_D |f(x) - \phi(x)| dx \leq r.$$

So we derive the continuity for the weak $L^2(D)$ topology. To reach the strong topology it sufficient now to consider the continuity of the mapping

$$t \to \int_D |\zeta(t,x)|^2 dx = \int_D \zeta(t,x) dx = ((\zeta(t), 1))_{L^2(D)}.$$

This continuity property enables us to define the connecting concept. Being given two measurable subsets (defined up to a zero measure subset)

$$\Omega_i \subset D, \mathrm{meas}(\Omega_i) = a, \;\; i = 0, 1\,,$$

we consider the family of connecting tubes

$$\mathbf{T}_r^p(\Omega_0, \Omega_1) = \{(\zeta, V) \in L^1(I, \mathbf{H}^r(D)) \times L^p_{\mathrm{div}}, \\ \text{verifying (1.1)},\; \zeta(i) = \chi_{\Omega_i},\; i = 0, 1\,\}.$$

2.2. Non empty family of connecting tubes

In order to handle non empty tubes, we consider a given measurable subset $\Omega \subset D, \mathrm{meas}(\Omega) = a$, and its connected family

$$\mathbf{B}_r^p(\Omega) = \{\omega \subset D \;s.t.\; \exists(\zeta, V) \in C^0(\bar{I}, L^1(D)) \cap L^1(I, \mathbf{H}^r(D)) \times L^p_{\mathrm{div}},\; s.t.\; \zeta = \zeta^2, \\ \zeta_t + \nabla\zeta.V = 0, \quad \zeta(0) = \chi_\Omega, \;\; \text{and} \;\; \chi_\omega = \zeta(\mathbf{1})\,\}. \tag{2.1}$$

It is important to notice that if $\Omega \in \mathbf{C}_r$ defined below at (3.1), the continuously moving domain Ω_t such that $\chi_{\Omega_t} = \zeta(t, .)$ is in $\mathbf{C}_r$ for almost every t, but not necessary for $t = 1$, so that the family $\mathbf{B}_r(\Omega)$ is not a subfamily of $\mathbf{C}_r$. Moreover as $V \in L^p_{\mathrm{div}}$, the moving connecting domain verifies $\mathrm{meas}(\Omega_t) = \int_D \zeta(t, x)dx = a > 0$ $a.e.t$, so it is not empty at $a.e.$ time.

By construction we have:

Theorem 2.3. *For any pair of sets $\Omega_i \in \mathbf{B}_r(\Omega)$, $i = 0, 1$, the connecting tube $\mathbf{T}_r(\Omega_0, \Omega_1)$ is non empty.*

Proof. Let $(\zeta^i, V^i) \in \mathbf{T}_r(\Omega, \Omega^i)$, then the piecewise defined element

$$(\zeta(t), V(t)) = (\zeta^0(1 - 2t), -2\,V^0(1 - 2t)), \quad 0 < t < 1/2, \\ = (\zeta^1(2t - 1), 2\,V^1(2t - 1)), \quad 1/2 < t < 1 \tag{2.2}$$

is an element of $\mathbf{T}_r^p(\Omega_0, \Omega_1)$.

3. Subsets in D with bounded Sobolev perimeter

We consider families of measurable subsets in D with perimeter-like properties: let $r \in [0, 1/2[$ and denote by $\mathbf{C}_r$ the family of measurable subsets in D with given measure a, $0 < a < \mathrm{meas}(D)$, defined as follows:

i) for $r = 0$, $\mathbf{C}_0 = \{\Omega \subset D \;s.t.\; \chi_\Omega \in BV(D),\; \mathrm{meas}(\Omega) = a\;\}$

ii) for $0 < r < 1/2$, as we know from [11] that $\{\zeta = \zeta^2 \in BV(D)\,\} \subset \mathbf{H}^r(D)$, we can relax the space $BV(D)$ by $\mathbf{H}^r(D)$, then we set:

$$\mathbf{C}_r = \left\{\zeta = \zeta^2 \in \mathbf{H}^r(D),\; \int_D \zeta(x)dx = a\right\}. \tag{3.1}$$

Theorem 3.1. *For $0 < r < 1/2$, $\mathbf{C}_r$ is weakly closed in $\mathbf{H}^r(D)$ and any bounded part is relatively compact in $\mathbf{C}_{r'}$ for any r', $0 < r' < r < 1/2$.*

For $r = 0$, $\mathbf{C}_0$ is weakly closed in $BV(D)$ and any bounded part is relatively compact in $L^1(D)$.

For given $h > 0$ we introduce

$$|\Omega|_{\mathrm{loc}(h,r)} = \int\int_{D\times D\cap\{|x-y|<h\}} \left(1 - \frac{|x-y|^2}{h^2}\right) \frac{|\zeta_\Omega(x) - \zeta_\Omega(y)|}{|x-y|^{N+2r}}\, dxdy$$
$$\leq ||\zeta_\Omega||^2_{\mathbf{H}^r(D)}. \tag{3.2}$$

With $\Omega^c = D \setminus \Omega$ we get:

$$|\Omega|_{\mathrm{loc}(h,r)} = 2\int\int_{\Omega\times\Omega^c\cap\{|x-y|<h\}} \left(1 - \frac{|x-y|^2}{h^2}\right) \frac{1}{|x-y|^{N+2r}}\, dxdy.$$

3.1. Sobolev perimeter

In order to define the Sobolev perimeter we first consider the smooth domain situation: if the boundary $\Gamma = \partial\Omega$ is a C^2 manifold then with $j_z^3(x) = 1 + zH + z^2K$ (where H and K are the mean and Gauss curvature of the surface Γ, we assume $N = 3$), we get

$$|\Omega|_{\mathrm{loc}(h,r)} = 2\int_\Gamma \left(\int_{-h}^0 \left(j_z^N(x)\left\{\int_{B_h(x+T_z(x))\cap\Omega^c} \times\frac{(1 - (|T_z(x) - y|^2/h^2)^+}{|T_z(x) - y|^{N+2r}} dy\right\}\right) dz\right)\, d\Gamma(x).$$

Assuming now that h is small enough compare to the curvatures we locally approximate in the ball $B_h(x)$ the piece of boundary $\Gamma \cap B(x + T_z(x))$ by a linear space. The term

$$m(h, x, z) = \int_{B_h(x+T_z(x))\cap\Omega^c} \frac{[1 - |T_z(x) - y|^2/h^2]^+}{|T_z(x) - y|^{N+2r}} dy$$

is no more depending on the point $x \in \Gamma$ so that we set

$$m(h, z) := \int_{B_h(0+T_z(0))\cap\Omega^c} \frac{[1 - ((z + y_2)^2 + y_1^2)/h^2]^+}{((z + y_2)^2 + y_1^2)^{N/2+r}} dy.$$

We set

$$M(h) = 2\int_{-h}^0 m(h, z)dz.$$

Then we get

$$|\Omega|_{\mathrm{loc}(h,r)} = M(h)\int_\Gamma d\Gamma(x) + o(h), \quad h \to 0.$$

Necessarily, as $||\zeta_\Omega||_{\mathbf{H}^r(D)} < \infty$, this term as a finite limit but this limit is zero:

Proposition 3.2. $$|\Omega|_{\mathrm{loc}(h,r)} \to 0, \quad h \to 0.$$

Proof. With $E_h = \{|x-y| \le h\}, \text{meas}(E_h) \to 0$ and $\zeta_{E_h} F \le F$ with

$$F = \frac{|\zeta_\Omega(x) - \zeta_\Omega(y)|}{|x-y|^{N+2r}} \in L^1(D \times D).$$

3.2. Asymptotic analysis when $h \to 0$

Proposition 3.3. *For any r, $0 < r < 1/2$, there exists a constant $a(r)$ such that*

$$M(h)/h^{1-2r} = a(r) \ + o(1), \quad h \to 0 \tag{3.3}$$

Proof for $N = 2$. We get:

$$m(h,z) = 2\int_0^{\sqrt{h^2-z^2}} du \Big(\int_0^{\sqrt{h^2-u^2}} \times [1-((z+v)^2+u^2)/h^2]^+ \, ((z+v)^2+u^2)^{-(N/2+r)} \, dv \Big)$$

$$M(h) = 2\int_{-h}^0 dz \Big\{ \int_0^{\sqrt{h^2-z^2}} du \Big(\int_0^{\sqrt{h^2-u^2}} \times \ [1-((z+v)^2+u^2)/h^2]^+ \, ((z+v)^2+u^2)^{-(1+r)} dv \Big) \Big\},$$

with $Z = 1/h\, z$, we get

$$M(h) = 2h\int_{-1}^0 dZ \Big\{ \int_0^{h\sqrt{1-Z^2}} du \Big(\int_0^{\sqrt{h^2-u^2}} \times \ [1-((hZ+v)^2+u^2)/h^2]^+ \, ((hZ+v)^2+u^2)^{-(1+r)} dv \Big) \Big\}.$$

With $U = 1/h\ u$ we get

$$M(h) = 2h^2\int_{-1}^0 dZ \Big\{ \int_0^{\sqrt{1-Z^2}} dU \Big(\int_0^{h\sqrt{1-U^2}} \times \ [1-((hZ+v)^2+h^2U^2)/h^2]^+ \, ((hZ+v)^2+h^2\,U^2)^{-(1+r)} dv \Big) \Big\}.$$

With $V = 1/h\ v$ we get

$$M(h) = 2h^{1-2r}\int_{-1}^0 dZ \Big\{ \int_0^{\sqrt{1-Z^2}} dU \Big(\int_0^{\sqrt{1-U^2}} \times [1-((Z+V)^2+U^2)]^+ ((Z+V)^2+U^2)^{-(1+r)} dV \Big) \Big\}.$$

Notice that as $0 < r < 1/2$ we have $\mu = 1-2r > 0$ and we consider

$$M(h)/h^{1-2r} = a(r) \ + o(1), \tag{3.4}$$

where the main part $a(r)$ is independent on $h,\ h \to 0$, is given by:

$$a(r) = \int_{-1}^{0} dZ \Big\{ \int_{0}^{\sqrt{1-Z^2}} dU \Big(\int_{0}^{\sqrt{1-U^2}} \times [1 - ((Z+V)^2 + U^2)]^+ \, ((Z+V)^2 + U^2)^{-(1+r)} dV \Big) \Big\}.$$

3.2.1. Perimeter. We set

$$P_{h,r}(\Omega) = \frac{1}{2a(r)\ h^{1-2r}} \quad |\Omega|_{\mathrm{loc}(h,r)}. \tag{3.5}$$

And ζ being the characteristic function of Ω we shall also denote this perimeter as being $P_{h,r}(\zeta)$.

Proposition 3.4. *For all r, $0 < r < 1/2$, and any open set $\Omega \subset D$ with C^2 boundary Γ, $\Gamma \subset \bar{D}$ the following asymptotic holds:*

$$P_{h,r}(\Omega) \to \int_{\Gamma \cap D} d\Gamma,\ h \to 0.$$

3.3. Perimeter estimate

For $r = 0$ we have

$$|\zeta_\Omega|_{BV(D)} = |\Omega| + |\nabla \zeta_\Omega|_{M^1(D,R^N)} \leq |D| + P_D(\Omega).$$

Let $0 < r < 1/2,\ h > 0$, consider $\rho_h(r) = (1 - r^2/h^2)^+$, so that

$$P_{h,r}(\zeta(t)) = \frac{1}{a(r)h^{1-2r}} \int\int_{\Omega_t \times \Omega_t^c} \frac{\rho_h(|x-y|)}{|x-y|^{N+2r}}\ dxdy \tag{3.6}$$

we have

Proposition 3.5. $\forall (r,p),\ 0 < r < 1/2,$

$$||\zeta_\Omega||^2_{H^r(D)} \leq \ |D| \ + (\sqrt{2}/h)^{N+2r} |D|^2 \ + \ a(r)h^{1-2r}\ P_{h,r}(\Omega). \tag{3.7}$$

Proof. Notice that

$$P_{h,r}(\zeta) = \frac{1}{a(r)h^{1-2r}} \int\int_{D\times D} \rho_h(|x-y|) \frac{|\zeta(x) - \zeta(y)|}{|x-y|^{N+2r}}\ dxdy.$$

Moreover

$$\begin{aligned} ||\zeta(t)||^2_{H^r(D)} &= |\Omega|^2 + \int\int_{D\times D} \frac{|\zeta(x) - \zeta(y)|}{|x-y|^{N+2r}}\ dxdy \\ &\leq |D|^p + \int\int_{\{|x-y| > h/\sqrt{2}\}} \frac{|\zeta(x) - \zeta(y)|}{|x-y|^{N+2r}}\ dxdy \\ &\vec{\mathbf{H}}_{h,r} + \int\int_{\{|x-y| \leq h/\sqrt{2}\}} \frac{|\zeta(x) - \zeta(y)|}{|x-y|^{N+Zr}}\ dxdy. \end{aligned}$$

As $\rho_h(r) > 1/2$ for $r < h/\sqrt{2}$ we get

$$\leq (\sqrt{2}/h)^{N+r}|D|^2 + \int_\Omega \int_{\Omega^c} \frac{\rho_h(|x-y|)}{|x-y|^{N+2r}} dxdy.$$

That is

$$||\zeta(t)||^2_{H^r(D)} \leq (\sqrt{2}/h)^{N+r}|D|^2 + |\Omega|_{\mathrm{loc}(h,r)}.$$

3.4. Sobolev mean curvature $\vec{\mathbf{H}}_{h,r}(\zeta)$

When $\zeta \in BV(D)$ the perimeter in D is given by

$$P_D(\Omega) = ||\nabla\zeta||_{M^1(D,R^N)}.$$

For a given smooth vector field Z the perimeter $P(\Omega_s)$ of the perturbed domain $\Omega_s = T_s(Z)(\Omega)$ is not differentiable with respect to s. When the boundary Γ is a smooth manifold then it is differentiable and we have:

$$\frac{\partial}{\partial s} P_D(\Omega_s)_{\{s=0\}} = \int_\Gamma \Delta b_\Omega \;< Z(0), n >\, d\Gamma.$$

Where $H = \Delta b_\Omega$ is the mean curvature of Γ, so that $H\vec{n}$ appears as the shape gradient of the perimeter (for smooth domains). In the general situation (for non smooth domains), the BV perimeter being not shape differentiable, we introduced the h-Sobolev perimeter which is shape differentiable, its shape gradient will be the h-Sobolev curvature. We first analyse the h-Sobolev-perimeter shape derivative; this term turns to be always differentiable with respect to the transverse perturbations as follows: let us consider some "small" parameter s (perturbation parameter) and any smooth vector field, $\mathbf{Z}(s,x)$, $\mathbf{Z} \in C^0([0,s_0[,\mathbf{D}(D,R^N))$ such that $\mathrm{div}_x \mathbf{Z}(s,.) = 0$. As usual we designate by $T_s(\mathbf{Z})$ its flow mapping and consider the Sobolev perimeter of the s-perturbed set:

$$\begin{aligned} &P_{h,r}(\zeta_\Omega \, o\, T_s(\mathbf{Z})^{-1}) \\ &\quad = 2\frac{1}{a(r)h^{1-2r}} \int_{\Omega\times\Omega^c} \frac{[1-|T_s(\mathbf{Z})(x) - T_s(\mathbf{Z})(y)|^2/h^2]^+}{||T_s(\mathbf{Z})(x) - T_s(\mathbf{Z})(y)||^{N+2r}} \, dxdy. \end{aligned}$$

So that, with $\zeta^s = \zeta_\Omega \, o\, T_s(\mathbf{Z})^{-1} = \zeta_{T_s(\mathbf{Z})(\Omega)}$, we have:

$$\begin{aligned} \frac{\partial}{\partial s} P_{h,r}(\zeta^s)_{s=0} &= -2\,\frac{1}{a(r)h^{1-2r}} \\ &\times \Bigg[(N+2r) \int_{\Omega\times\Omega^c} \frac{[1-||x-y||^2/h^2]^+}{||x-y||^{N+2r}} \left\langle \frac{x-y}{||x-y||}, \frac{Z(x)-Z(y)}{||x-y||} \right\rangle dxdy \\ &\quad - \int_{\Omega\times\Omega^c \cap \{|x-y|<h\}} \frac{1}{||x-y||^{N+2r}} \left\langle \frac{x-y}{h^2}, Z(x) - Z(y) \right\rangle dxdy \Bigg]. \end{aligned} \tag{3.8}$$

As $||x-y|| \leq h$ in the previous integrals we have:

$$||Z(x) - Z(y)|| \leq \; h\, ||DZ||_{L^\infty(D,R^{N^2})}.$$

Then there exists a measure $\vec{\mathbf{H}}_{h,r}(\zeta)$ such that

$$\langle \vec{\mathbf{H}}_{h,r}(\zeta),\ Z\rangle = \frac{\partial}{\partial s} P_{h,r}(\zeta^s)_{s=0}.$$

3.4.1. Smooth domains. If $\zeta = \zeta_\Omega$ the set Ω being a smooth domain with boundary Γ then the measure $\vec{\mathbf{H}}_{h,r}(\zeta)$ is supported by the tubular neighbourhood of the boundary:

$$\mathbf{U}_h(\partial\Omega) = \cup_{x\in\partial\Omega} B(x,h).$$

Moreover,

Lemma 3.6. *for all $r < 1/2$ and smooth "transverse field" Z, we have the following convergence*

$$\int_0^s \langle \vec{\mathbf{H}}_{h,r}(\zeta^\sigma), Z\rangle d\sigma \to \int_0^s \int_{\Gamma^\sigma} H^\sigma \vec{n}^\sigma.Z\, d\Gamma^\sigma\, d\sigma,\ \ as\ \ h \to 0.$$

Indeed

$$\int_{\Gamma^s} d\Gamma^s = \int_\Gamma d\Gamma + \int_0^s \int_{\Gamma^\sigma} H^\sigma \vec{n}^\sigma.Z\, d\Gamma^\sigma\, d\sigma.$$

Also

$$P_{h,r(\zeta^s)} = P_{h,r(\zeta)} + \int_0^s \langle \vec{\mathbf{H}}_{h,r}(\zeta^\sigma), Z\rangle d\sigma.$$

And

$$P_{h,r}(\zeta^s) \to \int_{\Gamma^s} d\Gamma^s,\ as\ h \to 0$$

and also

$$P_{h,r}(\zeta) \to \int_\Gamma d\Gamma,\ as\ h \to 0.$$

3.5. Time capacity term

An evident candidate for the surface tension like term $\theta_{h,r}$ would be $\int_0^1 P_{h,r}(\zeta(t))dt$. This term would perfectly be transversely shape differentiable and will lead as well to existence result for the minimization then to existence of solution to the Euler equation (4.4) but will not furnish a metric as it is never zero. Then the idea is to replace it by some term in the form $\int_0^1 \frac{\partial}{\partial t} P_{h,r}(\zeta(t))dt$ but as V is a non smooth vector field (conversely to the transverse fields Z) the time derivative of the Sobolev perimeter does not exists. For any tube

$$\zeta \in L^p(I, H^r(D))$$

form (3.2) we have

$$P_{h,r}(\zeta(t)) \le\ a(r)\, h^{1-r}\, ||\zeta(t)||_{H^r(D)}. \tag{3.9}$$

So that $P_{h,r}(\zeta(.)) \in L^p(I)$. We consider the closed convex set,

$$K^p_{h,r}(\zeta) = \Big\{ \nu \in W^{1,p}(I),\ \nu(i) \ge 0,\ i = 0,1, \tag{3.10}$$

$$\int_0^1 P_{h,r}(\zeta(t))\, dt\ \le\ \int_0^1 \nu(t)\ dt\ \nu(0)\ =\ P_{h,r}(\zeta(0)),\ \ \nu(1) = P_{h,r}(\zeta(1))\,)\Big\}$$

$$\theta^p_{h,r}(\zeta) = \mathrm{INF}_{\{\nu\in K^p_{h,r}(\zeta)\}} \int_0^1 |\nu'(t)|^p\,dt. \tag{3.11}$$

Obviously the convex set $K^p_{h,r}(\zeta)$ is never empty as the constraint is only on the mean value. As

$$\int_0^1 \nu(t)dt = \nu(0) + \int_0^1 (1-t)\nu'(t)dt \le \nu(0) + \left(\int_0^1 |\nu'(t)|^p dt\right)^{1/p}$$

then

$$\int_0^1 P_{h,r}(\zeta(t))\,dt \le \int_0^1 \nu(t)\,dt \le \nu(0) + \left(\int_0^1 |\nu'(t)|^p dt\right)^{1/p}.$$

From (3.7) we have:

Proposition 3.7. $\forall\nu\in K^p_{h,r}(\zeta)$,

$$\int_0^1 ||\zeta_{\Omega_t}||^2_{H^r(D)}\,dt \le |D| + (\sqrt{2}/h)^{N+2r}|D|^2 \tag{3.12}$$
$$+\, a(r)h^{1-2r}\left[P_{h,r}(\zeta(0)) + P_{h,r}(\zeta(1)) + \left(\int_0^1 |\nu'(t)|^p dt\right)^{1/p}\right]$$

4. Shape-morphing pseudo-metric on $\mathbf{B}^p_r(\Omega)$

The scaling parameter $h > 0$ and the Sobolev weight r, $0 < r < 1/2$ being fixed in this analysis, the metric associated to $p \ge 1$ and $\epsilon > 0$ is

$$d^{\epsilon,p}_{h,r}(\Omega_0,\Omega_1) \tag{4.1}$$
$$:= \mathrm{INF}_{\{(\zeta,V)\in\mathbf{T}^p_r(\Omega_0,\Omega_1)\}} \int_0^1 \left(\int_D (\alpha+\beta\zeta)||V(t,x)||^p dx\right)^{(1+\epsilon)/p} dt. + \theta^p_{h,r}(\zeta)$$

Theorem 4.1. *For $p \ge 1$, $\epsilon > 0$, $0 < r < 1/2$, ,$d^{p,\epsilon}_{h,r}$ is a ϵ-quasi metric on $\mathbf{B}^p_r(\Omega)$: $\forall(\Omega_0,\Omega_1,\Omega_1) \in B^p_r(\Omega)^3$,*

$$d^{p,\epsilon}_{h,r}(\Omega_0,\Omega_1) = 0 \;\; \textit{iff} \;\; \Omega_0 = \Omega_1, \quad d^{p,\epsilon}_{h,r}(\Omega_0,\Omega_1) = d^{p,\epsilon}_{h,r}(\Omega_1,\Omega_0)$$

$$d^{p,\epsilon}_{h,r}(\Omega_0,\Omega_2) \le 2^\epsilon\,(\,d^{p,\epsilon}_{h,r}(\Omega_0,\Omega_1) + d^{p,\epsilon}_{h,r}(\Omega_1,\Omega_2)\,).$$

Notice that with $p \ge 1$, $d^{p,0}_{h,r}$ is a metric on $\mathbf{B}(\Omega)$.

Theorem 4.2. *Let $p > 1$, $\epsilon > 0$, , $0 < r < 1/2$, equipped with $d^{p,\epsilon}_{h,r}$ the family $\mathbf{B}^p_r(\Omega)$ is a complete quasi-metric space. Moreover the geodesic (ζ,V) between to elements Ω_i, $i = 0,1$ solves the following Euler problem: there exist some "pressure" term*

$\mathbf{P} \in \mathbf{D}'(D)$ *and some surface tension* $\sigma \in R$ *such that, the Sobolev curvature measure* $\vec{\mathbf{H}}_{h,r} = \vec{\mathbf{H}}_{h,r}(\zeta)$ *being defined previously,*

$$\frac{\partial}{\partial t}\zeta + \nabla\zeta.V = 0, \;\; \zeta(0) = \chi_{\Omega_0}, \;\; \zeta(1) = \chi_{\Omega_1} \tag{4.2}$$

$$\operatorname{div} V = 0, \;\; \zeta = \zeta^2 \tag{4.3}$$

$$\frac{\partial}{\partial t}(\,(\alpha\zeta+\beta)||V||^{p-2}\,V) + D((\alpha\zeta+\beta)||V||^{p-2}V).V + \nabla\mathbf{P} = \sigma\,\vec{\mathbf{H}}_{h,r}. \tag{4.4}$$

4.1. This Euler equation does not reduce to Hamilton-Jacobi equation (for some scalar potential)

It is an important point that the right-hand sides in the previous euler flow equation is not curl free, so it does not derives from a potential and the geodesic field V does not reduces to a gradient term as in a incompressible perfect fluid. Indeed the support of $\operatorname{curl} V$ is included in the boundary of the moving set Ω_t In the very simple situation of $B(D) = BV(D)$ and Γ_t is a smooth surface we would get

$$\langle \operatorname{curl}(\vec{\mathbf{H}}_{h,r}), Z\rangle = \langle \vec{\mathbf{H}}_{h,r}, \operatorname{curl} Z\rangle = \int_0^1\int_{\Gamma_t} H_t n_t.\operatorname{curl} Z(t)\, d\Gamma_t(x)dt$$

$$= \int_0^1\int_{\Gamma_t} H_t \operatorname{div}_{\Gamma_t}(n_t \times Z(t)\,)\; d\Gamma_t(x)dt = -\int_0^1\int_{\Gamma_t} (\nabla_{\Gamma_t}H_t \times n_t).Z(t)\;\; d\Gamma_t(x)dt.$$

And γ_t being the trace operator on the manifold Γ_t:

$$\operatorname{curl}\mu(t) = \gamma^*_{\Gamma_t}.(\nabla_{\Gamma_t}H_t \times n_t).$$

Which is zero if and only if the surface Γ_t has a constant mean curvature. Still assuming the interface Γ_t to be a smooth manifold we would get the restrictions of V to the open domains Ω_t and Ω_t^c as gradients so that would be in the following form: $V = \chi_{\Omega_t}\nabla\phi_1(t) + (1-\chi_{\Omega_t})\nabla\phi_2(t)$. This morphic metric can be handled numerically. In this direction we developed several Galerkin approach based on level set parametrization for the moving domain, see [20], [23]. In several experiment the geodesic turns to be numerically stable [12], [13].

In Theorem 4.2 the real parameter σ can be a priori chose, $\sigma > 0$ by simply choosing $\theta^{p,\epsilon}_{h,r} = \int_0^1 P_{h,r}(\zeta(t))dt$. The minimum is reached and any minimizer ζ solves the Euler equation with,the chosen term σ but it fails to be a metric (or a quasi metric), see [22].

5. Proofs

The proof of Theorem 4.1 is similar the ones in [20], [23]. We concentrate on the proof of Theorem 4.2.

5.1. Existence for minimizer

We shall make use of the following compactness result, see [30], [14], [24].

5.1.1. Compactness result.

Theorem 5.1. *Let $p > 1$ and $0 \le r < 1/2$. Consider a sequence $\zeta_n \in C_r$, bounded in $L^1(I, \mathbf{H}^r(D))$ together with $\frac{\partial}{\partial t}\zeta_n$ bounded in $L^p(I, W^{-1,1}(D))$. Then there exists a subsequence and an element $\zeta \in C_r \cap L^1(I, B_r(D)) \cap W^{1,1}(I, W^{-1,1}(D)) \subset C^0(I, L^1(D))$ such that ζ_n strongly converges to ζ in $L^1(I, L^1(D))$ with $\frac{\partial}{\partial t}\zeta \in L^p(I, M^1(D, R))$ verifying*

$$||\zeta||_{L^1(I,\mathbf{H}^r(D))} \le \liminf\ ||\zeta_n||_{L^1(I,\mathbf{H}^r(D))}$$

and

$$||\frac{\partial}{\partial t}\zeta||_{L^p(I,W^{-1,1}(D))} \le \liminf\ ||\frac{\partial}{\partial t}\zeta_n||_{L^p(I,W^{-1,1}(D))}.$$

Moreover we defined the r-perimeters as being:

$$P_0(\zeta(t)) := ||\nabla_x \zeta(t)||_{M^1(D,R^N)},$$

$$r > 0, \quad P_{h,r}(\zeta(t)) = \int\int_{D\times D} \rho_h(|x-y|)|\zeta(x) - \zeta(y))|/|x-y|^{(N+2r)}\ \ dxdy$$

then:

$\zeta(t,x) = \zeta^2(t,x)$, a.e.$(t,x) \in I \times D$ and $\zeta \in C^0(I, L^1(D))$ imply that the mapping:

$$t \in \bar{I} \to P_{h,r}(\zeta(t)) \text{ is l.s.c.} \tag{5.1}$$

5.1.2. Existence of minimizing tube.

Proposition 5.2. *Let $\zeta_n \in C_r$ be strongly convergent to ζ in $L^1(I \times D)$ and weakly convergent in $L^2(I, H^r(D))$. Then*

$$\int_0^1 P_{h,r}(\zeta(t))dt \le \liminf\ \int_0^1 P_{h,r}(\zeta_n(t))dt.$$

We have

$$\int_0^1 P_{h,r}(\zeta(t))dt = \int_0^1 ||\zeta(t)||^2_{H^r(D)}dt - \int_0^1\int\int_{\{(x,y)\in D\times D,\, |x-y|>h\}} F(x,y)dxdydt.$$

The second term, correcting term, is continuous while the first one, being the square of a Hilbert norm, is l.s.c.

Proposition 5.3. *The mapping $\zeta \to \theta^p_{h,r}(\zeta)$ from $L^p(I, H^r(D))$ in R is weakly lower semicontinuous.*

Let ζ_n be a weakly convergent sequence to ζ, for each n let ν_n be a minimizing element in the closed convex set $K(\zeta_n)$, it remains bounded in $W^{1,p}(I)$ and we still denote by ν_n a weakly subsequence converging to ν in this linear space. From (3.12) the sequence ζ_n remain bounded in $L^2(I, H^r(D))$ and $\frac{\partial}{\partial t}\zeta_n$ remains bounded in $L^p(I, W^{-1,1}(D))$ and then, from Theorem 5.1, is strongly converges in $L^1(I \times D)$

to some element $\zeta = \zeta^2 \in C([0,1], L^1(D))$ and is also weakly converging to that element in $L^2(I, H^r(D))$ then, from Proposition 5.2, in the limit we get $\nu \in K(\zeta)$.

We consider a minimizing sequence $(\zeta_n, V_n) \in \mathbf{T}(\Omega_0, \Omega_1)$.there exists subsequences such that $V_n \rightharpoonup V$, weakly in $L^p(I \times D)$ and $\zeta_n \to \zeta$ strongly in $L^1(I \times D)$. Effectively as $(\zeta_n)_t = \mathrm{div}(\ -\zeta_n\ V_n\ )$, we have $p > 1$ and from previous estimate, with the notation $B_r(D) = H^r(D)$ for $0 < r < 1/2$ or $= BV(D)$ for $r = 0$, we get

$$||\zeta_n||_{L^1(I,B_r(D))} \leq\ M_1, \quad ||(\zeta_n)_t||_{L^p(I,W^{-1,1}(D))} \leq M_2.$$

The conclusion derives from the compactness result. From this strong L^1 convergence we derive that $\zeta^2 = \zeta$. We consider the weak formulation for the convection problem (1.1):

$$\forall \psi \in C^1(I \times \bar{D},\ R^N),\ \ \psi(0,.) = 0,$$

$$\int_0^1 \int_D \zeta_n\ (\ -\psi_t\ \ - \nabla\psi.V_n\ \ )\, dxdt = -\int_{\Omega_1} \psi(0,x)dx,$$

in which we can pass to the limit and we conclude that $(\zeta, V) \in \mathbf{T}(\Omega_0, \Omega_1)$. Moreover the element (ζ, V) is classically a minimizer as the two terms are weakly lower semi continuous respectively for each weak topologies, as we have

$$\int_0^1 \int_D \zeta(t,x)\, |V(t,x)|^p dxdt = \int_0^1 \int_D\ |\zeta(t,x)\, V(t,x)|^p dxdt.$$

And

$$\zeta_n\, V_n \ \ \text{weakly converges in } L^p(I \times D) \text{ to } \ \zeta\, V$$

Indeed, for any $\phi \in L^{p^*}(I \times D)$ we have $|\phi\,(\zeta_n - \zeta)|^{p^*} \leq 2^{p^*}\ |\phi|^{p^*} \in L^1(I \times D)$ while $\phi\,(\zeta_n(t,x) -\ \zeta(t,x)) \to 0,\ \ a.e.(t,x),$ so that $\phi\,\zeta_n \to \phi\,\zeta$ strongly in $L^{p^*}(I \times D)$. Now as V_n weakly converges to V we get

$$\int_0^1 \int_D \phi\ \ V_n\, \zeta_n\, dxdt\ \to\ \int_0^1 \int_D \phi\ \ V\zeta\, dxdt;$$

so that $V_n\, \zeta_n$ weakly converges in $L^p(I \times D)$ to $V\,\zeta$.

5.2. Convergence of Cauchy sequence

Consider a sequence of domains $\Omega_n \in \mathbf{B}(\Omega)$ such that $d^{p,\epsilon}_{h,r}(\Omega_p, \Omega_q) \to 0$ as $p, q \to \infty$. We consider $d^{p,\epsilon}_{h,r}(\Omega, \Omega_n) \leq d^{p,\epsilon}_{h,r}(\Omega, \Omega_{n_0}) + d^{p,\epsilon}_{h,r}(\Omega_{n_0}, \Omega_n) \leq M = 2d^{p,\epsilon}_{h,r}(\Omega, \Omega_{n_0})$ (For any n_0 large enough and $n \geq n_0$.) To begin with we obtain the existence of a minimizing elements $(\bar{\zeta}_n, \bar{V}_n)$ in $\mathbf{T}_r(\Omega, \Omega_n)$ and $\nu_n \in K(\zeta_n)$ which are uniformly bounded by M.

As $\nu_n(0) = P_{h,r}(\zeta_\Omega)$, we have ν_n bounded in $W^{1,p}(0,1) \subset C^0([0,1])$, then $\theta(\zeta_n) \leq M_1$ and then from (3.7), (3.12) we get $||\zeta_n||_{L^p(I,W^{r,p}(D))} \leq M$.

Then we have converging subsequences, still denoted ζ_n, V_n, ν_n to some element ζ, V, ν as $n \to \infty$. And as $\zeta = \zeta^2 \in C([0,1], L^1(D)) \cap L^2(0,1,H^r(D))$ we set $\zeta(1) = \zeta_{\Omega_\infty}$ where the measurable subset $\Omega_\infty \in \mathbf{B}(\Omega)$ is defined up to a zero measure subset in D and verify $\mathrm{meas}(\Omega_\infty) = a$.

5.3. Stability of geodesics

We consider now the perturbed set $\Omega^s = T_s(\Omega)$ and, for all s, $0 \le s < s^*$, the minimizing elements ζ^s, V^s, ν^s associated with the distance $d^{p,\epsilon}_{h,r}(\Omega^s, \Omega_1)$. We have $d^{p,\epsilon}_{h,r}(\Omega^s, \Omega_1) \le d^{p,\epsilon}_{h,r}(\Omega^s, \Omega) + d^{p,\epsilon}_{h,r}(\Omega, \Omega_1)$; the first term is uniformly bounded as $s \to 0$ from the smoothness of the vector field Z (and the possible use of its flow mapping $T_s(Z)$ for estimating this term). Then as previously we derive the convergence of ζ^s, V^s, ν^s to respective elements ζ, V, ν. As previously we get $\nu \in K(\zeta)$ and $(\zeta, V) \in \mathbf{T}_r(\Omega, \Omega_1)$ being a minimizer for the distance $d^{p,\epsilon}_{h,r}(\Omega, \Omega_1)$.

5.4. The Euler equation

5.4.1. Transverse field. Transverse field action preserving tubes and transverse tube analysis has been developed in [28], [29], [30], [32], [33], [14], in connection with optimization and optimal control in non cylindrical evolution problem (time-depending domain and geometry).

Let us consider a perturbation parameter $s \ge 0$ and any smooth horizontal non autonomous vector field over R^{N+1} (s being the evolution parameter for a dynamic in R^{N+1})

$$\mathbf{Z}(s,\ t, x) = (0,\ z(s,\ t, x)) \in R_t \times\ R^N, \quad \mathrm{div}_x\, z(s, t, .) = 0.$$

Such that

$$\mathbf{Z}(s,\ 0, x)\ =\ \mathbf{Z}(s,\ 1, x)\ =\ 0 \tag{5.2}$$

5.5. Transverse perturbed tube

For any element $(\zeta, V) \in \mathbf{T}(\Omega_0, \Omega_1)$, we consider the perturbed tube $(\zeta^s,\ V^s)$ where

$$\zeta^s(t, x) := \zeta o T_s(\,\mathbf{Z})(x))^{-1},$$

and

$$V^s(t, x)\ =\ (D[T_s(\mathbf{Z})^{-1}])^{-1}.\left(\ V(t) o T_s(\mathbf{Z})^{-1}\ -\ \frac{\partial}{\partial t}(T_s(\mathbf{Z})^{-1})\right). \tag{5.3}$$

Notice that $(D[T_s(\mathbf{Z})^{-1}])^{-1} = D(T_s(\mathbf{Z})) o T_s(\mathbf{Z})^{-1}$, so that

$$V^s(t, x) o T_s(\mathbf{Z})\ =\ D(T_s(\mathbf{Z})).V(t)\ +\ D(T_s(\mathbf{Z}))^{-1}.\frac{\partial}{\partial t}(T_s(\mathbf{Z})).$$

From classical calculus, see [4], [21], [10], [11], [5], using the strong flow mapping $T_s(\mathbf{Z})$ we get the following stability result for the connecting family:

Theorem 5.4. *Let be given $z \in C^0([0,\ s_1] \times [0,\ 1], C^1(\bar{D}, R^N))$, $z(s, t).n = 0$, on ∂D and Ω a measurable subset in D. Consider any pair Ω^i, $i = 0, 1$ in $\mathbf{B}(\Omega)$, then, with $\mathbf{Z} = (0, z)$, we have:*

$\forall(\zeta, V) \in \mathbf{T}(\Omega_0, \Omega_1)$, *the elements* (ζ^s, V^s) *defined at* (5.3) *verifies:* $(\zeta^s, V^s) \in \mathbf{T}(\Omega_0, \Omega_1)$

Remarks

1) this stability property does not require the function ζ to be a characteristic function. This property still hold true for example for probability measures.
2) As $V \in H_0^p$, the moving domain verifies $\text{meas}(\Omega_t) = \int_D \zeta(t,x)dx = a$ and the s-perturbed moving domain Ω_t^s such that $\chi_{\Omega_t^s} = \zeta(t)oT_s(\mathbf{Z}(t))^{-1}$ (or equivalently $\Omega_t^s = T_s(\mathbf{Z}(t))(\Omega_t)$), verifies $\text{meas}(\Omega_t^s) = a > 0$ if $\text{div}_x\, z(s,t,.) = 0$ in D.

5.6. Euler equation solved by the minimizer

In order to analyse the necessary conditions associated with any minimizer of $\mathbf{E}^p$ over the set $\mathbf{T}(\Omega_0, \Omega_1)$ we introduce transverse transformations of the tube. Without any loss of generality and in order to simplify the calculus we consider here the specific quadratic situation:

5.6.1. Transverse derivative, quadratic case ($p = 2$). Assume that $\text{div}_x\, \mathbf{Z}^t = 0$, then

$$\int_D (\alpha\zeta^s(t,x) + \beta)\, |V^s(t,x)|^2\, dx = \int_D (\alpha\zeta(t,x) + \beta)\, |V^s(t)oT_s(\mathbf{Z}^t)(x)|^2\, dx.$$

So that the optimality of the element (ζ, V) writes:

$$1/s\ (\ \mathbf{E}(\zeta^s,\, V^s oT_s) - \mathbf{E}(\zeta,\, V)\ )\ \geq\ 0.$$

Now the following quotient has a strong limit in $L^2(I \times D)$:

$$\begin{aligned}
\frac{V^s oT_s - V}{s} &= \frac{d}{ds}[V^s oT_s(\mathbf{Z}^t)\,]_{s=0} \\
&= \frac{d}{ds}\left[(D(T_s(\mathbf{Z}^t)^{-1})^{-1}.\left(\ V(t)\ -\ \frac{\partial}{\partial t}(T_s(\mathbf{Z}^t)^{-1}\,)oT_s(\mathbf{Z}^t)\right)\right]_{s=0} \\
&= \frac{d}{ds}\left[\,(D(T_s(\mathbf{Z}^t)oT_s(\mathbf{Z}^t)^{-1}.\left(\ V(t)\ -\ \frac{\partial}{\partial t}(T_s(\mathbf{Z}^t)^{-1}\,)oT_s(\mathbf{Z}^t)\right)\right]_{s=0} \\
&= \frac{\partial}{\partial t}Z(t)\ +\ DZ(t).V(t)\ \in L^2(I \times D, R^N).
\end{aligned}$$

Where we always denote $Z(t)(x) = Z(t,x) := \mathbf{Z}^t(0,x)$ (that is at $s = 0$). Indeed we know that if V was smoother, say $V \in L^2(H^1(\Omega))$ we would have:

$$\frac{\partial}{\partial s}[V^s\,]_{s=0} = \ Z_t\ + [Z(t), V(t)]\ :=\ H_V.Z.$$

Where the Lie bracket is $[Z,V] = DZ.V - DV.Z$, so we would get the previous expression for the derivative of $V^s oT_s(\mathbf{Z}^t)$, as $(V^s oT_s)_s = (V^s)_s + DV^s.DZ(t)$. This analysis in strong form is used in the non cylindrical shape analysis (or dynamical domains analysis) in several previous works, see for example [3], [6], [33], [7], [8].

5.6.2. Quadratic term $\mathbf{E^2}$ ($\boldsymbol{p=2}$). As

$$\int_0^1\int_D((\alpha\zeta^s+\beta)\,|V^s|^2\,-\,(\alpha\zeta+\beta)|V|^2)/s\;dxdt$$
$$=\int_0^1\int_D((\alpha\zeta+\beta)\,(|V^s oT_s|^2\,-\,|V|^2)/s\;dxdt$$
$$=\int_0^1\int_D((\alpha\zeta+\beta)\,(V^s oT_s+V)\,(V^s oT_s-V)/s\;dxdt$$
$$\to 2\int_0^1\int_D((\alpha\zeta+\beta)\,V.\left(\frac{\partial}{\partial t}Z(t)\,+\,DZ(t).V(t)\right)\,dxdt$$
$$=-2<\frac{\partial}{\partial t}((\alpha\zeta+\beta)\,V\;)\,+\,\vec{\mathrm{div}}(\,(\alpha\zeta+\beta)\,V).V\,)\;,\;Z\;>_{\mathbf{D'\times D}}$$

where, as $\operatorname{div} V=0$,

$$\vec{\mathrm{div}}(\,(\alpha\zeta+\beta)\,V).V)_i=\text{“}D(\,(\alpha\zeta+\beta)\,V).V\text{”}_i$$
$$=\partial_j(\;(\alpha\zeta+\beta)\;V_i\,V_j\;)\in W^{-1,1}(D)=\partial_j(\;(\alpha\zeta+\beta)\;V_i\,)\;V_j.$$

5.6.3. Transverse field preserving time mean value of the perimeter.

Proposition 5.5.

$$\forall Z(0,t,x)\in C([0,1],W_0^{1,\infty}(D)),\;\; s.t.\,,$$
$$\int_0^1\langle\vec{\mathbf{H}}_{h,r}(\partial\Omega_t)),Z(0,t)\rangle_{\mathbf{M}(D,R^N)\times C_c(D,R^N)}\,dt\;=0$$
$$\exists s^*>0,\;\exists Z(s,t,x)\in C([0,s^*[,C([0,1],W_0^{1,\infty}(D)))$$
$$\textit{such that, with}\;\;\Omega_s^t=T_s(Z(t))(\Omega_t),$$
$$\forall s,\;0\le s<s^*,\;\int_0^1\langle\vec{\mathbf{H}}_{h,r}(\partial\Omega_s^t),Z(s)\rangle_{\mathbf{M}(D,R^N)\times C_c(D,R^N)}\,dt\;=0.$$

Where we denote by $Z(t)$ the mapping $(s,x)\to Z(s,t,x)$ whose flow mapping builds the perturbed tube ζ^s;

Then for any such Z we have

$$\forall s,\;\;K^p_{h,r}(\zeta^s)=K^p_{h,r}(\zeta),$$

then we get

$$\frac{\partial}{\partial s}\theta^p_{h,r}(\zeta^s)=0.$$

5.6.4. Transverse derivative for $\boldsymbol{\theta^p_{h,r}}$. As the convex constraint K in the definition of $\theta^p_{h,r}$ is so simple we can easily verify that this term is itself differentiable in the direction of any smooth transverse vector field Z. Let us consider the function

$$\nu_0(t)=6\;t(1-t),\quad\text{verifying}\;\int_0^1\nu_0(t)dt\;=\;1,\;\;\nu_0(0)=\nu_0(1)=0.$$

With $a(s) = \int_0^1 P_{h,r}(\zeta^s(t))\,dt$ we have

$$\theta^p_{h,r}(\zeta^s) = \mathrm{Inf}\left\{ \int_0^1 |\nu'(t)|^p\,dt \;\mid\; \int_0^1 \nu(t)dt \,\geq\, a(s) \right\}.$$

Then

$$\theta^p_{h,r}(\zeta^s) = \mathrm{Inf}\left\{ \int_0^1 |\nu'(t)|^p\,dt \;\mid\; \int_0^1 (\nu(t) - a(s)\nu_0(t))dt \,\geq\, 0 \right\}.$$

Setting $w(s,t) = \nu(t) - a(s)\nu_0(t)$, we get

$$\theta^p_{h,r}(\zeta^s) = \mathrm{Inf}\left\{ \int_0^1 \left| \frac{\partial}{\partial t} w(s,t) + a(s)\nu_0'(t)|^p\,dt \right| \int_0^1 w(t)dt \,\geq\, 0 \right\}$$

the convex constraint is not depending on the parameter s then it is classical that this minimum (uniquely achieved on the convex set) is differentiable with respect to the parameter s, and at $s = 0$, for example with $p = 2$, we get

$$\frac{\partial}{\partial s}\theta^2_{h,r}(\zeta^s)|_{s=0} = \int_0^1 \left(\frac{\partial}{\partial t} w^*(0,t) + a(0)\nu_0'(t) \right)\, \dot{a}(0)\nu_0'(t)\,dt.$$

Where $w^* = \nu^* - a(0)\nu_0$ is the optimal solution while

$$\dot{a}(0) = \frac{\partial}{\partial s}\int_0^1 P_{h,r}(\zeta^s)|_{s=0} = \int_0^1 \langle \vec{\mathbf{H}}_{h,r}(\Gamma_t), Z(0,t)\rangle dt.$$

So that

$$\frac{\partial}{\partial s}\theta^2_{h,r}(\zeta^s)|_{s=0} = \sigma \;\int_0^1 \langle \vec{\mathbf{H}}_{h,r}(\Gamma_t), Z(0,t) rangledt,$$

where

$$\begin{aligned} \sigma_2 &= \int_0^1 \frac{\partial}{\partial t}\nu^*(t)\;\nu_0'(t)\,dt \\ &= 5\,P_{h,r}(\zeta(1)) - 6\,P_{h,r}(\zeta(0)) + 12\int_0^1 \nu^*(t)dt. \end{aligned}$$

And similarly we would get the expression for σ_p.

5.6.5. Variational solution to incompressible Euler-convection problem. We have

Theorem 5.6. *Let Ω be any given element in $\mathbf{B}$. Then any minimizer (ζ, V) to the functional $\mathbf{E}^2$ over the family of tubes $\mathbf{T}(\Omega_0, \Omega_1)$ solves the following problem:*

$$\frac{\partial}{\partial t}\zeta + \nabla\zeta.V = 0,\;\; \zeta(0) = \chi_{\Omega_0},\;\; \zeta(1) = \chi_{\Omega_1} \tag{5.4}$$

$$\mathrm{div}\,V = 0,\;\; \zeta = \zeta^2 \tag{5.5}$$

$$\exists \mathbf{P}\;s.t.\;\frac{\partial}{\partial t}((\alpha\zeta + \beta)\,V) + D((\alpha\zeta + \beta)\,V).V + \nabla\mathbf{P} = \vec{\mathbf{H}}_{h,r}. \tag{5.6}$$

Remark, see [25], equation (5.6) writes

$$(\alpha\zeta + \beta)\left(\frac{\partial}{\partial t}V + DV.V\right) + \nabla\mathbf{P} = 1/2\,\vec{\mathbf{H}}_{h,r}. \tag{5.7}$$

More generally we have:

Theorem 5.7. *Let Ω be any given element in $\mathbf{B}$. Then any minimizer (ζ, V) to the functional $\mathbf{E}^p$ over the family of tubes $\mathbf{T}(\Omega_0, \Omega_1)$ solves the following problem:*

$$\frac{\partial}{\partial t}\zeta + \nabla\zeta.V = 0,\ \ \zeta(0) = \chi_{\Omega_0},\ \ \zeta(1) = \chi_{\Omega_1} \tag{5.8}$$

$$\mathrm{div}\, V = 0,\ \ \zeta = \zeta^2 \tag{5.9}$$

$$\exists\mathbf{P}\ s.t.\ \frac{\partial}{\partial t}(\,(\alpha\zeta+\beta)||V||^{p-2}\,V) + D(\,(\alpha\zeta+\beta)||V||^{p-2}\,V).V + \nabla\mathbf{P} = 1/p\,\vec{\mathbf{H}}_{h,r}. \tag{5.10}$$

5.6.6. *h*-perimeter in E. In the interesting case where $\mathbf{H}^r(D) = H^r(D)$ we consider, for any given "small" $h > 0$ the $L^1(I)$ norm of the perimeter:

$$p_{h,r}(\zeta) := \int_0^1 \left(\int_{D\times D} \rho_h(||x-y||)\ \frac{|\,\zeta(x)-\zeta(y)\,|}{||x-y||^{N+2r}}\,dxdy\right) dt. \tag{5.11}$$

So that it is enough to chose the surface tension term in the form $\sigma\, p_h(\zeta)$. This term turns to be always differentiable with respect to the transverse perturbations as follows:

$$p_{h,r}(\zeta o T_s(\mathbf{Z})^{-1})$$
$$= \int_0^1\int_{D\times D} \rho_h(||T_s(\mathbf{Z})(x) - T_s(\mathbf{Z})(y)||)\,\frac{|\zeta(x)-\zeta(y)|}{||T_s(\mathbf{Z})(x) - T_s(\mathbf{Z})(y)||^{N+2r}}\,dxdydt.$$

So that a.e. t in I we have

$$\frac{\partial}{\partial s}p_{h,r}(\zeta^s(t))_{s=0} \tag{5.12}$$
$$= \int_{D\times D} \rho_h(||x-y||)\,\frac{|\zeta(x)-\zeta(y)|}{||x-y||^{N+2r}}\,\langle\frac{x-y}{||x-y||}, \frac{Z(t,x)-Z(t,y)}{||x-y||}\rangle dxdy$$
$$+ \int_{D\times D} \rho_h'(||x-y||)\,\frac{|\zeta(x)-\zeta(y)|}{||x-y||^{N+2r}}\langle x-y, Z(t,x)-Z(t,y)\rangle dxdy.$$

As $||x-y|| \le h$ in the previous integrals we have:

$$Z(t,x) - Z(t,y) = DZ(t,x) + \delta(t)(y-x)).(y-x)$$

there exists a measure $\vec{\mathbf{H}}_{h,r}(\Gamma(t))$ supported by

$$\Delta_h(\Sigma) = \cup_{0<t<1}\,\{t\} \times\ (\cup_{x\in\partial\Omega_t} B(x,h)\ )$$

such that

$$\langle\vec{\mathbf{H}}_{h,r},\ Z\rangle = \frac{\partial}{\partial s}P_{h,r}(\zeta^s(t))_{s=0}.$$

In some sense, when $h \to 0$, the measure converges to the mean curvature of the moving boundary Γ_t.

5.7. s-transverse derivative

ζ being a tube and $\mathbf{Z}(s,t,x)$ being a smooth *horizontal* vector field (i.e., $\mathbf{Z}(s,t,x) = (0, Z(s,t,x)) \in R^{N+1}$), we consider the derivative of the capacity term $\theta^p_{h,r}(\zeta^s)$ where $\zeta^s(t) = \zeta(t) o T_s(Z(t))^{-1}$ is the transversely s-perturbed tube. The important point is that the (h,r)-perimeter of this tube ζ^s is not differentiable with respect to t as the vector field V is not smooth but it will be differentiable with respect to the perturbation parameter s as the vector field Z, as a "test" function, is smooth.

An important point is that $Z(s,0,x) = 0$ and $Z(s,1,x) = 0$ are independent on s so that the inequalities in the convex definition are not perturbed by the parameter s (as $P_{h,r}(\zeta(i)) = P_{h,r}(\Omega_i),\ i = 0,1$). We consider

$$a^s(t) = |\Omega^s_t|_{\mathrm{loc}(h,r)},$$

$$a^s(t) = \int\int_{D\times D} \frac{\rho_h(|T_s(Z(s,t))(x) - T_s(Z(s,t))(y)|}{|T_s(Z(s,t))(x) - T_s(Z(s,t))(y)|^{N+r}}\ |\zeta(t,x) - \zeta(t,y)|\ dxdy.$$

And then $a^s(i)$ is independent on s; $i = 1,2$, $a^s(i) = P_{h,r}(\Omega_i)$.

At $s = 0$ we have:

$$\frac{\partial}{\partial s}(||T_s(Z(s,t))(x) - T_s(Z(s,t))(y)||)_{s=0} = \langle \frac{x-y}{||x-y||},\ Z(x) - Z(y)\rangle.$$

So that we get

$$\begin{aligned}\frac{\partial}{\partial s}\left[\int_0^1 a^s dt\right]_{s=0} &= \int_0^1 \Bigg[\int\int_{D\times D} \Bigg\{\ \nabla\rho_h(|x-y|)\left\langle \frac{x-y}{|x-y|}, Z(x) - Z(y)\right\rangle \\ &\qquad -\alpha\rho_h(|x-y|)\left\langle \frac{x-y}{|x-y|}, \frac{Z(x)-Z(y)}{|x-y|}\right\rangle\Bigg\} \\ &\qquad \times \frac{|\zeta(t,x) - \zeta(t,y)|}{|x-y|^{N+\alpha}}\ dxdy\Bigg]\ dt \\ &= \int_0^1 \langle \vec{\mathbf{H}}_{h,r}(\partial\Omega_t), Z(0,t)\rangle\ dt.\end{aligned}$$

6. Asymptotic analysis

An important issue is the asymptotic analysis when $\alpha + \beta \to 0$, see [20]. For $p = 1$ the vector field just appears through the speed boundary element:

$v(t) = \langle V(t), n_t\rangle$ on $\partial\Omega_t$, so that

$$\left\|\frac{\partial}{\partial t}\zeta\right\|_{L^1(I,M^1(D))} = \int_0^1\int_{\Omega_t} |v(t,x)|\, d\Gamma_t(x)dt.$$

So that the metric takes the following intrinsic form: the Eulerian vector field is no more necessary (in the limit it would solve, formally, some eikonal equation). We simply consider the set of characteristic functions

$$\mathbf{C}\ =\ \{\ \zeta = \zeta^2\ \in L^1(I\times D)\ \},\quad \mathbf{C}^0 = \mathbf{C}\cap C^0(I, L^1(D)) \tag{6.1}$$

the family of *connecting tubes*

$$\mathbf{T}^0(\Omega_0,\Omega_1) = \{\zeta \in \mathbf{C}^0 \ s.t. \ \zeta(i) = \chi_{\Omega_i}, \ i = 0,1 \ \}. \tag{6.2}$$

Considering the Banach space of bounded measure $M^1(D)$ we set

$$p \geq 1, \ \ \mathbf{C}^p \ = \left\{ \ \zeta \in \mathbf{C} \ s.t. \ \frac{\partial}{\partial t}\zeta \in L^p(I, M^1(D)) \right\}, \tag{6.3}$$

that is

$$\mathbf{C}^p = \mathbf{C} \cap L^p(I, BV(D)) \subset C^0(I, L^1(D)) \tag{6.4}$$

$$p \geq 1, \ \ \mathbf{C}^p = \left\{ \ \zeta \in \mathbf{C}^0 \ s.t. \ \frac{\partial}{\partial t}\zeta \in L^p(I, M^1(D)) \right\}. \tag{6.5}$$

Corollary 6.1. *Let $p \geq 1$, then*

$$d_p(\Omega_0,\Omega_1) = \text{Inf}_{\{\zeta \in \mathbf{C}^p, \, \zeta(i)=\chi_{\Omega_i} \ \}} \int_0^1 \left\| \frac{\partial}{\partial t}\,\zeta(t) \right\|^p_{M^1(D)} dt \tag{6.6}$$

is a quasi metric. When $p = 1$, d_1 is a metric.

In level set representation, let $\Omega_i = \{x \in D, \ \ \phi_i(x) > 0 \ \}$ then the moving domain Ω_t is search in the form $\Omega_t = \{x \in D, \ \ \phi(t,x) > 0 \ \}$ for some smooth function ϕ verifying the connection property: $\phi(i,x) = \phi_i(x), \ \ i = 1,2$ and it turns that

$$\left\| \frac{\partial}{\partial t}\,\zeta(t) \right\|_{M^1(D)} = \int_{\{x \in D, \ \phi(t,x)=t\}} \frac{\partial}{\partial t}\phi(t,x)| \ ||\nabla \phi(t,x)||^{-1} d\Gamma_t(x).$$

Using an “ad hoc” Galerkin approximation we obtain geodesic connecting domains with different topologies.

6.1. Conclusion: the morphic metric extends to images

We derived several complete quasi metrics associated with small parameters, $\epsilon > 0$ associated to the triangle inequality verified up to the factor 2^ϵ, and (h,r) associated to the local scaling when describing the curvature of the moving boundary in the geodesics. With $\epsilon = 0$ and the Soblev analysis replaced by the usual BV space we got a metric but not a complete metric space.

An extension of that morphic metric is for monochromatic images. This topic is developed in a forthcoming paper and we describe it formally. The basic idea is to consider the previous morphic metric on each level set of the image u. More precisely

Let $u \in L^1(D)$ we consider the monotone rearrangement

$$u^\#(t) \ = \text{meas}(\{y \in D \, s.t. \, u(y) < t \ \} \,)$$

and

$$\beta_{[u]} \in L^\infty(D), \ \ \ \beta_{[u]}(x) = u^\#(\,u(x)\,) = \text{meas}(\{y \in D \, s.t. \, u(y) < u(x) \ \} \,)$$

That is:

$$\beta_{[u]} = u^\# \circ u.$$

There exists a monotone function $u^* : [0, |D|] \longrightarrow R$, such that

$$u = u^* \circ \beta_{[u]}.$$

The morphic image distance consists in considering separately the two functions and set

$$d(u^1, u^2) := \text{Inf}_{(\beta,V)\in T(\beta^1,\beta^2)} \int_0^1 ||V(t)||^p \, dt \;)^{\frac{1+\epsilon}{p}} + \int_0^1 ||\beta(t)||_{BV(D)} dt + ||(u^1)^* - (u^2)^*||_{L^2(R)}.$$

The main property is that

$$\beta_{[\beta_{[u]}]} = \beta_{[u]}.$$

We shall always denote by β a function defined on D, verifying $\beta = \beta_{[\beta]}$, so that $0 \leq \beta \leq |D| = \text{meas}(D)$. And we denote by $\mathbf{\Theta}(D)$ the non convex set of such functions:

$$\mathbf{\Theta}(D) = \{\beta \in L^\infty(D),\, 0 \leq \beta \leq |D|, \quad \beta = \beta_{[\beta]} \;\}$$

The set $\Theta(D)$ has the following stability property: if $\beta \in \Theta(D)$ then $\beta \circ T_t \in \Theta(D)$ for any flow mapping T_t of any free divergence vector field V on D, more precisely:

Lemma 6.2. *Let* $\text{div}\, V(t) = 0$ *then*

$$\beta(u \circ T_t(V)) = \beta(u) \circ T_t(V), \;\; \beta(u \circ T_t(V)^{-1}) = \beta(u) \circ T_t(V)^{-1}.$$

6.1.1. $\boldsymbol{\beta}$-tube analysis. We designate by $\beta_{[u]}$ (or β_u, or simply β) the function associated with $u \in L^1(D)$.

The *Eulerian approach* consists in considering the connecting β-tubes $(\beta(t), V(t))$ as solutions to the weak convection (6.7) associated to a free divergence speed vector field V. Then we can repeat the previous analysis with couples (β, V) verifying

$$\beta_{[\beta]} = \beta, \;\; \frac{\partial}{\partial t}\beta + \nabla\beta.V = 0, \;\; \beta(i) = \beta_{[u^i]} \;\; i = 1, 2\;. \tag{6.7}$$

Such a solution $\beta(t)$ is continuous from I in $L^2(D)$ equipped with its weak topology.

The analysis of the steps of $\beta(t)$ is an important issue. They are enumerable and we show that if $\beta_{[u^i]}$ have no step (a function $f \in L^1(D)$ has a step r if $\text{meas}(\{x \in D \; s.t. \; f(x) = t\;\}) = 0$), then for all $t \in I$, $\beta(t)$ has none too. The morphic part of the image analysis relies on the morphic metric on the elements $\beta_{[u^i]}$.

References

[1] Delfour, Michel C. and Zolésio, Jean-Paul. Oriented distance function and its evolution equation for initial sets with thin boundary. SIAM J. Control Optim. 42 (2004), no. 6, 2286–2304

[2] M. Cuer and J.-P. Zolésio. Control of singular problem via differentiation of a min-max. Systems Control Lett. 11 (1988), no. 2, 151–158.

[3] R. Dziri and J.-P. Zolésio. Dynamical Shape Control in Non-cylindrical Navier-Stokes Equations. J. convex analysis, vol. 6, 2, 293–318, 1999.

[4] J. Sokolowski and J.-P. Zolésio. Introduction to shape optimization, sci, 16, Springer-Verlag, Heidelberg, N.Y., 1991

[5] B. Kawohl, O. Pironneau, L. Tartar, J.-P. Zolésio. Optimal Shape Design, Lecture Notes in Mathematical, 1740, Springer-Verlag, Heidelberg, N.Y., 1998

[6] P. Cannarsa, G. Da Prato and J.-P. Zolésio. The damped wave equation in a moving domain, Journal of Differential Equations, 1990, 85, 1–16.

[7] G. Da Prato and J.-P. Zolésio. Dynamical programming for non Cylindrical Parabolic Equation, Sys. Control Lett., 11, 1988.

[8] G. Da Prato and J.-P. Zolésio. Existence and Control for wave equation in moving domain, L.N.C.I.S Springer-Verlag, 144, 1988.

[9] L. Ambrosio. Lecture notes on optimal transport problems. Mathematical aspects of evolving interfaces (Funchal, 2000), 1–52, Lecture Notes in Math., 1812, Springer, Berlin, 2003. 49Q20 (49-02)

[10] F.R. Desaint and J.-P. Zolésio. *Manifold derivative in the Laplace-Beltrami equation. Journal of Functional Analysis*, 151(1): 234, 269, 1997.

[11] M. Delfour and J.-P. Zolésio. Shape and Geometry Advances in Design and Control, 04, SIAM, 2001, 2nd edition 2010.

[12] Louis Blanchard and Jean-Paul Zolesio. Galerkin Strategy for Level Set Shape Analysis: Application to Geodesic Tube. In "System Modeling and Optimization with Applications". Adam Korytowski, Kazimierz Malanowski, Wojciech Mitkowski and Maciej Szymkat eds. Springer-Verlag, IFIP series, 2010; pp. 183–200.

[13] L. Blanchard, J.P. Zolesio. Moving domain by Galerkin-Level Set strategy: Application to shape geodesics. enumath proc., 2006.

[14] M. Moubachir and J.-P. Zolésio. Moving Shape Analysis and Control: application to fluid structure interaction. Pure and Applied Mathematics series, CRC, 2006.

[15] Michel C. Delfour and Jean-Paul Zolésio. The new family of cracked sets and the image segmentation problem revisited.

[16] Michel C. Delfour and Jean-Paul Zolésio. Shape identification via metric constructed from oriented distance function. *Control and Cybernetic*, 34(1):137–164, 2005. *Commun. Inf. Syst.*, 4(1):29–52, 2004.

[17] Michel C. Delfour and Jean-Paul Zolésio. Shape analysis via oriented distance functions. *Journal of Functional Analysis*, 123, 1994.

[18] Delfour, Michel C. and Zolesio, Jean-Paul. Uniform fat segment and cusp properties for compactness in shape optimization. Appl. Math. Optim. 55 (2007), no. 3, 385–419.

[19] A.M. Micheletti. Metrica per famigilie di domini limitati e proprietà generiche degli autovalori, Ann. Scuola Norm. Sup. Pisa Ser. III, 26 (1972), 683–684.

[20] J.P. Zolesio. Complete Shape Metric and Geodesic. In "System Modeling and Optimization with Applications". Adam Korytowski, Kazimierz Malanowski, Wojciech Mitkowski and Maciej Szymkat, eds. Springer-Verlag, IFIP series, March 2009; pp. 155–179.

[21] Jean-Paul Zolésio. Introduction to shape optimization and free boundary problems. In Michel C. Delfour, editor, *Shape Optimization and Free Boundaries*, volume 380 of *NATO ASI, Series C: Mathematical and Physical Sciences*, pages 397, 457, 1992.

[22] ___ . Shape Topology by Tube Geodesic. In Information processing: Recent Mathematical Advances in Optimization and Control. Presses de l'Ecole des Mines de Paris, pages 185–204, 2004.

[23] ___ . Control of Moving Domains, Shape Stabilization and Variational Tube Formulations. International Series of Numerical Mathematics, Vol. 155, Birkhäuser Verlag, Basel, pages 329–382, 2007.

[24] ___ . Tubes Analysis. Lecture notes in pure and applied mathematics; v. 252 Boca Raton London New York, Chapman-Hall/CRC, 2007.

[25] ___ . Variational Formulation for Incompressible Euler equation by Weak Shape evolution, Intern. Series of Num. Math., vol. 133, 309–323, Birkhäuser, 1999.

[26] ___ . Control of Moving Domains, Shape Stabilization and Variational Tube Formulation, Intern. Series of Num. Math., vol. 153, 329–382, Birkhäuser, 2007.

[27] ___ . Variational Principle in the Euler Flow. In G. Leugering, editor, *proceedings of the IFIP-WG*7.2 *conference, Chemnitz*, volume 133 of *Int. Series of Num. Math.*, 1999.

[28] ___ . Shape Differential with Non Smooth Field. In Computational Methods for Optimal Design and Control. J. Borggard, J. Burns, E. Cliff and S. Schreck, eds., volume 24 of *Progress in Systems and Control theory*, pp. 426–460, Birkhäuser, 1998.

[29] ___ . Weak set evolution and variational applications. In *Shape optimization and optimal design*, lecture notes in pure and applied mathematics, vol. 216, pp. 415–442, Marcel Dekker, N.Y., 2001.

[30] ___ . Set Weak Evolution and Transverse Field, Variational Applications and Shape Differential Equation. INRIA report RR-464, 2002. (http://www-sop.inria.fr/rapports/sophia/RR-464).

[31] ___ . Optimisation de domaine par déformation. Thèse de doctorat d'état. University of Nice, 1979.

[32] Dziri, R. and Zolésio, J.-P. Dynamical shape control in non-cylindrical hydrodynamics, Inverse problem, 1999, 15, 1, 113–122.

[33] Dziri, R. and Zolésio, J.-P. Eulerian derivative for Non-Cylindrical shape functionals. Lecture notes in pure and applied mathematics; v. 252 Boca Raton London New York, Chapman-Hall/CRC, 2007.

Jean-Paul Zolésio
CNRS-INLN
1361 Routes des Lucioles, Sophia Antipolis
F-06560 Valbonne, France

and

INRIA
Route des Lucioles – BP 93
F-06902 Sophia Antipolis Cedex, France
e-mail: jean-paul.zolesio@inln.cnrs.fr